HOLOMORPHIC FUNCTIONS AND THE HEAT KERNEL MEASURE ON AN INFINITE DIMENSIONAL COMPLEX ORTHOGONAL GROUP

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Abstract. The heat kernel measure μ_t is constructed on an infinite dimensional complex group using a diffusion in a Hilbert space. Then it is proved that holomorphic polynomials on the group are square integrable with respect to the heat kernel measure. The closure of these polynomials, $\mathcal{H}L^2(SO_{HS}, \mu_t)$, is one of two spaces of holomorphic functions we consider. The second space, $\mathcal{H}L^2(SO(\infty))$, consists of functions which are holomorphic on an analog of the Cameron-Martin subspace for the group. It is proved that there is an isometry from the first space to the second one.

The main theorem is that an infinite dimensional nonlinear analog of the Taylor expansion defines an isometry from $\mathcal{H}L^2(SO(\infty))$ into the Hilbert space associated with a Lie algebra of the infinite dimensional group. This is an extension to infinite dimensions of an isometry of B. Driver and L. Gross for complex Lie groups.

All the results of this paper are formulated for one concrete group, the Hilbert-Schmidt complex orthogonal group, though our methods can be applied in more general situations.

1. Introduction

In this paper we will study Hilbert spaces of holomorphic functions over a particular infinite dimensional complex group G. We will choose an infinite dimensional Lie algebra naturally associated with the group, and fix a Hermitian inner product on it. As usual, one may view the universal enveloping algebra as a space of left-invariant differential operators on the group. A holomorphic function on G then determines an element, α, of the dual of the universal enveloping algebra by means of the identity $\langle \alpha, \beta \rangle = (\beta f)(e)$ for all left-invariant differential operators β, where e is the identity element of the group. Thus, α is just the set of Taylor coefficients of f at the identity. When G is finite dimensional, the Taylor map, $f \mapsto \alpha$, is known (cf. [7]) to be an isometry from the Hilbert space of holomorphic functions, square integrable with respect to a heat kernel measure on G, to a subspace of the dual of the universal enveloping algebra. An outline of these finite dimensional results of B. Driver and L. Gross [7] will be given later in the introduction. The objective of the present work is to establish a corresponding isometry for an infinite dimensional group.

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Two fundamental differences between the finite dimensional and infinite dimensional cases must be addressed. First, we need to have an analog of the heat kernel measure on an infinite dimensional, non locally compact, non commutative group. For a finite dimensional group the heat kernel measure has a density with respect to Haar measure, making it possible to apply techniques from partial differential equations. However, on the group to be considered in this paper there is no Haar measure to help analyze a heat kernel density. Partial differential equation techniques for studying such measures are not available. Instead, we will use stochastic differential equations in infinite dimensions to construct such a measure. The theory of such equations has been developed for Hilbert and Banach spaces, not infinite dimensional manifolds or groups. Moreover, since there are no previous examples to guide our intuition for a general theory, we will focus in this paper on one particular case. We will take the group to be the group of complex Hilbert-Schmidt orthogonal operators over an infinite dimensional Hilbert space. This Hilbert-Schmidt complex orthogonal group, SO_{HS}, will be described in Section 2. The fact that the group SO_{HS} can be embedded in a Hilbert space, namely, the space of Hilbert-Schmidt operators, plays an important role in the construction of the heat kernel measure. To construct the heat kernel measure we will use a diffusion on the group. We will construct the diffusion on G by first constructing a diffusion on this ambient Hilbert space. The stochastic differential equation determining the diffusion is nonlinear with Lipshitz nonlinearities, and its coefficients are chosen in such a way that the stochastic process actually lives in the group. Then we will use finite dimensional approximations to this diffusion to obtain some of our results. We can identify elements of the group with infinite matrices, which allows us to make significant use of the polynomials in the matrix entries. Using properties of the stochastic process we will prove that these naturally defined holomorphic polynomials are square integrable with respect to the measure determined by the process.

A second difference between the finite and infinite dimensional cases arises from the need to find a viable notion of holomorphic function on the group SO_{HS}. We will consider two spaces of holomorphic functions. One of the spaces, $\mathcal{H}L^2(SO_{HS}, \mu_t)$, is defined as the closure of the holomorphic polynomials in $L^2(\mu_t)$-norm. However, such a function might not have a version which is differentiable on the whole group. Therefore we also consider a Hilbert space $\mathcal{H}L^2(SO(\infty))$ of functions differentiable on a subset $SO(\infty)$ of the group. This subset plays the role of the Cameron-Martin subspace. The main theorem is that the Taylor map, $f \mapsto \alpha$, is an isometry from $\mathcal{H}L^2(SO(\infty))$ into a Hilbert space contained in the dual of the universal enveloping algebra. This will imply, in particular, that any function in $\mathcal{H}L^2(SO(\infty))$ is uniquely determined by its derivatives at the identity. In addition, there is a natural isometry from the closure of the holomorphic polynomials in $L^2(\mu_t)$-norm to $\mathcal{H}L^2(SO(\infty))$. This isometry is an extension of the inclusion of these polynomials into this second space.

When G is finite dimensional, the heat kernel measure is defined as follows. There is a unique probability measure satisfying $e^{t\Delta/4}f = \mu_t * f$, where the Laplacian Δ is a naturally defined second order differential operator associated with a given Hermitian inner product on the Lie algebra of G. In the finite dimensional case the measure $\mu_t = \mu_t(z)dz$ has the heat kernel $\mu_t(z)$ as its density with respect to Haar measure on G. Note that if $G = \mathbb{C}^n$ then the heat kernel measure μ_t is a Gaussian measure. Denote by $\mathcal{H}L^2(G, \mu_t(z)dz)$ the space of holomorphic functions over G which are square integrable with respect to the heat kernel measure μ_t. B. Driver and L. Gross [7] have shown that the Taylor map, $f \mapsto \alpha$, is an isomorphism between $\mathcal{H}L^2(G, \mu_t(z)dz)$ and a subspace, J_t^0, of the algebraic dual of the universal enveloping algebra of the Lie algebra of G, when G is a connected simply connected complex (finite dimensional) Lie group. J_t^0 is a Hilbert space with respect to a norm which is naturally associated with the given inner product on the Lie algebra of G.

In finite dimensions, when G happens to be the complexification of a compact Lie group K, there is, in addition to the two Hilbert spaces, $\mathcal{H}L^2(G, \mu_t(z)dz)$ and J_t^0, discussed above, a third naturally arising Hilbert space. The third space is $L^2(K, \rho_t(x)dx)$, where $\rho_t(x)dx$ is the heat kernel measure associated to a bi-invariant Laplacian on K. B. Hall [11], [12] has described a natural unitary isomorphism from $L^2(K, \rho_t(x)dx)$ to $\mathcal{H}L^2(G, \mu_t(z)dz)$. It is worth noting that one should not expect an analog of Hall's transform for our group because his proof requires Ad-invariance of the inner product on $Lie(K)$, and this condition seems to be really essential. By contrast, the isomorphism from $\mathcal{H}L^2(G, \mu_t(z)dz)$ to J_t^0 does not require Ad-invariance of the Hermitian inner product on the Lie algebra of G. This is important in infinite dimensions since the norm we need to use is not Ad-invariant. Though there is an invariant inner product on the Lie algebra of SO_{HS}, namely, the Hilbert-Schmidt inner product, J_t^0 turns out to be trivial in this case. The same happens for some other infinite dimensional groups whose Lie algebras have Ad-invariant inner product. We will show this in Section 8 for the Hilbert-Schmidt complex orthogonal group, SO_{HS}.

L. Gross proved in [9] the isomorphism between $L^2(K, \rho_t(x)dx)$ and J_t^0 for a compact connected Lie group K using probabilistic methods. Later O. Hijab in [14], [15] found an analytical proof of this isomorphism. This isometry also depends on the AdK-invariance of the inner product on $Lie(K)$ and seems unlikely to have a natural analog for our infinite dimensional orthogonal group. An extensive exposition of the subject can be found in [10].

In conclusion, we should note that in the case when G is replaced by a separable Hilbert space, J_t^0 is a bosonic Fock space. In this situation the isomorphisms between these three spaces give three different representations of the Fock space. The isomorphisms were studied by V.Bargmann, I. Segal, P. Krée and others (see, for example, [1], [2], [17], [24], [25]). Recently such spaces have been studied in case of a complex abstract Wiener space by several authors including I. Shigekawa [26] and H. Sugita [28], [29]. Our results are precise analogs of some of the linear results of I. Shigekawa and H. Sugita, though the methods are entirely different.

Not much is known for infinite dimensional groups. However, B. Hall and A. Sengupta [13] have extended the isomorphism between $L^2(K, \rho_t(x)dx)$ and $\mathcal{H}L^2(G, \mu_t(x)dx)$ to the group of paths in a compact (finite dimensional) Lie group.

There have been a number of works about properties of infinite dimensional orthogonal and unitary groups. The representation theory of these groups has been studied, for example, in [16], [20], [27]. P. da la Harpe, R.J. Plymen and R.F. Streater have described topological properties of infinite dimensional orthogonal groups in [4], [5], [19] in connection with spinors in Hilbert space. One of the groups they have considered is the Hilbert-Schmidt orthogonal group.

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2. Notation and main results

First of all, we describe the group we will consider in this paper. This group is represented as a group of operators in a Hilbert space. Let H_r be a real separable Hilbert space, $H = H_r \oplus iH_r$ its complexification and I the identity operator on H. Denote by HS_r the Hilbert-Schmidt operators on H_r. Then HS_r is a real Hilbert space equipped with the Hilbert-Schmidt inner product $\langle A, B\rangle_{HS_r} = Tr(B^*A)$ for any $A, B \in HS_r$.

Consider the complexification of HS_r denoted by $HS = HS_r \oplus iHS_r$. HS can be identified with the space of complex linear Hilbert-Schmidt operators on H. Any element $A \in HS$ can be written as a sum of ReA and $i\,ImA$, where ReA and $ImA \in HS_r$. The product of elements of HS is defined naturally by

$$A\,B = (ReA\,ReB - ImA\,ImB) + i\,(ReA\,ImB + ImA\,ReB).$$

The space HS is a real Hilbert space equipped with the following inner product:

$$\langle A, B\rangle_{HS} \stackrel{\text{def}}{=} ReTr(A^*B) = \langle ReA, ReB\rangle_{HS_r} + \langle ImA, ImB\rangle_{HS_r}.$$

Definition 2.1. *The Hilbert-Schmidt complex orthogonal group SO_{HS}* is the connected component containing the identity I of the group $O_{HS} = \{B : B - I \in HS, B^TB = BB^T = I\}$, where B^T means the transpose of the operator B, i. e. $B^T = (ReB)^* + i(ImB)^*$.

