STOCHASTIC DIFFERENTIAL EQUATIONS ON NONCOMMUTATIVE L^2

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ABSTRACT. We prove that a class of stochastic differential equations with multiplicative noise has a unique solution in a noncommutative L^2 space associated with a von Neumann algebra. As examples we consider usual L^2 on a measure space, Hilbert-Schmidt operators and a hyperfinite II_1-factor. A problem of finding an inverse of the solution is then discussed. Finally, we explain how a stochastic differential equation can be used to construct a heat kernel measure on an infinite dimensional group.

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1. INTRODUCTION

The main goal of this paper is to describe how a certain class of stochastic differential equations can be solved on noncommutative L^2 spaces. Our results concern two different fields of mathematics: von Neumann algebras and stochastic differential equations on infinite dimensional spaces.

In Section 2 we give an overview of noncommutative L^p spaces. These spaces are completions of von Neumann algebras with respect to a norm induced by a trace. The idea of such spaces can be traced back to I.E.Segal in [12] for L^2 and L^1, and then extended and simplified by E.Nelson in [11]. Later U. Haagerup defined more general noncommutative L^p spaces (for example, in [9]). In this paper we will use construction which is closer to Segal's original one. These abstract completions can be realized as (unbounded) operators, and one of these realizations can be used to describe examples of noncommutative L^p spaces in Section 2. In particular, the standard L^p on a measurable space can be viewed as a particular case of this construction for an Abelian von Neumann algebra. Another example is the space

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of Hilbert-Schmidt operators realized as a noncommutative L^2 space with respect to the ordinary trace.

The fact that a noncommutative L^2 space is a Hilbert space allows us to use stochastic differential equations in a Hilbert space. These equations are with multiplicative noise, though in the case of the ordinary trace we consider an equation with both multiplicative and additive noises. We refer readers to [2] for a comprehensive review of what is known about stochastic differential equations in infinite dimensions. The main difference from the case of a general Hilbert space is that we use the operator composition as a multiplication on noncommutative L^p spaces. In general a Hilbert space is not an algebra, therefore the stochastic differential equations we consider are different from ones in [2]. One of applications of stochastic differential equations on noncommutative L^2 spaces is the construction of a heat kernel measure carried out by the author in [4], [5], [6]. This heat kernel measure lives in some infinite dimensional Lie group, and its Lie algebra is a subspace of a certain noncommutative L^2.

Acknowledgement. The subject of this paper combines two different branches of mathematics: noncommutative integration and infinite dimensional stochastic analysis. Both of these topics appeared in L.Gross' work. I thank him for introducing me to the field and encouraging my efforts. I also thank B. Driver for his help throughout the process of preparation of this work and an anonymous referee for useful suggestions.

2. Noncommutative L^p spaces

Let H be a separable Hilbert space, and $B(H)$ be the space of bounded linear operators on H. Denote by M a von Neumann algebra on the Hilbert space H, that is, $M \subseteq B(H)$ is a C^*-algebra closed in the weak-operator topology and contains the identity operator I. **The commutant** M' of M is the set of those elements of $B(H)$ that commute with all elements of M.

Now let us define a trace on M^+, the set of all positive elements of M.

Definition 2.1. Let M be a von Neumann algebra.

(1) A **trace** τ on M^+ is a function $\tau : M^+ \to [0, \infty]$ such that
   (a) $\tau(A + B) = \tau(A) + \tau(B)$ for any $A, B \in M^+$.
   (b) $\tau(cA) = c\tau(A)$ for any $0 \leqslant c \leqslant \infty$ and $A \in M^+$ with the usual convention $0\infty = 0$.
   (c) $\tau(UAU^{-1}) = \tau(A)$ for any unitary operator U in M and $A \in M^+$. This condition is equivalent to the usual trace property of being central $\tau(AB) = \tau(BA)$ for any $A, B \in M^+$.
(2) A trace τ is called **faithful** if $A \in M^+$, $\tau(A) = 0$ imply $A = 0$.
(3) A trace τ is called **finite** if $\tau(A) < \infty$ for any $A \in M^+$.
(4) A trace τ is called **normal** if for any $A \in M^+$ and any increasing net A_α converging to A in the strong operator topology $\tau(A_\alpha) \to \tau(A)$.
(5) A trace τ is called **semifinite** if $\tau(A) = \sup\{\tau(B) : B \in M^+, B \leqslant A, \tau(B) < \infty\}$ for any $A \in M^+$.

Throughout this paper we assume that the von Neumann algebra M is equipped with a faithful normal semifinite trace τ. Let

$$\|A\|_p = (\tau((A^*A)^{p/2}))^{1/p}$$

for $1 \leqslant p < \infty$. Then we can define a noncommutative L^p-spaces in the following way.

Definition 2.2. For $1 \leqslant p < \infty$ we denote by $L^p(M, \tau)$ the Banach space completion of the two-sided ideal $I = \{A \in M : \|A\|_1 < \infty\}$. The algebra M is equipped with the norm $\|\cdot\|_\infty = \|\cdot\|$, the operator norm.

Elements of the space $L^p(M, \tau)$ may be identified with certain (unbounded) operators. In what follows $\mathcal{D}(A)$ denotes the domain of an operator A on H.