Proposition 4.4 shows why O_{HS} is not connected. The fact that $SO_{HS} - I \subset HS$ gives two essential advantages: first of all, it helps to define an inner product on a Lie algebra of SO_{HS}, and second, it will allow us to construct the heat kernel measure on SO_{HS}. The Lie algebra of skew-symmetric Hilbert-Schmidt operators $so_{HS} = \{A : A \in HS, A^T = -A\}$ plays the role of a Lie algebra of SO_{HS}. The group SO_{HS} is the complexification of the Hilbert-Schmidt orthogonal group, SO_{HS_r}, and so_{HS} is the complexification of its Lie algebra. Note that $\langle\cdot,\cdot\rangle_{HS}$ is an Ad-HS_r-invariant inner product on so_{HS}. Later we will consider another inner product on a dense subspace of so_{HS} (which is not Ad-invariant) since the isometry we are trying to establish is trivial in the invariant case.

We will consider three Hilbert spaces $\mathcal{H}L^2(SO_{HS}, \mu_t)$, $\mathcal{H}L^2(SO(\infty))$ and J_t^0. The first two spaces are spaces of functions, while J_t^0 has an algebraic nature.

The first space, $\mathcal{H}L^2(SO_{HS}, \mu_t)$, is the closure in $L^2(SO_{HS}, \mu_t)$ of the space of holomorphic polynomials (in the matrix entries) on SO_{HS}. Here μ_t is the heat kernel measure on SO_{HS} which will be defined in Section 3 as the transition probability of a diffusion on SO_{HS}. Note that this definition uses the fact that all holomorphic polynomials are square-integrable with respect to the heat kernel measure. The latter will be proved in Section 5.

To define $\mathcal{H}L^2(SO(\infty))$ we consider $SO(\infty)$, a subset of SO_{HS}, which is the closure of the direct limit of a sequence of finite dimensional subgroups of SO_{HS} in a Riemannian metric. More precisely, fix an orthonormal basis $\{f_m\}_{m=1}^{\infty}$ of H_r. Let $HS_{n\times n} = \{A : \langle Af_m, f_k\rangle = 0 \text{ if } \max(m, k) > n\}$. Take a basis $\{e_k\}_{k=1}^{\infty}$ of HS such that $\{e_k\}_{k=1}^{2n^2}$ is a basis of $HS_{n\times n}$. Note that $SO_{HS}(n) = \{B \in SO_{HS}, B - I \in HS_{n\times n}\}$ is a group isomorphic to the complexification of the special orthogonal group of $\mathbb{R}^n$. Then $\mathfrak{g}_n = Lie(SO_{HS}(n)) = so_{HS}(n) = \{A \in HS_{n\times n}, A^T = -A\}$ is its Lie algebra. Note that $\mathfrak{g} = \cup_n g_n$ is a Lie subalgebra of so_{HS}. We will choose a new inner product $\langle\langle\cdot,\cdot\rangle\rangle$ on $\mathfrak{g}$ so that the corresponding norm is stronger than the Hilbert-Schmidt norm. This new inner product defines a Riemannian metric on $\cup_n SO_{HS}(n)$. $SO(\infty)$ is the closure of $\cup_n SO_{HS}(n)$ in this Riemannian metric. Then $\mathcal{H}L^2(SO(\infty))$ denotes a certain space of holomorphic functions on $SO(\infty)$ with a certain direct limit-type norm. More detailed definitions of these objects are given in Section 7.

Finally, we need the following notation to describe J_t^0.

Notation 2.2. *Let $\mathfrak{g}$ be a complex Lie algebra with a Hermitian inner product on it. Then $T(\mathfrak{g})$ will denote the algebraic tensor algebra over $\mathfrak{g}$ as a complex vector space and $T'(\mathfrak{g})$ will denote the algebraic dual of $T(\mathfrak{g})$. Define a norm on $T(\mathfrak{g})$ by*

$$|\beta|_t^2 = \sum_{k=0}^{n} \frac{k!}{t^k} |\beta_k|^2, \beta = \sum_{k=0}^{n} \beta_k, \beta_k \in \mathfrak{g}^{\otimes k}, k = 0, 1, 2, ..., t > 0 \tag{2.1}$$

Here $|\beta_k|$ is the cross norm on $\mathfrak{g}^{\otimes k}$ arising from the inner product on $\mathfrak{g}^{\otimes k}$ determined by the given inner product on $\mathfrak{g}$. The coefficients $\frac{k!}{t^k}$ are related to the heat kernel. $T_t(\mathfrak{g})$ will denote the completion of T in this norm. The topological dual of $T_t(\mathfrak{g})$ may be identified with the subspace $T_t^*(\mathfrak{g})$ of $T'(\mathfrak{g})$ consisting of such $\alpha \in T'(\mathfrak{g})$ that the norm

$$|\alpha|_t^2 = \sum_{k=0}^{\infty} \frac{t^k}{k!} |\alpha_k|^2, \alpha = \sum_{k=0}^{\infty} \alpha_k, \alpha_k \in (\mathfrak{g}^{\otimes k})^*, k = 0, 1, 2, ..., t > 0 \tag{2.2}$$

is finite. Here $|\alpha_k|$ is the norm on $(\mathfrak{g}^{\otimes k})^*$ dual to the cross norm on $\mathfrak{g}^{\otimes k}$.

There is a natural pairing for any $\alpha \in T'(\mathfrak{g})$ and $\beta \in T(\mathfrak{g})$ denoted by

$$\langle \alpha, \beta \rangle = \sum_{k=0}^{\infty} \langle \alpha_k, \beta_k \rangle, \alpha = \sum_{k=0}^{\infty} \alpha_k, \beta = \sum_{k=0}^{n} \beta_k, \alpha_k \in (\mathfrak{g}^{\otimes k})^*, \beta_k \in \mathfrak{g}^{\otimes k}, k = 0, 1, 2, ...$$

Denote by $J(\mathfrak{g})$ the two-sided ideal in $T(\mathfrak{g})$ generated by $\{\xi \otimes \eta - \eta \otimes \xi - [\xi, \eta], \xi, \eta \in \mathfrak{g}\}$. Let $J^0(\mathfrak{g}) = \{\alpha \in T'(\mathfrak{g}) : \alpha(J) = 0\}$. Finally, let $J_t^0(\mathfrak{g}) = T_t^*(\mathfrak{g}) \cap J^0(\mathfrak{g})$.

The first natural choice for $\mathfrak{g}$ for our group is so_{HS} with the Hilbert-Schmidt inner product, which is Ad-HS_r-invariant. The problem with this inner product is that in this case J_t^0 is trivial (see Section 8). Therefore we will use another Lie algebra, $\mathfrak{g} = \cup \mathfrak{g}_n = \cup so_{HS}(n)$, as described above together with an appropriate inner product on $\mathfrak{g}$.

The inner product on $\mathfrak{g}$ will be non Ad-invariant. Namely, the inner product is defined by $\langle\langle A, B \rangle\rangle = \langle Q^{-1/2}A, Q^{-1/2}B \rangle_{HS}$, where Q is a symmetric positive trace class operator on so_{HS} such that all $so_{HS}(n)$ are invariant subspaces of Q. Actually, the closure of $\mathfrak{g}$ in the corresponding norm is the subspace of so_{HS} defined as $U_0 = Q^{1/2}(so_{HS})$. This norm is denoted by $\|X\|_{U_0} = \|Q^{-1/2}X\|_{HS}$.

In some sense U_0 determines the directions in which derivatives of functions on SO_{HS} will be taken. If we choose too large a set of directions we might not have nonconstant holomorphic functions. Moreover, if we define informally the Laplacian as

$$\Delta_Q = \frac{1}{2} \sum_{n=1}^{\infty} (\partial_{\xi_n})^2$$

where $\{\xi_n\}_{n=1}^{\infty}$ is an orthonormal (in $\langle\langle \cdot, \cdot \rangle\rangle$) basis of U_0, then the stronger inner product gives a weaker Laplacian. In Section 3.3 we show that the heat kernel measure μ_t solves the heat equation with the Laplacian Δ_Q. The weaker Laplacian allows μ_t to live on SO_{HS}. Note that $SO(\infty)$ is an analog of the Cameron-Martin subspace associated with any infinite dimensional Wiener space.

For the Lie algebra $\mathfrak{g}$ we denote $J_t^0 = J_t^0(\mathfrak{g})$. The latter can be called a space of Taylor coefficients for functions in $\mathcal{H}L^2(SO(\infty))$. Indeed, our main result is that the Taylor series determines an isometry from $\mathcal{H}L^2(SO(\infty))$ into J_t^0. The fact that J_t^0 is not trivial (for the Lie algebra $\mathfrak{g}$) follows from the existence of this isometry from $\mathcal{H}L^2(SO_{HS}, \mu_t)$ to J_t^0 and the fact that $\mathcal{H}L^2(SO_{HS}, \mu_t)$ contains all holomorphic polynomials on SO_{HS}.

For each function f in $\mathcal{H}L^2(SO(\infty))$ we use a natural notation

$$(1-D)_e^{-1} f = \sum_{n=1}^{\infty} D_e^n f$$

for the series of its derivatives at the identity considered as an element of T'. Then our results (Theorem 7.6, Theorem 7.5 and Theorem 7.4) can be summarized as follows.

1. $\mathcal{H}L^2(SO(\infty))$ is a Hilbert space.
2. $(1-D)_e^{-1}$ is an isometry from $\mathcal{H}L^2(SO(\infty))$ into J_t^0.
3. The embedding of the space of holomorphic polynomials $\mathcal{HP}$ into the Hilbert space $\mathcal{H}L^2(SO(\infty))$ can be extended to an isometry from $\mathcal{H}L^2(SO_{HS}, \mu_t)$ into $\mathcal{H}L^2(SO(\infty))$.

The following diagram describes these isometries:

$$\mathcal{HP} \xrightarrow{\text{inclusion}} \mathcal{H}L^2(SO_{HS}, \mu_t) \xrightarrow{\text{Theorem 7.4}} \mathcal{H}L^2(SO(\infty)) \xrightarrow{\text{Theorem 7.5}} J_t^0$$

3. Construction of the heat kernel measure

The goal of this section is to define the heat kernel measure on SO_{HS}. To achieve it we will use a diffusion on the group. If we consider a stochastic process in SO_{HS} $X_t : \Omega \times \mathbb{R} \longrightarrow SO_{HS}$, then $Y_t = X_t - I$ is an element of HS. This space is a Hilbert space, which enables us to use the machinery of the stochastic differential equations in Hilbert spaces developed in recent years.

3.1. **Existence and uniqueness of a stochastic process.** We begin with the definition of the process Y_t. Denote by U the space so_{HS} with the inner product $\langle \cdot, \cdot \rangle_{HS}$. Let W_t be a U-valued Wiener process with a covariance operator $Q : U \longrightarrow U$. We assume that Q is a symmetric positive trace-class operator. Sometimes we will identify Q with its extension by zero to the orthogonal complement of U in HS. Let $U_0 = Q^{\frac{1}{2}}(U)$ as before with the inner product $\langle u, v \rangle_0 = \langle Q^{-1/2}u, Q^{-1/2}v \rangle_{HS}$.

Denote by $L_2^0 = L_2(U_0, HS)$ the space of the Hilbert-Schmidt operators from U_0 to HS with the (Hilbert-Schmidt) norm $\|\Psi\|^2_{L_2^0} = Tr[\Psi Q \Psi^*]$.

We need to define the coefficients of the stochastic differential equation before we formulate Theorem 3.3.

Notation 3.1. *$B : HS \longrightarrow L_2^0$, $B(Y)U = U(Y+I)$ for $U \in U_0$, $F : HS \longrightarrow HS$, $F(Y) = -\frac{1}{2}\sum_n (Q^{1/2}e_n)^T (Q^{1/2}e_n)(Y+I)$, where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of HS as a real space.*

Proposition 3.2. *$\sum_m (Q^{1/2}e_m)^T(Q^{1/2}e_m)$ does not depend on the choice of the basis $\{e_m\}_1^{\infty}$. In addition, $\sum_m (Q^{1/2}e_m)^T(Q^{1/2}e_m) = \sum_m e_m^T Q e_m$.*

Proof. Define a bilinear real form on $HS \times HS$ by $L(f,g) = \Lambda((Q^{1/2}f)^T(Q^{1/2}g))$, where Λ is a real bounded linear functional on HS. Then $f \mapsto L(f,g)$ is a bounded linear functional on HS and so $L(f,g) = \langle f, \tilde{g}\rangle_{HS}$ for some $\tilde{g} \in HS$. There exists a linear operator on HS such that $L(f,g) = \langle f, Bg\rangle_{HS}$. Now we can see that since trace is independent of a basis, $TrB = \sum_m \langle e_m, Be_m\rangle_{HS} = \sum_m \Lambda(Q^{1/2}e_m)^T(Q^{1/2}e_m)$ does not depend on the choice of $\{e_m\}_{m=1}^{\infty}$. To prove the second half of the statement, we note that $\sum_m e_m^T Q e_m$ is independent of $\{e_m\}_{m=1}^{\infty}$ by similar reasons. At the same time, if we choose $\{e_m\}_{m=1}^{\infty}$ in which Q is diagonal, say, $Qe_m = \lambda_m e_m$, then $\sum_m (Q^{1/2}e_m)^T(Q^{1/2}e_m) = \sum_m \lambda_m e_m^T e_m = \sum_m e_m^T Q e_m$. Since both sums are independent of the choice of the basis, they are equal for any basis $\{e_m\}_{m=1}^{\infty}$. □

Theorem 3.3. 1. *The stochastic differential equation*

$$\begin{aligned} dY_t &= B(Y_t)dW_t + F(Y_t)dt, \\ Y_0 &= 0 \end{aligned} \tag{3.1}$$

has a unique solution, up to equivalence, among the processes satisfying

$$\boldsymbol{P}\left(\int_0^T \|Y_s\|_{HS}^2 ds < \infty\right) = 1.$$

2. *For any $p > 2$ there exists a constant $C_{p,T} > 0$ such that*

$$\boldsymbol{E} \sup_{t\in[0,T]} \|Y_t\|_{HS}^p \leqslant C_{p,T}.$$

Proof of Theorem 3.3. To prove this theorem we will use Theorem 7.4, p.186 from the book by G.DaPrato and J.Zabczyk [3]. It is enough to check that

1. F is a measurable mapping from HS to HS.
2. $\|F(Y_1) - F(Y_2)\|_{HS} \leqslant C\|Y_1 - Y_2\|_{HS}$ for any $Y_1, Y_2 \in HS$.
3. $\|F(Y)\|_{HS}^2 \leqslant K(1 + \|Y\|_{HS}^2)$ for any $Y \in HS$.
4. $B(Y)$ is a measurable mapping from HS to L_2^0.
5. $\|B(Y_1) - B(Y_2)\|_{L_2^0} \leqslant C\|Y_1 - Y_2\|_{HS}$ for any $Y_1, Y_2 \in HS$.
6. $\|B(Y)\|_{L_2^0}^2 \leqslant K(1 + \|Y\|_{HS}^2)$ for any $Y \in HS$.

Proof of 1. Let us check that $F(Y)$ is in HS for any $Y \in HS$.

$$\|F(Y)\|_{HS} \leqslant \frac{1}{2}\|\sum_n (Q^{1/2}e_n)^T(Q^{1/2}e_n)\|_{HS}(\|Y\|_{HS} + 1) < \infty,$$

since

$$\begin{aligned} \|\sum_n (Q^{1/2}e_n)^T(Q^{1/2}e_n)\|_{HS} &\leqslant \sum_n \|(Q^{1/2}e_n)^T(Q^{1/2}e_n)\|_{HS} \leqslant \\ &\leqslant \sum_n \|Q^{1/2}e_n\|_{HS}^2 = TrQ < \infty \end{aligned}$$

by the assumption. The measurability is trivial.

Proof of 2.

$$\begin{aligned} \|F(Y_1) - F(Y_2)\|_{HS} &\leqslant \frac{1}{2}\|\sum_n (Q^{1/2}e_n)^T Q^{1/2}e_n\|_{HS}\|Y_1 - Y_2\|_{HS} \leqslant \\ &\leqslant TrQ\|Y_1 - Y_2\|_{HS} \end{aligned}$$

as in the proof of 1.

Proof of 3.

$$\|F(Y)\|_{HS} \leqslant \frac{1}{2}\|\sum_n (Q^{1/2}e_n)^T(Q^{1/2}e_n)Y\|_{HS} + \\ + \frac{1}{2}\|\sum_n (Q^{1/2}e_n)^T(Q^{1/2}e_n)\|_{HS} \leqslant \frac{1}{2}TrQ(\|Y\|_{HS}+1).$$

Finally, $\|F(Y)\|_{HS}^2 \leqslant \frac{1}{2}(TrQ)^2(1+\|Y\|_{HS}^2)$.

Proof of 4. We want to check that $B(Y)$ is in L_2^0 for any Y from HS. First of all, $B(Y)U \in HS$, for any $U \in U_0$. Indeed, $B(Y)U = U(Y+I) = UY + U \in HS$, since U and V are in HS.

Now let us verify that $B(Y) \in L_0^2$. Consider the Hilbert-Schmidt norm of B as an operator from U_0 to HS. Take an orthonormal basis $\{u_m\}_{m=1}^{\infty}$ of U_0. Then $\{Q^{-1/2}u_m\}_{m=1}^{\infty}$ is an orthonormal basis of U and the Hilbert-Schmidt norm of B can be found as follows:

$$\|B(Y)\|_{L_0^2}^2 = \sum_{m=1}^{\infty}\langle B(Y)u_m, B(Y)u_m\rangle_{HS} = \sum_{m=1}^{\infty}\langle u_m(Y+I), u_m(Y+I)\rangle_{HS} \leqslant \\ \|Y+I\|^2\sum_{m=1}^{\infty}\langle u_m, u_m\rangle_{HS} = \|Y+I\|^2 TrQ < \infty,$$

since the operator norm $\|Y+I\|$ is finite.

Proof of 5. Similarly to the previous proof we have
$\|B(Y_1) - B(Y_2)\|_{L_2^0} \leqslant \|Y_1 - Y_2\|_{HS}(TrQ)^{1/2}$.

Proof of 6. Use the estimate we have got in the proof of 4:
$\|B(Y)\|_{L_2^0} \leqslant (TrQ)^{1/2}\|Y+I\| \leqslant (TrQ)^{1/2}(\|Y\|_{HS}+1)$, so
$\|B(Y)\|_{L_2^0}^2 \leqslant 2(TrQ)(1+\|Y\|_{HS}^2)$. □

3.2. **Process $Y_t + I$ lives in SO_{HS}.**

Theorem 3.4. *$Y_t + I$ lies in SO_{HS} for any $t > 0$ with probability 1.*

Proof of Theorem 3.4. We need to check that $(Y_t+I)^T(Y_t+I) = (Y_t+I)(Y_t+I)^T = I$ with probability 1 for any $t > 0$. To achieve this goal, we will apply Itô's formula to $G(Y_t)$, where G is defined as follows: $G(Y) = \Lambda(Y^TY + Y^T + Y)$, Λ is a nonzero linear real bounded functional from HS to $\mathbb{R}$. Then $(Y+I)^T(Y+I) = I$ if and only if $\Lambda(Y^TY + Y^T + Y) = 0$ for any Λ. In order to use Itô's formula we must verify several properties of the process Y_t and the mapping G:

1. $B(Y_s)$ is an L_2^0-valued process stochastically integrable on $[0,T]$
2. $F(Y_s)$ is an HS-valued predictable process Bochner integrable on $[0,T]$ $\boldsymbol{P}$-a.s.
3. G and the derivatives G_t, G_Y, G_{YY} are uniformly continuous on bounded subsets of $[0,T] \times HS$.

Proof of 1. See 4 in the proof of Theorem 3.3.

Proof of 2. See 1 in the proof of Theorem 3.3.

Proof of 3. Let us calculate G_t, G_Y, G_{YY}. First of all, $G_t = 0$.

$$G_Y(Y)(S) = \lim_{\varepsilon\to 0}\frac{G(Y+\varepsilon S)-G(Y)}{\varepsilon} = \Lambda(S^TY+Y^TS+S^T+S) = \\ = \Lambda(S^T(Y+I)+(Y+I)^TS)$$

for any $S\in HS$.

$$G_{YY}(Y)(S\otimes T) = \lim_{\varepsilon\to 0}\frac{G_Y(Y+\varepsilon T)(S)-G_Y(Y)(S)}{\varepsilon} = \Lambda(S^TT+T^TS)$$

for any $S, T\in HS$. Thus condition 3 is satisfied.

We will use the following notation:

$$G_Y(Y)(S) = \langle \bar{G}_Y(Y), S\rangle_{HS}, \\ G_{YY}(Y)(S\otimes T) = \langle \bar{G}_{YY}(Y)S, T\rangle_{HS},$$

where $\bar{G}_Y$ is an element of HS and $\bar{G}_{YY}$ is an operator on HS corresponding to the functionals $G_Y\in HS^*$ and $G_{YY}\in (HS\otimes HS)^*$.