Definition 2.3.
(1) A linear operator A on H is called **affiliated with** M if $UA \subseteq AU$ for any unitary $U \in M'$. We will write $A \ \eta \ M$ if A is affiliated with M. The set of all closed densely defined operators affiliated with M will be denoted by $\overline{M}$.
(2) A subspace S of H is called **τ-measurable** if for any $\varepsilon \geqslant 0$ there exists a projection $P \in M$ such that

$$PH \subseteq S, \ \tau(P^\perp) \leqslant \varepsilon.$$

(3) An operator $A \in \overline{M}$ on H is called **τ-measurable** or **strongly measurable** if $\mathcal{D}(A)$ is τ-dense. The set of all τ-measurable operators will be denoted by $\widetilde{M}$.

Note that we can extend the trace to any positive self-adjoint operator A affiliated with M by

$$\tau(A) = \sup_n \tau\left(\int_0^n \lambda dE_\lambda\right),$$

where $A = \int_0^\infty \lambda dE_\lambda$ is the spectral resolution of A. Then the following statement holds. A proof can be found in [7], [11], [14].

Statement 2.4. *For $1 \leqslant p < \infty$*

$$L^p(M, \tau) \cong \{A \in \overline{M} : \tau(|A|^p) < \infty\} \subseteq \widetilde{M}.$$

Example 2.5. Abelian case. Let $M = L^\infty(X, \Omega, \mu)$ for a measure space (X, Ω, μ), where μ is a finite measure. Note that M can be identified with the multiplicative algebra $\mathcal{A} = \{M_f, f \in L^\infty(X, \mu)\} \subseteq B(L^2(X, \mu))$, where $M_f g = fg, g \in L^2(X, \mu)$. Define $\tau(f) = \int_X f d\mu$. Then τ is a faithful finite normal trace on M, and $L^p(M, \tau) \cong L^p(X, \mu)$.

Example 2.6. Ordinary trace. Let $M = B(H)$ and $\tau(A) = trA$ is the ordinary operator trace. Then $L^1(M, \tau)$ is the space of trace-class operators, and $L^2(M, \tau)$ is the space of Hilbert-Schmidt operators. Note that if H is finite-dimensional, then tr is just the matrix trace.

Example 2.7. A hyperfinite II$_1$-factor. We use a representation of a II$_1$-factor as the weak closure of a subalgebra of the CAR-algebra. In this description we follow, in particular, L. Gross [8] and I. E. Segal [13]. Let H be a complex Hilbert space with an inner product $(\cdot, \cdot)$. Denote by $\Lambda^n(H)$ the space of skew symmetric tensors of rank n over H. Put $\Lambda^0(H) = \mathbb{C}$ and write $\Lambda(H) = \oplus_{n=0}^\infty \Lambda^n(H)$. The "bare vacuum" $\Omega = 1$ is the element in $\Lambda^0(H) \subset \Lambda(H)$. For any x in H there exists a bounded operator C_x such that $C_x u = (n+1)^{\frac{1}{2}} x \wedge u, u \in \Lambda^n(H)$, where $x \wedge u$ denotes $P_a(x \otimes u)$ and P_a is the antisymmetric projection. If we denote $A_x = C_x^*$, then $C_x A_y + A_y C_x = (x, y)I$, which are the canonical anti-commutation relations.

We fix a conjugation $\mathcal{J}$ on H, that is, $\mathcal{J}$ is antilinear, antiunitary and idempotent. We will call an element x in H real with respect to $\mathcal{J}$ if $x = \mathcal{J}x$. Let us define $B_x \overset{\text{def}}{=} C_x + A_{\mathcal{J}x}$ for any x in H. Note that $B_x B_y + B_y B_x = 2\langle x, y\rangle I$, where $\langle x, y\rangle = (x, \mathcal{J}y)$.

Let $\mathcal{C}$ be the smallest weakly closed algebra of operators on $\Lambda(H)$ containing all the operators $B_x, x \in H$. We consider the trace on $\mathcal{C}$ given by $tr(u) = (u\Omega, \Omega)$. By Corollary 3.4 of [13] the space $\mathcal{C}$ with this trace is a hyperfinite II_1-factor, and there is just one such a factor due to Connes' result in [1]. By Theorem 5 of [8] the map $u \longmapsto u\Omega$ extends to a unitary map from $L^2(\mathcal{C})$ onto $\Lambda(H)$. In particular if we choose a real (with respect to $\mathcal{J}$) orthonormal basis $\{x_1, x_2, ..., x_n, ...\}$ in H, then $\{B_{x_{i_1}} B_{x_{i_2}} ... B_{x_{i_n}}, i_1 < i_2 < ... < i_n, 0 \leqslant n\}$ is an orthonormal basis in $L^2(\mathcal{C})$.

3. Preliminary remarks

Let $Q : L^2(M, \tau) \to L^2(M, \tau)$ be a bounded linear symmetric nonnegative operator, and W_t be an $L^2(M, \tau)$-valued Wiener process with Q as its covariance operator. We assume that Q is a trace class operator on $L^2(M, \tau)$. Let $Q^{1/2} L^2(M, \tau) = L^2_Q(M, \tau)$ be a Hilbert space equipped with the norm

$$\|A\|_{L^2_Q(M,\tau)} = \|Q^{-1/2} A\|_{L^2(M,\tau)}.$$

In what follows let $\{\xi_n\}_{n=1}^{\infty}$ denote an orthonormal basis of $L^2_Q(M, \tau)$ as a real space. We assume that $L^2_Q(M, \tau)$ is a subspace of M. We can think of W_t in terms of the orthonormal basis $\{\xi_n\}_{n=1}^{\infty}$ in the following way

$$W_t = \sum_{i=1}^{\infty} W_t^i \xi_i,$$

where W_t^i are one-dimensional independent real Wiener processes.