Now we can apply Itô's formula to $G(Y_t)$:

$$G(Y_t) = \int_0^t \langle \bar{G}_Y(Y_s), B(Y_s)dW_s\rangle_{HS} + \int_0^t \langle \bar{G}_Y(Y_s), F(Y_s)\rangle_{HS}ds + \\ + \int_0^t \frac{1}{2}Tr[G_{YY}(Y_s)(B(Y_s)Q^{1/2})(B(Y_s)Q^{1/2})^*]ds \tag{3.2}$$

Let us calculate the three integrands in (3.2) separately.
The first integrand is

$$\langle \bar{G}_Y(Y_s), B(Y_s)dW_s\rangle_{HS} = \langle \bar{G}_Y(Y_s), dW_s(Y_s+I)\rangle_{HS} = \\ \Lambda((dW_s(Y_s+I))^T(Y_s+I)+(Y_s+I)^TdW_s(Y_s+I)) = \\ \Lambda(-(Y_s+I)^TdW_s(Y_s+I)+(Y_s+I)^TdW_s(Y_s+I)) = 0,$$

since W_t is an so_{HS}-valued process.
The second integrand is

$$\langle \bar{G}_Y(Y_s), F(Y_s)\rangle_{HS} = \Lambda((F(Y_s))^T(Y_s+I)+(Y_s+I)^TF(Y_s)) = \\ = -\frac{1}{2}\Lambda\big(\sum_n (Y_s+I)^T(Q^{1/2}e_n)^T(Q^{1/2}e_n)(Y_s+I) + \\ + (Y_s+I)^T(Q^{1/2}e_n)^T(Q^{1/2}e_n)(Y_s+I)\big) = \\ = -\sum_n \Lambda\big((Y_s+I)^T(Q^{1/2}e_n)^T(Q^{1/2}e_n)(Y_s+I)\big).$$

The third integrand is

$$\frac{1}{2}Tr[G_{YY}(Y_s)(B(Y_s)Q^{1/2})(B(Y_s)Q^{1/2})^*] =$$
$$= \frac{1}{2}\sum_n \langle \bar{G}_{YY}(Y_s)B(Y_s)Q^{1/2}e_n, B(Y_s)Q^{1/2}e_n)\rangle_{HS} =$$
$$= \frac{1}{2}\sum_n \Lambda((B(Y_s)Q^{1/2}e_n)^T(B(Y_s)Q^{1/2}e_n) + (B(Y_s)Q^{1/2}e_n)^T(B(Y_s)Q^{1/2}e_n)) =$$
$$= \sum_n \Lambda(Q^{1/2}e_n(Y_s+I))^T(Q^{1/2}e_n(Y_s+I)) =$$
$$= \sum_n \Lambda(Y_s+I)^T(Q^{1/2}e_n)^T(Q^{1/2}e_n)(Y_s+I).$$

Therefore $\langle \bar{G}_Y(Y_s), F(Y_s)\rangle_{HS} + \frac{1}{2}Tr[\bar{G}_{YY}(Y_s)(B(Y_s)Q^{1/2})(B(Y_s)Q^{1/2})^*] = 0$. This shows that the stochastic differential of G is zero, so $G(Y_t) = 0$ for any $t > 0$.

By the Fredholm alternative $Y_t + I$ has an inverse, therefore it has to be $(Y_t+I)^T$. Thus $(Y_t+I)^T(Y_t+I) = (Y_t+I)(Y_t+I)^T = I$ for any $t > 0$. □

3.3. **Definition of the heat kernel measure.** Let us define μ_t as follows:

$$\int_{SO_{HS}} f(X)\mu_t(dX) = Ef(X_t(I)) = P_{t,0}f(I)$$

for any bounded Borel function f on SO_{HS}.

Definition 3.5. μ_t is called *the heat kernel measure on SO_{HS}.*

Now we will present a motivation for such a name for μ_t. What follows will not be used to prove the main results of this paper. Note that F and B depend only on $Y \in HS$, therefore $P_{s,t}f(Y) = Ef(Y(t,s;Y)) = P_{t-s}f(Y)$. According to Theorem 9.16 from the book by G.DaPrato and J.Zabczyk [3], p. 258, the following is true:

For any $\varphi \in C_b^2(HS)$ and $Y \in HS$ function $v(t,Y) = P_t\varphi(Y)$ is a unique strict solution from $C_b^{1,2}(HS)$ for the parabolic type equation called Kolmogorov's backward equation:

$$\begin{aligned} &\frac{\partial}{\partial t}v(t,Y) = \frac{1}{2}Tr[v_{YY}(t,Y)(B(Y)Q^{1/2})(B(Y)Q^{1/2})^*] + \langle F(Y), v_Y(t,Y)\rangle_{HS} \\ &v(0,Y) = \varphi(Y), t > 0, Y \in HS. \end{aligned} \tag{3.3}$$

Here $C_b^n(HS)$ denotes the space of all functions from HS to $\mathbb{R}$ that are n-times continuously Frechet differentiable with all derivatives up to order n bounded and $C_b^{k,n}(HS)$ denotes the space of all functions from $[0,T] \times HS$ to $\mathbb{R}$ that are k-times continuously Frechet differentiable with respect to t and n-times continuously Frechet differentiable with respect to Y with all partial derivatives continuous in $[0,T] \times HS$ and bounded.

Let us rewrite Equation (3.3) as the heat equation. First of all

$$
\begin{aligned}
Tr[v_{YY}(t,Y)(B(Y)Q^{1/2})(B(Y)Q^{1/2})^*] = \\
\sum_n \langle v_{YY}(t,Y)(B(Y)Q^{1/2})e_n, (B(Y)Q^{1/2})e_n\rangle_{HS} = \\
= \sum_n \langle v_{YY}(t,Y)(Q^{1/2}e_n)(Y+I), (Q^{1/2}e_n)(Y+I)\rangle_{HS}
\end{aligned}
$$

and

$$
\langle F(Y), v_Y(t,Y)\rangle = -\frac{1}{2}\sum_n \langle (Q^{1/2}e_n)^T(Q^{1/2}e_n)(Y+I), v_Y(t,Y)\rangle_{HS}.
$$

Now change Y to $X-I$. Then we get that for any smooth bounded function $\varphi(X): HS+I \to \mathbb{R}$, function $v(t,X) = P_t\varphi(X)$ satisfies this equation which can be considered as the heat equation:

$$
\begin{aligned}
&\frac{\partial}{\partial t} v(t,X) = L_1 v(t,X) \\
&v(0,X) = \varphi(X), t>0, X \in HS+I,
\end{aligned}
\tag{3.4}
$$

where the differential operator L_1 on the space $C_b^{1,2}(HS+I)$ is defined by

$$
\begin{aligned}
L_1 v \overset{\text{def}}{=} \frac{1}{2}\sum_n [\langle v_{XX}(X)(Q^{1/2}e_n)X, (Q^{1/2}e_n)X\rangle_{HS} - \\
- \langle (Q^{1/2}e_n)^T(Q^{1/2}e_n)X, v_X(X)\rangle_{HS}].
\end{aligned}
$$

Our goal is to show that L_1 is a Laplacian on SO_{HS} in a sense. More precisely, L_1 is a half of sum of second derivatives in the directions of an orthonormal basis of an analog of Lie algebra of SO_{HS}. We begin with description of an analog of a Lie algebra for SO_{HS} and an inner product on it. Recall that $U_0 = Q^{1/2}(so_{HS})$ with the inner product $\langle\langle A,B\rangle\rangle = \langle Q^{-1/2}A, Q^{-1/2}B\rangle_{HS}$. Note that U_0 is a subspace of so_{HS}, but it is not a Lie subalgebra. As we will see later though, under some assumptions on Q space U_0 has a dense (in the norm determined by $\langle\langle\cdot,\cdot\rangle\rangle$) subspace which is a Lie algebra. Let $\{\xi_n\}_{n=1}^{\infty}$ be an orthonormal basis of U_0 and define the Laplacian by

$$
(\Delta v)(X) = \frac{1}{2}\sum_{n=1}^{\infty} (\tilde{\xi_n}\tilde{\xi_n} v)(X),
\tag{3.5}
$$

where $(\tilde{\xi_n} v)(X) = \frac{d}{dt}\mid_{t=0} v(\exp(t\xi_n)X)$ for a function $v: SO_{HS} \to \mathbb{R}$ and so $\tilde{\xi_n}$ is the right-invariant vector field on SO_{HS} corresponding to ξ_n

Let us calculate derivatives of $v: I+HS \to \mathbb{R}$ in the direction of ξ_n:

$$
(\tilde{\xi_n} v)(X) = v_X(X)\frac{d}{dt}\mid_{t=0}(\exp(t\xi_n)X) = v_X(X)(\xi_n X)
$$

and therefore

$$
(\tilde{\xi_n}\tilde{\xi_n} v)(X) = v_{XX}(X)(\xi_n X, \xi_n X) + v_X(\xi_n^2 X).
$$

Using $\xi_n^T = -\xi_n$ we can rewrite the Laplacian

$$(\Delta v)(X) = \frac{1}{2}\sum_n [v_{XX}(X)(\xi_n X, \xi_n X) + v_X(\xi_n^2 X)] =$$

$$= \frac{1}{2}\sum_n [v_{XX}(X)(\xi_n X, \xi_n X) - v_X(\xi_n^T \xi_n X)].$$

Define $L_2 v(X) = (\Delta v)(X)$. Since $\tilde{\xi_n}$ is a right-invariant vector field, L_2 is a right-invariant differential operator.

It is known that L_2 does not depend on choice of the basis $\{\xi_n\}_{n=1}^{\infty}$. Let $\xi_n = Q^{1/2} e_n$, where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of U with $\langle \cdot, \cdot \rangle_{HS}$ as a inner product. Then we see that $L_1 v = L_2 v$ on $C_b^2(HS + I)$.

4. Approximation of the process

Let $\mathcal{F}$ be a subspace of $U_0 = Q^{1/2}(so_{HS})$ and $P_{\mathcal{F}}$ a projection onto $\mathcal{F}$. Then $\mathcal{F} \cap KerQ = \{0\}$, and therefore $P_{\mathcal{F}} Q P_{\mathcal{F}}$ is positive and invertible on $\mathcal{F}$. Indeed, for any $f \in \mathcal{F}$ we have $\langle P_{\mathcal{F}} Q P_{\mathcal{F}} f, f \rangle_{HS} = \langle Qf, f \rangle_{HS} = \langle Q^{1/2} f, Q^{1/2} f \rangle_{HS} > 0$ if $Qf \neq 0$.

Consider an equation

$$dY_{\mathcal{F}} = B_{\mathcal{F}}(Y_{\mathcal{F}})dW_t + F_{\mathcal{F}}(Y_{\mathcal{F}})dt, Y_{\mathcal{F}}(0) = 0,$$

where

$$\begin{aligned} &F_{\mathcal{F}}(Y) = -\frac{1}{2}\sum_m ((P_{\mathcal{F}} Q P_{\mathcal{F}})^{1/2} e_m)^T (P_{\mathcal{F}} Q P_{\mathcal{F}})^{1/2} e_m (Y + I), \\ &B_{\mathcal{F}}(Y)U = (P_{\mathcal{F}} U)(Y + I). \end{aligned}$$

This equation has a unique solution by the same arguments as in Section 3.1. Denote $Q_{\mathcal{F}} = P_{\mathcal{F}} Q P_{\mathcal{F}}$.