Several operators on H defined by the orthonormal basis $\{\xi_n\}_{n=1}^{\infty}$ of $L^2_Q(M, \tau)$ play a significant role in the proof of existence and uniqueness of solutions of stochastic differential equations in Section 4. The next statement about these operators follows a similar statement from [6], where it was proved for a hyperfinite II_1-factor.

Lemma 3.1. *Let us assume that* $\sum_{n=1}^{\infty} \xi_n^* \xi_n$, $\sum_{n=1}^{\infty} \xi_n \xi_n^*$ *and* $\sum_{n=1}^{\infty} \xi_n^2$ *are bounded operators (and the series are convergent in* M*), that is,*

$$\| \sum_{n=1}^{\infty} \xi_n^* \xi_n \| < \infty, \ \| \sum_{n=1}^{\infty} \xi_n^2 \| < \infty, \ \| \sum_{n=1}^{\infty} \xi_n \xi_n^* \| < \infty. \tag{3.1}$$

Then the operators $\sum_{n=1}^{\infty} \xi_n^* \xi_n$, $\sum_{n=1}^{\infty} \xi_n \xi_n^*$ *and* $\sum_{n=1}^{\infty} \xi_n^2$ *are independent of the choice of the basis* $\{\xi_n\}_{n=1}^{\infty}$.

Proof. Define a bilinear real form on $L^2(M, \tau) \times L^2(M, \tau)$ by

$$L(f, g) = \tau(h^* Q^{1/2} f Q^{1/2} g) = \langle Q^{1/2} g, (Q^{1/2} f)^* h\rangle_{L^2(M,\tau)},$$

for some $h \in M$, $f, g \in L^2(M, \tau)$. Then $f \mapsto L(f, g)$ is a bounded linear functional on $L^2(M, \tau)$ and

$$L(f,g) = \langle Q^{1/2}f, h(Q^{1/2}g)^*\rangle_{L^2(M,\tau)} = \langle f, Q^{1/2}(h(Q^{1/2}g)^*)\rangle_{L^2(M,\tau)}.$$

Let $Bg = Q^{1/2}(h(Q^{1/2}g)^*)$. Then B is a trace class operator from $L^2(M,\tau)$ to $L^2(M,\tau)$ since $Q^{1/2}$ is a Hilbert-Schmidt operator on $L^2(M,\tau)$, and multiplication by h is a bounded operator on $L^2(M,\tau)$. The trace of B over $L^2(M,\tau)$ is

$$TrB = \sum_{n=1}^{\infty}\langle e_n, Be_n\rangle_{L^2(M,\tau)} = \sum_{n=1}^{\infty} L(e_n,e_n) = \sum_{n=1}^{\infty}\tau(h^*\xi_n^2),$$

and it does not depend on the choice of $\{\xi_n\}_{n=1}^{\infty}$ for any $h \in M$. Use the forms $M_1(f,g) = \tau(h^*(Q^{1/2}f)^*Q^{1/2}g)$ and $M_2(f,g) = \tau(h^*Q^{1/2}f(Q^{1/2}g)^*)$ to verify that $\sum\limits_{n=1}^{\infty}\xi_n^*\xi_n$ and $\sum\limits_{n=1}^{\infty}\xi_n\xi_n^*$ are independent of the choice of the basis. □

Remark 3.2. If there exists an orthonormal basis of $L^2(M,\tau)$ such that its elements are uniformly bounded in the operator norm, and the operator Q is diagonal in this basis, then condition 3.1 is satisfied (see Example 3.1).

Example 3.1. Abelian case. Let $L^2(M,\tau) = L^2(0,1)$ be the Abelian L^2-space with the Lebesgue measure. Then we can choose an orthonormal basis in such a way that the conditions on the series described above can be satisfied. For example, take an orthonormal (real) basis of $L^2(0,1)$ to be $e_n = \sqrt{2}\sin(\pi nx)$ for $n = 1,2,\ldots$. Let Q be diagonal in this basis, that is, $\xi_i = \sqrt{\lambda_i}e_i$, and define a function $g(x)$ on $[0,1]$ by $g(x) = \sum\limits_{n=1}^{\infty}\xi_n^*\xi_n = \sum\limits_{n=1}^{\infty}\xi_n\xi_n^* = \sum\limits_{n=1}^{\infty}\xi_n^2$. Then for any x in $[0,1]$ we have $0 \leqslant \xi_n^*\xi_n = \xi_n\xi_n^* = \xi_n\xi_n = \lambda_n e_n^2 \leqslant \lambda_n$ and therefore

$$0 \leqslant \sum_{n=1}^{\infty}\xi_n^2 \leqslant \sum_{n=1}^{\infty}\lambda_n I = trQI \in B(L^2(0,1)),$$

if Q is a trace-class operator.

In particular, if we choose $Q^{-1} = -\Delta$, where Δ is the Dirichlet Laplacian, then $\lambda_n = 1/(\pi n)^2$, and the condition 3.1 is satisfied. Moreover, in this case we can compute the series in 3.1

$$g(x) = \sum_{n=1}^{\infty}\xi_n^2 = \sum_{n=1}^{\infty}\frac{\sin(\pi nx)^2}{\pi^2n^2} = \frac{1}{2}x(1-x).$$

This follows from Parseval's identity for the function $f(s)$ defined by $f(s) = 1, |s| \leqslant |x|$ and $f(s) = 0, |x| \leqslant |s| \leqslant 1$. Another way is to look at Green's function $G(x,y)$ for the Dirichlet problem on $[0,1]$

$$G(x,y) = \begin{cases} x(1-y) & x \leqslant y \\ y(1-x) & y \leqslant x \end{cases},$$

and expand $G(x,y)$ as follows

$$G(x,y) = 2\sum_{n=1}^{\infty}\langle G(x,\cdot), \sin(n\pi\cdot)\rangle_{L^2(0,1)}\sin(n\pi y) = 2\sum_{n=1}^{\infty}\frac{\sin(\pi nx)\sin(n\pi y)}{\pi^2n^2}.$$

Note that $\sum\limits_{n=1}^{\infty} \xi_n^* \xi_n = G(x,x)$ is the values on the diagonal of the integral kernel $G(x,y)$ for $Q = (-\Delta)^{-1}$.

Example 3.2. Ordinary trace. In the case when $L^2(M,\tau) = HS$ is the space of Hilbert-Schmidt operators (see Section 2) we can actually see that the conditions on the series described above can be satisfied. Indeed, elements of this $L^2(M,\tau)$ space can be viewed as infinite matrices whose entries form an l^2-series. Take an orthonormal basis of HS to be $\{e_{i,j}\}_{i,j=1}^{\infty}$, the matrix with the only nonzero entry equal to 1 at the ijth place. We can consider Q to be diagonal in this basis. To simplify calculations, let $Qe_{i,j} = \lambda_i \lambda_j e_{i,j}$, where $\lambda_i > 0$ and $\sum_{i=1}^{\infty} \lambda_i < \infty$. The last conditions makes Q a trace-class operator. In this setting $\xi_{i,j} = \sqrt{\lambda_i \lambda_j} e_{i,j}$ and

$$\sum_{i,j=1}^{\infty} \xi_{i,j}^* \xi_{i,j} = \sum_{i,j=1}^{\infty} \xi_{i,j} \xi_{i,j}^* = \sum_{i,j=1}^{\infty} \lambda_i \lambda_j e_{i,i} \in HS, \quad \sum_{i,j=1}^{\infty} \xi_{i,j}^2 = \sum_{i=1}^{\infty} \lambda_i^2 e_{i,i} \in HS.$$

and therefore these operators are bounded.

4. Stochastic differential equations

Let $B : L^2(M,\tau) \longrightarrow L^2(M,\tau)$ and $F : L^2(M,\tau) \longrightarrow L^2(M,\tau)$.

Theorem 4.1. *Assume that* $\sum\limits_{n=1}^{\infty} \xi_n^* \xi_n$, *is a bounded operator, the series is convergent in M) and denote*

$$\| \sum_{n=1}^{\infty} \xi_n^* \xi_n \| = K_1 < \infty.$$

Suppose that F and B are Lipschitz continuous on $L^2(M,\tau)$. Then for $\xi \in L^2(M,\tau)$ the stochastic differential equation

$$\begin{aligned} dX_t &= F(X_t)dt + dW_t B(X_t), \\ X_0 &= \xi \end{aligned} \tag{4.1}$$

has a unique solution, up to equivalence, among the processes satisfying

$$\boldsymbol{P}\left(\int_0^T \|X_s\|_{L^2(M,\tau)}^2 ds < \infty \right) = 1.$$

Proof of Theorem 4.1. Denote by $HS_Q = HS(L_Q^2(M,\tau), L^2(M,\tau))$ the space of the Hilbert-Schmidt operators from $L_Q^2(M,\tau)$ to $L^2(M,\tau)$ with the (Hilbert-Schmidt) norm

$$\|\Psi\|_{HS_Q}^2 = \sum_n \|\Psi \xi_n\|_{L^2(M,\tau)}^2,$$

where $\{\xi_n\}_{n=1}^{\infty}$ is an orthonormal basis of $L_Q^2(M,\tau)$ as a real space. We will use Theorem 7.4 from the book by DaPrato and Zabczyk [2]. We will abuse notation by treating B and C as maps from $L_Q^2(M,\tau)$ to $L^2(M,\tau)$ by multiplication. Here are the conditions that we need to check to apply the results from [2]

(1) $B(X)(\cdot)$ is a measurable mapping from $L^2(M,\tau)$ to HS_Q.
(2) $\|B(X_1) - B(X_2)\|_{HS_Q} \leqslant C\|X_1 - X_2\|_{L^2(M,\tau)}$ for any $X_1, X_2 \in HS$.
(3) $\|B(X)\|_{HS_Q}^2 \leqslant K(1 + \|X\|_{L^2(M,\tau)}^2)$ for any $X \in L^2(M,\tau)$.

(4) F is Lipschitz continuous on $L^2(M,\tau)$ and
$\|F(X)\|^2_{L^2(M,\tau)} \leqslant L(1+\|X\|^2_{L^2(M,\tau)})$ for any $X \in L^2(M,\tau)$.

By the assumption B and F are Lipschitz continuous on $L^2(M,\tau)$ with some positive constants B and F

$$\|B(X_1)-B(X_2)\|_{L^2(M,\tau)} \leqslant B'\|X_1-X_2\|_{L^2(M,\tau)},$$

$$\|F(X_1)-F(X_2)\|_{L^2(M,\tau)} \leqslant F'\|X_1-X_2\|_{L^2(M,\tau)}$$

for any X_1 and X_2 in $L^2(M,\tau)$.