Lemma 4.1. *$Y_{\mathcal{F}}$ is a solution of the equation*

$$\begin{aligned} &dY_{\mathcal{F}} = B_{\mathcal{F}}(Y_{\mathcal{F}})dW_{\mathcal{F},t} + F_{\mathcal{F}}(Y_{\mathcal{F}})dt, \\ &Y_{\mathcal{F}}(0) = 0, \end{aligned} \tag{4.1}$$

where $W_{\mathcal{F},t} = P_{\mathcal{F}} W_t$. In addition, $I + Y_{\mathcal{F},t} \in SO_{HS}$ a.s.

Proof. The first part is easy to check. Now check that $P_{\mathcal{F}} Q P_{\mathcal{F}}$ is the covariance operator of $W_{\mathcal{F}}$. By the definition of a covariance operator we know that $\langle f, Qg \rangle_{HS} = E\langle f, W_1 \rangle_{HS} \langle g, W_1 \rangle_{HS}$, therefore

$$\langle f, Q_{\mathcal{F}} g \rangle_{HS} = \langle P_{\mathcal{F}} f, Q P_{\mathcal{F}} g \rangle_{HS} = E\langle P_{\mathcal{F}} f, W_1 \rangle_{HS} \langle P_{\mathcal{F}} g, W_1 \rangle_{HS} =$$

$$= E\langle f, P_{\mathcal{F}} W_1 \rangle_{HS} \langle g, P_{\mathcal{F}} W_1 \rangle_{HS}.$$

Thus Equation 4.1 is actually the same as Equation 3.1 with the Wiener process that has the covariance $Q_{\mathcal{F}}$ instead of Q. Recall that we chose F and B so that the solution of Equation 3.1 plus the identity is in SO_{HS}. Therefore $I + Y_{\mathcal{F},t} \in SO_{HS}$ a.s. ∎

Choose a sequence of subspaces $\mathfrak{g}_n$ of $U_0 = Q^{1/2}so_{HS}$ so that $\mathfrak{g}_n \subset \mathfrak{g}_{n+1}$ and the closure of $\cup_n \mathfrak{g}_n$ in the metric of U_0 is equal to U_0. Denote $Y_{n,t} = Y_{\mathfrak{g}_n,t}, B_n = B_{\mathfrak{g}_n}, F_n = F_{\mathfrak{g}_n}, Q_n = Q_{\mathfrak{g}_n}, P_n = P_{\mathfrak{g}_n}$.

Lemma 4.2.

1. $TrQ_{\mathcal{F}} \leqslant TrQ$.
2. $P_nQ \xrightarrow[n\to\infty]{} Q$,
3. $QP_n \xrightarrow[n\to\infty]{} Q$,
4. $P_nQP_n \xrightarrow[n\to\infty]{} Q$,

where convergence is in the trace class norm.

Proof of 1. First of all, $Q_{\mathcal{F}}$ is nonnegative. Thus

$$TrQ_{\mathcal{F}} = \sum_n \langle Q_{\mathcal{F}} e_n, e_n\rangle_{HS} \leqslant \sum_n \langle Qe_n, e_n\rangle_{HS} = TrQ.$$

Proof of 2. According to [21],v.1, Theorems $VI.17, VI.21$ there are orthonormal systems $\{\phi_m\}_{m=1}^{\infty}, \{\psi_m\}_{m=1}^{\infty}$ in HS and positive numbers λ_m such that

$$\sum_m \lambda_m < \infty,$$
$$Q(\cdot) = \sum_n \lambda_m \langle \cdot, \psi_m\rangle_{HS}\phi_m.$$

Consider an operator $Ay = \langle x, \psi\rangle_{HS}\phi$. Then $\|A\|_{TrCl} = \|\psi\|_{HS}\|\phi\|_{HS}$. Assume first that $\|\psi\|_{HS} = 1, \|\phi\|_{HS} = 1$. Note that $|A|y = \langle y, \psi\rangle_{HS}\psi$, since $A^*y = \langle \phi, y\rangle_{HS}\psi$ and $A^*A = |A|^2$. Take such an orthonormal basis $\{e_n\}_{m=1}^{\infty}$ of HS that $e_1 = \psi$. Then $\|A\|_{TrCl} = \sum_n \langle |A|e_n, e_n\rangle_{HS} = \sum_n \langle e_n, \psi\rangle_{HS}^2 = \langle \psi, \psi\rangle_{HS}^2 = 1$. Using this we see that

$$\|\langle \cdot, \psi_m\rangle_{HS}\phi_m\|_{TrCl} = 1,$$
$$\|\langle \cdot, \psi_m\rangle_{HS}P_n\phi_m\|_{TrCl} \leqslant 1,$$
$$\|\langle \cdot, \psi_m\rangle_{HS}(P_n\phi_m - \phi_m)\|_{TrCl} \xrightarrow[n\to\infty]{} 0.$$

In addition, $\|P_nQ - Q\|_{TrCl} \leqslant \sum_m \lambda_m < \infty$. Then $\|P_nQ - Q\|_{TrCl} \xrightarrow[n\to\infty]{} 0$ by the Dominated Convergence Theorem.

The proof of 3 and 4 is similar. □

Theorem 4.3. *Denote by $\mathcal{H}_2$ the space of equivalence classes of HS-valued predictable processes with the norm:*

$$|\!|\!| Y |\!|\!|_2 = (\sup_{t\in[0,T]} E\|Y(t)\|_{HS}^2)^{1/2}.$$

Then

$$|\!|\!| Y_n - Y |\!|\!|_2 \xrightarrow[n\to\infty]{} 0.$$

Proof of Theorem 4.3. Let us apply the local inversion theorem (see, for example, Lemma 9.2, from the book by G.DaPrato and J.Zabczyk [3], p. 244) to $K(y, Y) = y + \int_0^t B(Y)dW_t + \int_0^t F(Y)dt$, where y is the initial value of Y and $Y = Y(y, t)$ is an HS-valued predictable process. Analogously we define $K_n(y, Y) =$

$y+\int_0^t B_n(Y)dW_t+\int_0^t F_n(Y)dt$. To apply this lemma we need to check that K and K_n satisfy the following conditions:

1. For any $Y_1(t)$ and $Y_2(t)$ from $\mathcal{H}_2$
$$\sup_{t\in[0,T]} E\|K(y,Y_1)-K(y,Y_2)\|_{HS}^2 \leqslant \alpha \sup_{t\in[0,T]} E\|Y_1(t)-Y_2(t)\|_{HS}^2,$$
where $0\leqslant\alpha<1$
2. For any $Y_1(t)$ and $Y_2(t)$ from $\mathcal{H}_2$
$$\sup_{t\in[0,T]} E\|K_n(y,Y_1)-K_n(y,Y_2)\|_{HS}^2 \leqslant \alpha \sup_{t\in[0,T]} E\|Y_1(t)-Y_2(t)\|_{HS}^2,$$
where $0\leqslant\alpha<1$
3. $\lim_{n\to\infty} K_n(y,Y)=K(y,Y)$ in $\mathcal{H}_2$.

Proof of 1. In what follows we use (5.2) to estimate the part with the stochastic differential.

$$\begin{aligned}
&E\|K(y,Y_1)-K(y,Y_2)\|_{HS}^2=\\
&=E\|\int_0^t (F(Y_1)-F(Y_2))ds+\int_0^t (B(Y_1)-B(Y_2))dW_s\|_{HS}^2\leqslant\\
&\leqslant 2E(\int_0^t \|F(Y_1)-F(Y_2)\|_{HS}ds)^2+2E\|\int_0^t B(Y_1)-B(Y_2)dW_s\|_{HS}^2\leqslant\\
&\leqslant 2(TrQ)^2E(\int_0^t \|Y_1-Y_2\|_{HS}ds)^2+8E\int_0^t \|B(Y_1)-B(Y_2)\|_{L_2^0}^2ds\leqslant\\
&\leqslant 2(TrQ)^2E(\int_0^t \|Y_1-Y_2\|_{HS}ds)^2+8TrQE\int_0^t \|Y_1-Y_2\|_{HS}^2ds\leqslant\\
&\leqslant 2(TrQ)^2tE\int_0^t \|Y_1-Y_2\|_{HS}^2ds+8TrQE\int_0^t \|Y_1-Y_2\|_{HS}^2ds\leqslant\\
&\leqslant (2(TrQ)^2t+8TrQ)t\sup_{t\in[0,T]} E\|Y_1-Y_2\|_{HS}^2
\end{aligned}$$

Note that for small t we can make $(2(TrQ)^2t+8TrQ)t$ as small as we wish, therefore 1 holds.

Proof of 2.
$$\|F_n(Y_1)-F_n(Y_2)\|_{HS}=\frac{1}{2}\|\sum_m (Q_n^{1/2}e_m)^T(Q_n^{1/2}e_m)(Y_1-Y_2)\|_{HS}\leqslant$$

$$\leqslant\frac{1}{2}\sum_m \|(Q_n^{1/2}e_m)^TQ_n^{1/2}e_m\|_{HS}\|Y_1-Y_2\|_{HS}\leqslant\frac{1}{2}\sum_m \|Q_n^{1/2}e_m\|_{HS}^2\|Y_1-Y_2\|_{HS}=$$

$$=TrQ_n\|Y_1-Y_2\|_{HS}\leqslant TrQ\|Y_1-Y_2\|_{HS}$$

by Lemma 4.2.

Similarly to the proof of condition 5 in the proof of Theorem 3.3 we have that

$$\begin{aligned}\|B_n(Y_1)-B_n(Y_2)\|_{L_0^2} &= \|P_n(\cdot)(Y_1-Y_2)\|_{U_0\to HS} \leqslant \\ Tr((P_nQ^{1/2})^*P_nQ^{1/2})\|Y_1-Y_2\|_{HS} &= Tr(Q^{1/2}P_nQ^{1/2})\|Y_1-Y_2\|_{HS} = \\ &= Tr(QP_n)\|Y_1-Y_2\|_{HS} \leqslant TrQ\|Y_1-Y_2\|_{HS}.\end{aligned}$$

Now use the same estimates as in 1 to see that 2 holds.