First of all, $B(X)(U) \in L^2(M,\tau)$ for any $U \in L^2_Q(M,\tau)$, since $U \in L^2_Q(M,\tau) \subseteq M$ and $B(X) \in L^2(M,\tau)$. Now let us verify that $B(X) \in HS_Q$. The Hilbert-Schmidt norm of B as an operator from $L^2_Q(M,\tau)$ to $L^2(M,\tau)$ can be found as follows

$$\|B(X)\|^2_{HS_Q} = \sum_{n=1}^{\infty}\langle \xi_n B(X), \xi_n B(X)\rangle_{L^2(M,\tau)} = \sum_{n=1}^{\infty}\tau(B(X)^*\xi_n^*\xi_n B(X)) \leqslant$$
$$\left(\sum_{n=1}^{\infty}\xi_n^*\xi_n\right)\tau(B(X)B(X)^*) = K_1\|B(X)\|^2_{L^2(M,\tau)} < \infty.$$

Then the Lipschitz continuity of B implies that

$$\|B(X_1)-B(X_2)\|_{HS_Q} \leqslant K_1\|B(X_1)-B(X_2)\|_{L^2(M,\tau)} \leqslant \sqrt{K_1}B'\|X_1-X_2\|_{L^2(M,\tau)}.$$

and

$$\|B(X)\|_{L^2(M,\tau)} \leqslant \max\{B', \|B(0)\|_{L^2(M,\tau)}\}(\|X\|_{L^2(M,\tau)}+1).$$

Thus conditions 1, 2 and 3 are satisfied. Condition 4 can verified similarly to the proof of 3. □

The following Proposition gives an estimate of the moments of the L^2-norm of X_t. An interesting question whether X_t is in $L^p(M,\tau)$ is still open.

Proposition 4.2. *For any $p \geqslant 2, t > 0$*

$$E\|X_t\|^p_{L^2(M,\tau)} < \frac{1}{C_{p,t}}(e^t C_{p,t}-1),$$

where $C_{p,t} = C_{\frac{p}{2}}M^{\frac{p}{2}}t^{\frac{p}{2}-1} + L^p t^{p-1}2^{\frac{p}{2}-1}, C_p = (p(2p-1))^p(\frac{2p}{2p-1})^{2p^2}$.

Proof. 1. First of all, let us estimate $E\|\int_0^t dW_s B(X_s)\|^p_{L^2(M,\tau)}$. From part 3 of the proof of Theorem 4.1 we know that $\|B(X)\|^2_{HS_Q} \leqslant M(\|X\|^2_{L^2(M,\tau)}+1)$ for a positive constant M. In addition we will use Lemma 7.2 from the book by DaPrato and Zabczyk [2], p.182: for any $r \geqslant 1$ and for an arbitrary HS_Q-valued predictable process $\Phi(t)$,

$$E(\sup_{s\in[0,t]}\|\int_0^s dW(u)\Phi(u)\|^{2r}_{L^2(M,\tau)}) \leqslant C_r E(\int_0^t \|\Phi(s)\|^2_{HS_Q}ds)^r, t \in [0,T], \tag{4.2}$$

where $C_r = (r(2r-1))^r(\frac{2r}{2r-1})^{2r^2}$. Thus

$$E\|\int_0^t dW_s B(X_s)\|^p_{L^2(M,\tau)} \leqslant C_{\frac{p}{2}} E(\int_0^t \|B(X_s)\|^2_{HS_Q} ds)^{\frac{p}{2}} \tag{4.3}$$

$$\leqslant C_{\frac{p}{2}} M^{\frac{p}{2}} E(\int_0^t ((\|X\|^2_{L^2(M,\tau)}+1)ds)^{\frac{p}{2}} \leqslant C_{\frac{p}{2}} M^{\frac{p}{2}} t^{\frac{p}{2}-1} E\int_0^t (\|X\|^2_{L^2(M,\tau)}+1)^{\frac{p}{2}} ds$$

Now we can use inequality $(x+y)^q \leqslant 2^{q-1}(x^q+y^q)$ for any $x, y \geqslant 0$ for the estimate (4.3)

$$E\|\int_0^t dW_s B(X_s)\|^p_{L^2(M,\tau)} \leqslant C_{\frac{p}{2}} M^{\frac{p}{2}} t^{\frac{p}{2}-1} 2^{\frac{p}{2}-1} E\int_0^t (1+\|X_s\|^p_{L^2(M,\tau)})ds$$
$$= C_{\frac{p}{2}} M^{\frac{p}{2}} 2^{\frac{p}{2}-1} t^{\frac{p}{2}-1}(t+E\int_0^t \|X\|^p_{HS} ds).$$

2. Second, estimate

$$E\|\int_0^t F(X_s)ds\|^p_{L^2(M,\tau)} \leqslant E\left(\int_0^t \|F(X_s)\|_{L^2(M,\tau)} ds\right)^p \leqslant$$
$$t^{p-1} E\int_0^t \|F(X_s)\|^p_{L^2(M,\tau)} ds \leqslant L^p t^{p-1} E\int_0^t (1+\|X\|^2_{L^2(M,\tau)})^{\frac{p}{2}} ds \leqslant$$
$$L^p t^{p-1} 2^{\frac{p}{2}-1} E\int_0^t (1+\|X_s\|^p_{L^2(M,\tau)})ds \leqslant L^p t^{p-1} 2^{\frac{p}{2}-1}(t+E\int_0^t \|X\|^p_{HS} ds).$$

Finally,

$$E\|X_t\|^p_{L^2(M,\tau)} \leqslant E\|\int_0^t B(X_s)dW_s\|^p_{HS} \leqslant C_{p,t}(t+E\int_0^t \|X_s\|^p_{HS} ds),$$

where $C_{p,t} = C_{\frac{p}{2}} M^{\frac{p}{2}} t^{\frac{p}{2}-1} + L^p t^{p-1} 2^{\frac{p}{2}-1}$.