Proof of 3. Here again we will use (5.2) to estimate the part with the stochastic differential.

$$\begin{aligned}&|||K_n(y,Y)-K(y,Y)|||_2^2 = |||\int_0^t (B_n(Y)-B(Y))dW_t + \int_0^t (F_n(Y)-F(Y))dt|||_2^2 = \\ &\sup_{t\in[0,T]} E\|\int_0^t (B_n(Y)-B(Y))dW_s + \int_0^t (F_n(Y)-F(Y))ds\|_{HS}^2 \leqslant \\ &\sup_{t\in[0,T]} 2E(\int_0^t \|F_n(Y)-F(Y)\|_{HS}ds)^2 + 2E\|\int_0^t B_n(Y)-B(Y)dW_s\|_{HS}^2 \leqslant \\ &\leqslant \sup_{t\in[0,T]} 2E(\int_0^t \|F_n(Y)-F(Y)\|_{HS}ds)^2 + 8E\int_0^t \|B_n(Y)-B(Y)\|_{L_2^0}^2 ds \xrightarrow[n\to\infty]{} 0.\end{aligned}$$

Indeed,

$$\begin{aligned}F_n(Y)-F(Y) &= \frac{1}{2}\sum_m [(Q_n^{1/2}e_m)^T(Q_n^{1/2}e_m) - (Q^{1/2}e_m)^T(Q^{1/2}e_m)](Y+I) = \\ &= \frac{1}{2}\sum_m [e_m{}^TQ_ne_m - e_m{}^TQe_m](Y+I) = \frac{1}{2}\sum_m (e_m{}^T(Q_n-Q)e_m)(Y+I)\end{aligned}$$

by Proposition 3.2. Note that $Q_n - Q$ is a self-adjoint trace class operator, thus there exists a basis $\{e_m^{(n)}\}_{m=1}^\infty$ in which Q_n-Q is diagonal: $(Q_n-Q)e_m^{(n)} = \lambda_m^{(n)}e_m^{(n)}$. Thus by Lemma 4.2

$$\|F_n(Y)-F(Y)\|_{HS} \leqslant \|Y+I\|_{HS}\frac{1}{2}\sum_m |\lambda_m^{(n)}| = \frac{1}{2}Tr(Q-Q_n)\|Y+I\|_{HS} \xrightarrow[n\to\infty]{} 0.$$

Now let us estimate $\|B(Y)-B_n(Y)\|_{L_2^0}^2$. Choose an orthonormal basis $\{u_m\}_{m=1}^\infty$ in U_0. Then

$$\begin{aligned}\|B(Y)-B_n(Y)\|_{L_2^0}^2 &= \sum_{m=1}^\infty \|(B(Y)-B_n(Y))u_m\|_{HS}^2 = \\ &= \sum_{m=1}^\infty \|(I-P_n)u_m(Y+I)\|_{HS}^2 \leqslant \|Y+I\|^2\sum_{m=1}^\infty \|(I-P_n)u_m\|_{HS}^2 \leqslant \\ &\leqslant \|Y+I\|^2 Tr[Q^{1/2}(I-P_n)Q^{1/2}] = \|Y+I\|^2 Tr[(I-P_n)Q^{1/2}Q^{1/2}] = \\ &= \|Y+I\|^2 Tr[(I-P_n)Q] \xrightarrow[n\to\infty]{} 0.\end{aligned}$$

by Lemma 4.2.

We know that there are unique elements Y, Y_n in the space $\mathcal{H}_2$ such that $Y = K(y, Y)$, $Y_n = K_n(y, Y_n)$ and therefore $\lim_{n\to\infty} Y_n = Y$ for any y by the local inversion lemma. $\square$

Now we will consider a concrete sequence of $\mathfrak{g}_n$ depending on an orthonormal basis $\{f_m\}_{m=1}^{\infty}$ of H_r. Recall that $HS_{n\times n} = \{A : \langle Af_m, f_k\rangle = 0 \text{ if } \max(m,k) > n\}$. Take a basis $\{e_k\}_{k=1}^{\infty}$ of HS such that $\{e_k\}_{k=1}^{2n^2}$ is a basis of $HS_{n\times n}$.

We also defined groups $SO_{HS}(n) = \{B \in SO_{HS}, B - I \in HS_{n\times n}\}$ isomorphic to the complexification of the special orthogonal group of $\mathbb{R}^n$ and Lie algebras $\mathfrak{g}_n = Lie(SO_{HS}(n)) = so_{HS}(n) = \{A \in HS_{n\times n}, A^T = -A\}$ with an inner product $\langle\langle A, B\rangle\rangle_n = \langle (P_nQP_n)^{-1/2}A, (P_nQP_n)^{-1/2}B\rangle_{HS}$. Here $P_n = P_{so_{HS}(n)}$ and P_nQP_n is considered as an operator on $\mathfrak{g}_n$, where it is invertible and positive.

Proposition 4.4. *Define an operator K on HS by $Kf_1 = -f_1, Kf_m = f_m$, for $m > 1$. Then for any $A, B \in SO_{HS}(n)$ we have $\|A - KB\| \geqslant \|B\|$, where $\|\cdot\|$ is the operator norm.*

Proof. $\|A - KB\| \geqslant \|AB^{-1} - K\|\|B^{-1}\|^{-1} = \|AB^{-1} - K\|\|B\|^{-1}$, since $B \in SO_{HS}(n)$. We know that $AB^{-1} \in SO_{HS}(n)$, therefore we can use the Cartan decomposition to write AB^{-1} in the form UV, where $U, V \in SO_{HS}$, U is real orthogonal and V is self-adjoint and positive. We have $\|AB^{-1} - K\| = \|UV - K\| = \|V - U^*K\|$. Note that U^*K is real orthogonal and $detU^*K = -1$. This means that at least one eigenvalue of U^*K is equal to -1. All the eigenvalues of V are real and positive. Therefore $\|V - U^*K\| \geqslant 1$ which completes the proof. $\square$

Corollary 4.5. *The group O_{HS} consists of two connected components, namely $O_{HS} = SO_{HS} \cup K(SO_{HS})$.*

From now we assume that $so_{HS}(n)$ is an invariant subspace of operator Q for all n. There is a plenty of such operators Q. For example, all nonnegative (positive on so_{HS}) trace class operators which are diagonal in the basis $\{e_k\}_{k=1}^{\infty}$. Under this condition, in particular, $\langle\langle A, B\rangle\rangle_n = \langle Q^{-1/2}A, Q^{-1/2}B\rangle_{HS}$.

Denote $P_t^n f(Y) = Ef(Y_n(t, Y))$, $Y \in HS_{n\times n}$ and $v^n(t, Y) = P_t^n f(Y)$. Then the following theorem holds.

Theorem 4.6. *For any $f \in C_b^2(HS_{n\times n})$ the function $v^n(t, Y)$ is a unique strict solution from $C_b^{1,2}(HS_{n\times n})$ for the parabolic type equation:*

$$\frac{\partial}{\partial t} v^n(t, Y) = \frac{1}{2} Tr[v^n_{YY}(B_n(Y)Q^{1/2})(B_n(Y)Q^{1/2})^*] + \langle F_n(Y), v^n_Y\rangle_{HS}$$
$$v^n(0, Y) = f(Y), t > 0, Y \in HS_{n\times n}.$$

We want to show that P_t^n corresponds to the heat kernel measure defined on $SO_{HS}(n)$ as on a Lie group (i. e. as in the finite dimensional case).

Note that for any $Y \in HS_{n\times n}$

$$Tr[v^n_{YY}(t, Y)(B_n(Y)Q^{1/2})(B_n(Y)Q^{1/2})^*] =$$
$$\sum_m \langle v^n_{YY}(t, Y)P_n(Q^{1/2}e_m)\ (Y + I), P_n(Q^{1/2}e_m)\ (Y + I)\rangle_{HS}.$$

$$\langle F_n(Y), v^n_Y(t,Y)\rangle_{HS} =$$
$$-\frac{1}{2}\sum_m \langle ((P_nQP_n)^{1/2}e_m)^T((P_nQP_n)^{1/2}e_m)(Y+I), v^n_Y(t,Y)\rangle_{HS}.$$

Note that $\{(P_nQP_n)^{1/2}e_m\}_{m=1}^{2n^2}$ is an orthonormal basis of $\mathfrak{g}_n$ since all $\mathfrak{g}_n$ are invariant subspaces of Q, so

$$L_1 v = \frac{1}{2}Tr[v^n_{YY}(t,Y)(B_n(Y)Q^{1/2})(B_n(Y)Q^{1/2})^*] + \langle F_n(Y), v^n_Y(t,Y)\rangle_{HS}$$

is equal to the Laplacian Δ^n on $SO_{HS}(n)$ defined similarly to (3.5). Thus, the transition probability $P^n_t = \mu^n_t(dX)$, where the latter is the heat kernel measure on $SO_{HS}(n)$ defined in the usual way.

Notation 4.7. $\|f\|_{L^2(SO_{HS}(n),\mu^n_t)} = \|f\|_{t,n}, \|f\|_{L^2(SO_{HS},\mu_t)} = \|f\|_t$.

5. Properties of the stochastic process and holomorphic polynomials

The following Proposition is a refinement of the second part of Theorem 3.3. It might be useful for the estimates of the L^2-norms of holomorphic polynomials on SO_{HS}.

Proposition 5.1. *For any $p > 2, t > 0$*

$$E\|Y_t\|^p_{HS} < \frac{1}{C_{p,t}}(e^t C_{p,t} - 1),$$

where $C_{p,t} = 2^{p-1}\max\{\frac{1}{2}(TrQ)^p t^{p-1}, C_{\frac{p}{2}}2^{p-1}(TrQ)^{\frac{p}{2}}t^{\frac{p}{2}-1}\}$.