Thus, $E\|X_t\|^p_{L^2(M,\tau)} < \frac{1}{C_{p,t}}(e^t C_{p,t} - 1)$ by Gronwall's lemma. □

The next theorem explains how to find a left inverse to solutions of a certain class of stochastic differential equations. The particular coefficients of these equations insure that the solution to the second stochastic equation is a left inverse of the solution of the first equation. Note that the solutions are elements of a certain space of unbounded operators which makes the task of finding inverses very difficult. This inverse is double-sided if $L^2(M,\tau)$ is the space of the Hilbert-Schmidt operators.

Theorem 4.3. *Assume that* $\sum_{n=1}^{\infty} \xi_n^*\xi_n$, $\sum_{n=1}^{\infty} \xi_n\xi_n^*$ *and* $\sum_{n=1}^{\infty} \xi_n\xi_n$ *are bounded operators (and the series are convergent in* M*) and denote*

$$\|\sum_{n=1}^{\infty} \xi_n^*\xi_n\| = K_1 < \infty, \ \|\sum_{n=1}^{\infty} \xi_n\xi_n^*\| = K_2 < \infty.$$

Let X_t *and* Z_t *be the solutions of the stochastic differential equations*

$$dX_t = TX_t dt + dW_t X_t,$$
$$dZ_t = Z_t T dt - Z_t dW_t,$$
$$X_0 = x, \ Z_0 = z,$$

where x and z are in $L^2(M,\tau)$ and $T = 1/2\sum_{n=1}^{\infty}\xi_n^2$. Then $Z_tX_t = zx$ with probability 1 for any $t > 0$.

Proof of Theorem 4.3. First of all, note that the first equation has a unique solution in $L^2(M,\tau)$ by Theorem 4.1, and the second one has a unique solution in $L^2(M,\tau)$ by very similar arguments. To prove the statement of Theorem 4.3 we will apply Itô's formula to $G(Z_t,X_t)$, where G is defined as follows: $G(Z,X) = \lambda(ZX)$, where $Z,X \in L^2(M,\tau)$ and λ is a nonzero linear real bounded functional from $L^2(M,\tau)\times L^2(M,\tau)$ to $\mathbb{R}$. Here we view G as a function on a Hilbert space $L^2(M,\tau)\times L^2(M,\tau)$. Then $Z_tX_t = xz$ if and only if $\lambda(Z_tX_t - xz) = 0$ for any such λ. In order to use Itô's formula we must verify several properties of the processes Z_t and X_t and the mapping G:

(1) $\tilde{B}(\tilde{X}_s)$ is an HS_Q-valued process stochastically integrable on $[0,T]$
(2) G and the derivatives G_t, G_Y, G_{YY} are uniformly continuous on bounded subsets of $[0,T]\times L^2(M,\tau)\times L^2(M,\tau)$, where $Y = (Z,X)$.

Proof of 1. See 1 in the proof of Theorem 4.1.

Proof of 2. Let us calculate G_t, G_Y, G_{YY}. First, $G_t = 0$. For any $S = (S_1,S_2), V = (V_1,V_2) \in L^2(M,\tau)\times L^2(M,\tau)$ and

$$G_Y(Y)(S) = \lambda(S_1X + ZS_2), \quad G_{YY}(Y)(S\otimes T) = \lambda(S_1V_2 + V_1S_2).$$

Thus condition 2 is satisfied. We will use the following notation

$$G_Y(Y)(S) = \langle \bar{G}_{YY}, S\rangle_{L^2(M,\tau)},$$
$$G_{YY}(Y)(S\otimes V) = \langle \bar{G}_{YY}(Y)S, V\rangle_{L^2(M,\tau)},$$

where $\bar{G}_Y$ is an element of $L^2(M,\tau)\times L^2(M,\tau)$ and $\bar{G}_{YY}$ is an operator on $L^2(M,\tau)\times L^2(M,\tau)$ corresponding to the functionals $G_Y \in (L^2(M,\tau)\times L^2(M,\tau))^*$ and $G_{YY} \in ((L^2(M,\tau)\times L^2(M,\tau))\otimes(L^2(M,\tau)\times L^2(M,\tau)))^*$.

Now we can apply Itô's formula to $G(Z_t,X_t)$

(4.4)
$$G(Z_t,X_t) = \int_0^t \langle \bar{G}_Y(Z_s,X_s), (-Z_s dW_s, dW_s X_s)\rangle_{L^2(M,\tau)} +$$
$$\int_0^t \langle \bar{G}_Y(Z_s,X_s), (Z_sT, TX_s)\rangle_{L^2(M,\tau)} ds +$$
$$\frac{1}{2}\int_0^t Tr_{L^2(M,\tau)}[\bar{G}_{YY}(Z_s,X_s)(-Z_sQ^{1/2}(\cdot), Q^{1/2}(\cdot)X_s)(-Z_sQ^{1/2}(\cdot), Q^{1/2}(\cdot)X_s)^*)]ds.$$

Let us calculate the integrands in (4.4) separately. The first integrand is

$$\langle \bar{G}_Y(Z_s,X_s), (-Z_s dW_s, dW_s X_s)\rangle_{L^2(M,\tau)} = G_Y(Z_s,X_s)(-Z_s dW_s, dW_s X_s)$$
$$= \lambda(-Z_s dW_s X_s + Z_s dW_s X_s) = 0.$$

The second integrand is

$$\langle \bar{G}_Y(Z_s,X_s), (Z_sT, TX_s)\rangle_{L^2(M,\tau)} = \lambda(Z_sTX_s + Z_sTX_s)) = 2\lambda(Z_sTX_s).$$

The third integrand is

$$\frac{1}{2}\int_0^t Tr_{L^2(M,\tau)}[\bar{G}_{YY}(Z_s,X_s)(-Z_sQ^{1/2}(\cdot),Q^{1/2}(\cdot)X_s)(-Z_sQ^{1/2}(\cdot),Q^{1/2}(\cdot)X_s)^*)]=$$

$$\frac{1}{2}\sum_{n=1}^{\infty}G_{YY}(Z_s,X_s)\left(\left(-Z_sQ^{1/2}e_n,Q^{1/2}(e_n)X_s\right)\otimes\left(-Z_sQ^{1/2}e_n,Q^{1/2}(e_n)X_s\right)\right)=$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\lambda\left(-Z_s\xi_n^2X_s-Z_s\xi_n^2X_s\right)=-\sum_{n=1}^{\infty}\lambda\left(Z_s\xi_n^2X_s\right).$$

This shows that the stochastic differential of G is zero, so $G(Z_t,X_t)=G(Z_0,X_0)=zx$ for any $t>0$. □

Remark 4.4. As we described earlier, if we choose $M=B(H)$ and $\tau(A)=trA$ to be the ordinary operator trace, then $L^1(M,\tau)$ is the space of trace-class operators, and $L^2(M,\tau)$ is the space of Hilbert-Schmidt operators. Consider

$$dX_t=T(X_t+I)dt+dW_t(X_t+I),X_0=0.$$

Note that this equation has the additive noise as well as the multiplicative one. It is possible to solve this equation since $L^2(M,\tau)\subseteq M=B(H)$ in this case. Then X_t is in $GL(H)$, the invertible elements of $B(H)$, with probability 1 for any $t>0$ as was shown in [5]. This means that in this case X_t has a double-sided inverse. The shift by the identity operator is necessary to ensure that X_t lives in $GL(H)$ since I is not a Hilbert-Schmidt operator.

Remark 4.5. This lemma has been proved in the case of a hyperfinite II_1-factor in[6].

Remark 4.6. In [5] and [6] the stochastic differential equations had no drift terms. The reason for that is that in those cases Q was complex linear, and therefore $T=1/2\sum_{n=1}^{\infty}\xi_n^2=0$. Indeed, Lemma 3.1 shows that $\sum\limits_{n=1}^{\infty}\xi_n^2$ is independent of the choice of the basis $\{\xi_n\}_{n=1}^{\infty}$, and this basis can be chosen so that the sum is 0. Namely, we can choose such a basis that $\xi_{2k}=i\xi_{2k-1}$, where $i=\sqrt{-1}$.

5. Kolmogorov's backward equation

First of all, the coefficient B depends only on $X\in L^2(M,\tau)$, therefore the transition probability satisfies $P_{s,t}f(X)=Ef(X(t,s;X))=P_{t-s}f(X)$. Later in this section we give Kolomogorov's backward equation (Equation 5.2) for a stochastic differential equation of the form 4.1. At the moment though let us look at Kolomogorov's equation in the particular case of $B(X)=X$ and $F(X)=1/2\sum_{n=1}^{\infty}\xi_n^2X$. Then Equation 5.2 is the heat equation with the Laplacian which is a half of the sum of second exponential derivatives in the directions of an orthonormal basis of $L_Q^2(M,\tau)$. This Lapalcian is an infinite dimensional analogue of the Laplacian on a finite-dimensional Lie group. In the case of the Hilbert-Schmidt operators, this is a Laplacian on an infinite-dimensional (Lie) group, but for a hyperfinite II_1-factor this is not so.

Let $v:L^2(M,\tau)\to\mathbb{R}$ be a function and $\partial_nv(X)=(\tilde{\xi_n}v)(X)=\frac{d}{dt}\mid_{t=0}v(e^{t\xi_n}X)$. Here $\tilde{\xi_n}$ is the right-invariant vector field corresponding to ξ_n. Then the Laplacian is

$$\Delta v = \frac{1}{2}\sum_{n=1}^{\infty}\partial_n^2 v = \frac{1}{2}\sum_{n=1}^{\infty}\tilde{\xi_n}\tilde{\xi_n}v, \tag{5.1}$$

Let us calculate derivatives ∂_n of $v : L^2(M,\tau) \to \mathbb{R}$

$$(\partial_n v)(X) = v_X(X)\frac{d}{dt}\mid_{t=0}(e^{t\xi_n}X) = v_X(X)(\xi_n X)$$

and therefore

$$(\tilde{\xi_n}\tilde{\xi_n}v)(X) = v_{XX}(X)(\xi_n X \otimes \xi_n X) + v_X(\xi_n^2 X).$$

Thus the Laplacian is

$$(\Delta v)(X) = \frac{1}{2}\sum_{n=1}^{\infty}[v_{XX}(X)(\xi_n X \otimes \xi_n X) + v_X(\xi_n^2 X)] =$$
$$\frac{1}{2}Tr[v_{XX}(t,X)(Q^{1/2}(\cdot)X)(Q^{1/2}(\cdot)X)^*] + \frac{1}{2}\sum_{n=1}^{\infty}(\xi_n^2 X, v_X(t,X))_{L^2(M,\tau)}.$$

Therefore the Lie group Laplacian is the same differential operator that appears in Kolmogorov's backward equation (Equation 5.2), and so this equation can be viewed as a heat equation. This justifies the following definition.