Proof. First of all, let us estimate $E(\int_0^t \|F(Y_s)\|_{HS}ds)^p$. From the proof of condition 3 of Theorem 3.3 we have $\|F(Y)\|_{HS} \leqslant \frac{1}{2}TrQ(\|Y\|_{HS}+1)$. Use Hölder's inequality $\int_0^t f(s)ds \leqslant (\int_0^t f^p ds)^{\frac{1}{p}}(t^{\frac{1}{p'}})$ for $f(s) = \|F(Y_s)\|_{HS}$ to get:

$$E(\int_0^t \|F(Y_s)\|_{HS}ds)^p \leqslant t^{p-1}E\int_0^t \|F(Y_s)\|^p_{HS}ds \leqslant t^{p-1}(\frac{1}{2}TrQ)^p E\int_0^t (\|Y\|_{HS}+1)^p ds \tag{5.1}$$

Now estimate $E\|\int_0^t B(Y_s)dW_s\|^p_{HS}$. From part 6 of the proof of Theorem 3.3 we know that $\|B(Y)\|^2_{L_2^0} \leqslant 2TrQ(\|Y\|^2_{HS}+1)$. In addition we will use Lemma 7.2, p.182 from the book by G.DaPrato and J.Zabczyk [3]: for any $r \geqslant 1$ and for arbitrary L_2^0-valued predictable process $\Phi(t)$,

$$E(\sup_{s\in[0,t]}\|\int_0^s \Phi(u)dW(u)\|^{2r}_{HS}) \leqslant C_r E(\int_0^t \|\Phi(s)\|^2_{L_2^0}ds)^r, t\in[0,T], \tag{5.2}$$

where $C_r = (r(2r-1))^r(\frac{2r}{2r-1})^{2r^2}$. Then

$$(5.3)\quad E\|\int_0^t B(Y_s)dW_s\|_{HS}^p \leqslant C_{\frac{p}{2}} E(\int_0^t \|B(Y_s)\|_{L_2^0}^2 ds)^{\frac{p}{2}} \leqslant$$

$$C_{\frac{p}{2}}(2TrQ)^{\frac{p}{2}} E(\int_0^t ((\|Y\|_{HS}^2+1)ds)^{\frac{p}{2}} \leqslant C_{\frac{p}{2}} 2^{\frac{p}{2}}(TrQ)^{\frac{p}{2}} t^{\frac{p}{2}-1} E\int_0^t (\|Y\|_{HS}^2+1)^{\frac{p}{2}} ds$$

Now we can use inequality $(x+1)^q \leqslant 2^{q-1}(x^q+1)$ for any $x \geqslant 0$ for the estimates (5.1) and (5.3):

$$E(\int_0^t \|F(Y_s)\|_{HS} ds)^p \leqslant t^{p-1}(\frac{1}{2}TrQ)^p 2^{p-1} E\int_0^t (1+\|Y\|_{HS}^p)ds =$$

$$= \frac{1}{2} t^{p-1}(TrQ)^p (t + E\int_0^t \|Y\|_{HS}^p ds)$$

and

$$E\|\int_0^t B(Y_s)dW_s\|_{HS}^p \leqslant C_{\frac{p}{2}} 2^{\frac{p}{2}}(TrQ)^{\frac{p}{2}} t^{\frac{p}{2}-1} 2^{\frac{p}{2}-1} E\int_0^t (1+\|Y_s\|_{HS}^p)ds =$$

$$= C_{\frac{p}{2}}(TrQ)^{\frac{p}{2}} 2^{p-1} t^{\frac{p}{2}-1}(t + E\int_0^t \|Y\|_{HS}^p ds)$$

Finally,

$$E\|Y_t\|_{HS}^p \leqslant 2^{p-1}[E(\int_0^t \|F(Y_s)\|_{HS} ds)^p + E\|\int_0^t B(Y_s)dW_s\|_{HS}^p]$$

$$\leqslant C_{p,t}(t + E\int_0^t \|Y_s\|_{HS}^p ds),$$

where $C_{p,t} = 2^{p-1}\max\{\frac{1}{2}(TrQ)^p t^{p-1}, C_{\frac{p}{2}} 2^{p-1}(TrQ)^{\frac{p}{2}} t^{\frac{p}{2}-1}\}$.

Thus, $E\|Y_t\|_{HS}^p < \frac{1}{C_{p,t}}(e^t C_{p,t} - 1)$ by Gronwall's lemma. □

Notation 5.2. *Suppose f is a function from SO_{HS} to $\mathbb{C}$. Let $D_X f = (Df)(X)$ denote a unique element of U_0^* such that*

$$(D_X f)(\xi) = (\tilde{\xi} f)(X) = \frac{d}{dt}\Big|_{t=0} f(\exp(t\xi)X), \qquad \xi \in U_0 = Q^{1/2} so_{HS}, X \in SO_{HS},$$

if the derivative exists. Similarly $D_X^k f = (D^k f)(X)$ denotes a unique element of $(U_0^{\otimes k})^*$ such that

$$(D_X^k f)(\beta) = (\tilde{\beta} f)(X), \qquad \beta \in U_0^{\otimes k}, X \in SO_{HS}$$

and $D_{n,X}^k f = (D_n^k f)(X)$ denotes a unique element of $(\mathfrak{g}_n^*)^{\otimes k}$ such that

$$(D_{n,X}^k f)(\beta) = (\tilde{\beta} f)(X), \qquad \beta \in \mathfrak{g}_n{}^{\otimes k}, X \in SO_{HS}(n).$$

Later we will use the following notation

$$(1-D)_X^{-1} f = \sum_{k=0}^{\infty} D_X^k f \qquad \text{and} \qquad (1-D)_{n,X}^{-1} = \sum_{k=0}^{\infty} D_{n,X}^k f.$$

Recall that we have fixed $\{f_k\}_{k=1}^{\infty}$, an orthonormal basis of H_r.

Definition 5.3. Let $p : SO_{HS} \longrightarrow \mathbb{C}$. p is called *a holomorphic polynomial*, if p is a complex linear combination of finite products of monomials $p_k^{m,l}(X) = (\langle ReXf_m, f_l\rangle + i\langle ImXf_m, f_l\rangle)^k$. We will denote the space of all such polynomials by $\mathcal{HP}$.

Theorem 5.4. 1. *$\mathcal{HP} \subset L^p$, for any $p > 1$.*
2. *The kth derivative $(D_X^k p)(\beta)$ exists and is complex linear for any $p \in \mathcal{P}$, $X \in SO_{HS}$, $\beta \in (U_0)^{\otimes k} = (Q^{1/2} so_{HS})^{\otimes k}$.*

Proof.

$$\int_{SO_{HS}} |p_k^{m,l}(X)|^p \mu_t(dX) = E|p_k^{m,l}(Y_t + I)|^p = E|(Y_t)_{ml} + \delta_{ml})|^{pk} \leqslant$$

$$E2^{pk-1}(|(Y_t)_{ml}|^{pk} + 1) \leqslant 2^{pk-1} E(\|Y_t\|_{HS}^{pk} + 1) < \infty.$$

Which means that $p_k^{m,l} \in L^p$ for any $p > 1$. Thus $\mathcal{HP} \subset L^p(SO_{HS}, \mu_t)$.

To prove the second part is enough to check that $p_k^{m,l}$ is holomorphic. Let ξ be any element of U_0, then the derivative of $p_k^{m,l}$ in the direction of ξ can be calculated by the formula

$$(\tilde{\xi} p_k^{m,l})(X) = \frac{d}{dt}\bigg|_{t=0} p_k^{m,l}(\exp(t\xi)X) =$$
$$= k(\langle Re(\xi X)f_m, f_l\rangle + i\langle Im(\xi X)f_m, f_l\rangle) p_{k-1}^{m,l}(X) = kp_1^{m,l}(\xi X)p_{k-1}^{m,l}(X).$$

To prove that $p_k^{m,l}$ is holomorphic all we need is to check that $Dp_k^{m,l}(\xi) = (\tilde{\xi} p_k^{m,l})$ is complex linear. Indeed, for any $\alpha \in \mathbb{C}$ we have

$$((\tilde{\alpha\xi}) p_k^{m,l})(X) = kp_1^{m,l}(\alpha\xi X)p_{k-1}^{m,l}(X) = k\alpha p_1^{m,l}(\xi X)p_{k-1}^{m,l}(X) = \alpha(\tilde{\xi} p_k^{m,l})(X)$$

□

Remark 5.5. Any polynomial $p \in \mathcal{HP}$ can be written in the following form $p(X) = \sum_{k=1}^{m} \prod_{l=1}^{n_m} Tr(A_{kl}X)$ for some $A_{kl} \in HS$. The converse is not true in general, but the closure in $L^2(SO_{HS}, \mu_t)$ of all functions of the form $\sum_{k=1}^{m} \prod_{l=1}^{n_m} Tr(A_{kl}X)$ coincides with the closure of holomorphic polynomials. Therefore the next definition is basis-independent, though the definition of $\mathcal{HP}$ depends on the choice of $\{f_k\}_{k=1}^{\infty}$.

Definition 5.6. The closure of all holomorphic polynomials in $L^2(SO_{HS}, \mu_t)$ is called $\mathcal{H}L^2(SO_{HS}, \mu_t)$.

Lemma 5.7. *Let $f : SO_{HS} \longrightarrow [0, \infty]$ be a continuous function in $L^2(SO_{HS}, \mu_t)$. If $\|f\|_{t,n} \leqslant C < \infty$, then $Ef(X_n) \xrightarrow[n\to\infty]{} Ef(X)$.*

Proof. Note that there exists a subsequence $\{X_{n_k}\}$ such that $X_{n_k} \xrightarrow[k\to\infty]{} X$ a.s. We will prove first that if $\|f\|_{t,n_k} \leqslant C < \infty$, then $Ef(X_{n_k}) \xrightarrow[k\to\infty]{} Ef(X)$. Denote $g_k(\omega) = f(X_{n_k}(\omega)), g(\omega) = f(X(\omega)), \omega \in \Omega$.
Our goal is to prove that $\int_\Omega g_k(\omega)dP \longrightarrow \int_\Omega g(\omega)dP$ as $k \to \infty$. Define $f_l(X) = \min\{f(X), l\}$ for $l > 0$ and $g_{k,l}(\omega) = f_l(X_{n_k}(\omega)), g_l(\omega) = f_l(X(\omega))$. Then $g_{k,l} \leqslant$

$l, g_l \leqslant l$ for any $\omega \in \Omega$, so $\int_\Omega g_{k,l}(\omega)dP \xrightarrow[k\to\infty]{} \int_\Omega g_l(\omega)dP$ by the Dominated Convergence Theorem since f is continuous.

$$\int_\Omega (g_k(\omega) - g_{k,l}(\omega))dP = \int_{\{\omega: f(X_{n_k})\geqslant l\}} (f(X_{n_k}) - l)dP \leqslant$$

$$\leqslant \int_\Omega f(X_{n_k}) \mathbb{1}_{\{\omega: f(X_{n_k})\geqslant l\}} dP \leqslant \|f\|_{t,n_k} (P\{\omega : f(X_{n_k}) \geqslant l\})^{\frac{1}{2}} \leqslant$$

$$\leqslant \|f\|_{t,n_k} \frac{E|f(X_{n_k})|}{l}$$

by Chebyshev's inequality. Thus

$$0 \leqslant \int_\Omega (g_k(\omega) - g_{k,l}(\omega))dP \leqslant \frac{C^2}{l}.$$

Similarly

$$0 \leqslant \int_\Omega (g(\omega) - g_l(\omega))dP \leqslant \frac{1}{l}\|f\|_t E|f(X)|.$$

Therefore $Ef(X_{n_k}) \longrightarrow Ef(X)$ as $k \to \infty$.

To complete the proof suppose that the conclusion is not true. Then there is a subsequence X_{n_k} such that $|Ef(X_{n_k} - Ef(X)| > \varepsilon$ for any k. However, we always can choose a subsequence $X_{n_{k_m}}$ such that $X_{n_{k_m}} \xrightarrow[m\to\infty]{} X$ a.s. and therefore $Ef(X_{n_{k_m}}) \xrightarrow[m\to\infty]{} Ef(X)$. Contradiction. □

Corollary 5.8. $\|f\|_{t,n} \xrightarrow[n\to\infty]{} \|f\|_t$ *for any* $f \in \mathcal{HP}$.