Definition 5.1. We define a Borel measure μ_t by

$$\int_{L^2(M,\tau)} f(X)\mu_t(dX) = Ef(X_t(I)) = P_{t,0}f(I)$$

for any bounded Borel function f on $L^2(M,\tau)$. Then μ_t is called *the heat kernel measure on $L^2(M,\tau)$.*

Now let us look at the general case. According to Theorem 9.16 from [2] for any $\varphi \in C_b^2(L^2(M,\tau))$ and $X \in L^2(M,\tau)$ the function $v(t,X) = P_t\varphi(X)$ is a unique strict solution from $C_b^{1,2}(L^2(M,\tau))$ for the parabolic type equation (Kolmogorov's backward equation)

$$\begin{aligned}&\frac{\partial}{\partial t}v(t,X) = \frac{1}{2}Tr[v_{XX}(t,X)(Q^{1/2}(\cdot)B(X))(Q^{1/2}(\cdot)B(X))^*] + \\ &\qquad\qquad (F(X), v_X(t,X))_{L^2(M,\tau)} \\ &v(0,X) = \varphi(X), t > 0, X \in L^2(M,\tau).\end{aligned} \tag{5.2}$$

Here $C_b^n(L^2(M,\tau))$ denotes the space of all functions from $L^2(M,\tau)$ to $\mathbb{R}$ that are n-times continuously Frechet differentiable with all derivatives up to order n bounded and $C_b^{k,n}(L^2(M,\tau))$ denotes the space of all functions from $[0,T] \times L^2(M,\tau)$ to $\mathbb{R}$ that are k-times continuously Frechet differentiable with respect to t and n-times continuously Frechet differentiable with respect to X with all partial derivatives continuous in $[0,T] \times L^2(M,\tau)$ and bounded.

Let us rewrite Equation (5.2) in the following way

$$Tr[v_{XX}(t,X)(Q^{1/2}(\cdot)B(X))(Q^{1/2}(\cdot)B(X))^*]$$
$$= \sum_{n=1}^{\infty} v_{XX}(t,X)(Q^{1/2}e_nB(X) \otimes Q^{1/2}e_nB(X))$$
$$= \sum_{n=1}^{\infty} v_{XX}(t,X)(\xi_nB(X) \otimes \xi_nB(X)),$$

where $v_{XX}(t,X)$ is viewed as a functional on $L^2(M,\tau) \otimes L^2(M,\tau)$. Thus Kolmogorov's backward equation is

$$\frac{\partial}{\partial t}v(t,X) = \frac{1}{2}\sum_{n=1}^{\infty} v_{XX}(t,X)(\xi_nB(X) \otimes \xi_nB(X)) + (F(X), v_X(t,X))_{L^2(M,\tau)}$$
$$v(0,X) = \varphi(X), t > 0, X \in L^2(M,\tau),$$

which is a heat equation with a first order term.

References

[1] A. Connes, *Classification of injective factors. Cases II_1, II_∞, III_λ, $\lambda \neq 1$*, Ann. of Math. (2) 104, 1976, **1**, 73–115.

[2] G.DaPrato and J.Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, 1992.

[3] J. Dixmier, *von Neumann algebras*, North-Holland Mathematical Library, 27, 1981.

[4] M. Gordina, *Holomorphic functions and the heat kernel measure on an infinite dimensional complex orthogonal group*, Potential Analysis, **12**, 2000, 325–357.

[5] M. Gordina, *Heat kernel analysis and Cameron-Martin subgroup for infinite dimensional groups*, J. Funct. Anal., **171**, 2000, 192–232.

[6] M. Gordina, *Taylor map on groups associated with a II_1-factor*, to appear in Infin. Dimens. Anal. Quantum Probab. Relat. Top., 2002.

[7] M. Gordina, *Noncommutative integration in examples*, notes, 2002.

[8] L. Gross, *Existence and uniqueness of physical ground states*, J. Funct. Anal., **10**, 1972, 52–109.

[9] U. Haagerup, *L^p-spaces associated with an arbitrary von Neumann algebra*, Algbres d'oprateurs et leurs applications en physique mathmatique (Proc. Colloq., Marseille, 1977), pp. 175–184, Colloq. Internat. CNRS, 274, CNRS, Paris, 1979.

[10] R. Kadison, J. Ringrose, *Fundamentals of the theory of operator algebras*, Vol.I-IV. Graduate Studies in Mathematics, **16**, American Mathematical Society, Providence, RI, 1997.

[11] E. Nelson, *Notes on non-commutative integration*, Journal of Functional Analysis, 1974, **15**, 103–116.

[12] I. E. Segal, *A non-commutative extension of abstract integration*, Ann. of Math., 1953, **57**, 401–457.

[13] I. E. Segal, *Tensor algebras over Hilbert spaces. II*, Ann. of Math., **63**, 1956, 160–175.

[14] M. Terp, *L^p spaces associated with von Neumann algebras*, manuscript.

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