Proof. From the estimates on $E\|Y_t\|^p_{HS}$ we can find $C(p,t)$ such that $\||p|^2\|_{t,n(k)} \leqslant C(p,t)$ for any k. Then apply Lemma 5.7 to $f = |p|^2$. □

6. Estimates of derivatives of holomorphic functions

Let $SO(\infty)$ denote the closure of $\cup_n SO_{HS}(n)$ in the following Riemannian metric: $d(A,B) = \inf\{\int_0^1 \|h(s)^{-1}\frac{dh}{ds}\|_{U_0} ds\}$, where $h : [0,1] \to SO_{HS}, h(0) = A, h(1) = B\}$. Note that $\mathfrak{g} = \cup_n \mathfrak{g}(n)$ is a Lie algebra and U_0 is the closure of $\mathfrak{g}$ in the inner product on U_0.

The following estimate was proved by B. Driver and L. Gross in [7] for $f \in \mathcal{H}L^2(SO_{HS}(n), \mu_t^n)$:

$$|(\tilde{\beta} f)(g)|^2_{(\mathfrak{g}_n^*)^{\otimes k}} \leqslant \|f\|^2_{t,n} \frac{k!|\beta|^2}{r^k} e^{|g|^2/s}, \text{for } \ g \in SO_{HS}(n), r > 0, s + r \leqslant t, \beta \in \mathfrak{g}_n^{\otimes k},$$

where $|(\tilde{\beta} f)(g)|_{(\mathfrak{g}_n^*)^{\otimes k}}$ is $(T_g G(n)^*)^{\otimes k}$-norm (which can be identified with $(\mathfrak{g}_n^*)^{\otimes k}$). We will need a slight modification of this estimate. Taking supremum over all $\beta \in \mathfrak{g}_n^{\otimes k}, |\beta| = 1$, we get

$$|(D_n^k f)(g)|^2 \leqslant \|f\|^2_{t,n} \frac{k!}{r^k} e^{|g|^2/s}, \tag{6.1}$$

where D_n^k is defined for $SO_{HS}(n)$ and $\mathfrak{g}_n$ by Notation 5.2. Note that if $\|f\|_{t,n}$ are uniformly bounded, (6.1) gives us a uniform bound, i. e. independent of n. The following estimates can be proved for D_n^k:

Lemma 6.1. *Let $r > 0, q + r \leqslant t, X, Y \in SO_{HS}(n), f \in \mathcal{H}L^2(SO_{HS}(n), \mu_t^n)$. Then*

$$|(D_n^k f)(X) - (D_n^k f)(Y)|_{(\mathfrak{g}_n^*)^{\otimes k}} \leqslant \|f\|_{t,n} K_{k+1} d(X, Y),$$

where $K_k = K_k(X, Y) = \left(\frac{k!}{r^k}\right)^{1/2} e^{|X|^2 + d(X,Y)^2/q}$.

Proof. Take $h : [0, 1] \longrightarrow SO_{HS}(n)$ such that $h(0) = X, h(1) = Y$. Then by (6.1)

$$|D_n^k f(X_i) - D_n^k f(Y)|_{(\mathfrak{g}_n^*)^{\otimes k}} = |\int_0^1 \frac{d}{ds}(D_n^k f)(h(s))ds|_{(\mathfrak{g}_n^*)^{\otimes k}} \leqslant$$

$$\int_0^1 |\frac{d}{ds}(D_n^k f)(h(s))|_{(\mathfrak{g}_n^*)^{\otimes k}} ds \leqslant \int_0^1 |D(D_n^k f)(h(s))(\frac{d}{ds}h(s))|_{(\mathfrak{g}_n^*)^{\otimes k}} ds \leqslant$$

$$\int_0^1 |D((D_n^k f)(h(s))|_{(\mathfrak{g}_n^*)^{\otimes k+1}} |\frac{d}{ds}h(s)|_{T_{h(s)}SO_{HS}(n)} ds \leqslant$$

$$\|f\|_{t,n} \Big(\frac{(k+1)!}{r^{k+1}} \sup_{u \in [0,1]} e^{|h(u)|^2/q} \Big)^{1/2} \int_0^1 |\frac{d}{ds}h(s)|_{T_{h(s)}SO_{HS}(n)} ds \leqslant$$

$$\|f\|_{t,n} \Big(\frac{(k+1)!}{r^{k+1}} \Big)^{1/2} \sup_{u \in [0,1]} e^{|h(0)|^2 + d(h(0), h(u))^2/q} \int_0^1 |\frac{d}{ds}h(s)|_{T_{h(s)}SO_{HS}(n)} ds$$

Taking infimum over all such h we see that

$$|D_n^k f(X) - D_n^k f(Y)|_{(\mathfrak{g}_n^*)^{\otimes k}} \leqslant \|f\|_{t,n} \Big(\frac{(k+1)!}{r^{k+1}} \Big)^{1/2} e^{|X|^2 + d(X,Y)^2/q} d(X, Y).$$

□

Lemma 6.2. *Let $X \in SO_{HS}(n), f, g \in \mathcal{H}L^2(SO_{HS}(n), \mu_t^n), t > 0$. Then*

$$|D_n^k f(X) - D_n^k g(X)|_{(\mathfrak{g}_n^*)^{\otimes k}} \leqslant M_k \|f - g\|_{t,n},$$

where $M_k = M_k(X, t) = \left(\frac{k!}{(t/2)^k}\right)^{1/2} e^{|X|^2/t}$.

Proof. From (6.1) we have that for $r > 0, q + r \leqslant t$

$$|D_n^k f(X) - D_n^k g(X)|^2_{(\mathfrak{g}_n^*)^{\otimes k}} \leqslant \|f - g\|^2_{t,n} \frac{k!}{r^k} e^{|X|^2/q}.$$

Now take $q = r = t/2$ to get what we claimed. □

Lemma 6.3. *Let $X \in SO_{HS}(n), \xi \in \mathfrak{g}_n, f \in \mathcal{H}L^2(SO_{HS}(n), \mu_t^n)$. Then there is a constant $C = C(X, \xi, t) > 0$ such that*

$$|\frac{f(e^{u\xi}X) - f(X)}{u} - (D_X f)(\xi)| \leqslant \|f\|_{t,n} C u$$

for small enough $u > 0$.

Proof. Let $h(s)=e^{s\xi}X, 0\leqslant s\leqslant u$.

$$\frac{f(e^{u\xi}X)-f(X)}{u}-(D_Xf)(\xi)=\frac{1}{u}\int_0^u\frac{d}{ds}f(h(s))ds-(D_Xf)(\xi)=$$

$$=\frac{1}{u}\int_0^u(\frac{d}{ds}f(h(s))-(D_Xf)(\xi))ds=\frac{1}{u}\int_0^u(D_{h(s)}f)(\frac{d}{ds}h(s))-(D_Xf)(\xi))ds=$$

$$=\frac{1}{u}\int_0^u(D_{h(s)}f)(\xi)-(D_Xf)(\xi))ds.$$

Thus by Lemma 6.1 for any $r>0, q+r\leqslant t$

$$|\frac{f(e^{u\xi}X)-f(X)}{u}-(D_Xf)(\xi)|\leqslant\frac{1}{u}\int_0^u|(D_{h(s)}f)(\xi)-(D_Xf)(\xi))|ds\leqslant$$

$$\frac{1}{u}\int_0^u\|f\|_{t,n}K_2(h(s),X)d(h(s),X)\|\xi\|_{U_0}ds\leqslant$$

$$\leqslant\frac{1}{u}\int_0^u\|f\|_{t,n}\frac{\sqrt{2}}{r}e^{(|X|^2+\|\xi\|_{U_0}^2s^2)/q}s\|\xi\|_{U_0}^2ds=$$

$$=\|f\|_{t,n}\frac{\sqrt{2}}{r}e^{|X|^2/q}\|\xi\|_{U_0}^2\frac{1}{u}\int_0^u e^{(\|\xi\|_{U_0}^2/q)s^2}sds=$$

$$=\|f\|_{t,n}\frac{q}{\sqrt{2}r}e^{|X|^2/q}\frac{e^{(\|\xi\|_{U_0}^2/q)u^2}-1}{u}\leqslant\|f\|_{t,n}Cu$$

for small u. $\square$

Theorem 6.4. *Let f be a function on $\cup_n SO_{HS}(n)$. Suppose that $f|_{SO_{HS}(n)}$ is holomorphic for any n and $\|f\|_{t,n}\leqslant C_t<\infty$. Then there exists a unique continuous function g on $SO(\infty)$ such that $g|_{SO_{HS}(n)}=f$ for any n.*

Proof. Take $X\in SO(\infty)$. We would like to define g by

$$g(X)=\lim_{n\to\infty}f(X_n),\qquad X_n\in SO_{HS}(n), X_n\xrightarrow[n\to\infty]{d}X.$$

Let us check that the limit exist. Assume that $l\leqslant n$. By Lemma 6.1 for $X_n, X_l\in SO_{HS}(n)$

$$|D^kf(X_n)-D^kf(X_l)|_{(\mathfrak{g}_n^*)^{\otimes k}}\leqslant\|f\|_{t,n}K_{k+1}(X_n,X_l)d(X_n,X_l)$$
$$\leqslant C_tK_{k+1}(X_n,X_l)d(X_n,X_l).$$

Thus $f(X_n)$ is a Cauchy sequence and therefore the limit exists.

The uniqueness of the extension follows from this simple argument:

$$|g_1(X)-g_2(X)|\leqslant|g_1(X)-g_1(X_n)|+|g_1(X_n)-g_2(X_n)|+|g_2(X_n)-g_2(X)|=$$

$$=|g_1(X)-g_1(X_n)|+|g_2(X_n)-g_2(X)|\xrightarrow[n\to\infty]{}0$$

for $X_n\in SO_{HS}(n)$, $X\in SO(\infty), X_n\xrightarrow[n\to\infty]{d}X$. $\square$

7. The isometries

In Section 6 we noted that $\mathfrak{g} = \cup_n \mathfrak{g}_n = \cup_n so(n)$ is a Lie algebra which is dense in U_0. For this Lie algebra (with the inner product $\langle\langle \cdot, \cdot \rangle\rangle$) and its Lie subalgebras $\mathfrak{g}_n$ we can consider $T, T', T_t^*, J, J^0, J_t^0$ defined in Section 2.

Notation 7.1.

$$T(\mathfrak{g}) = T, T'(\mathfrak{g}) = T', T_t^*(\mathfrak{g}) = T_t^*, T(\mathfrak{g}(n)) = T_n, T'(\mathfrak{g}(n)) = T_n', T_t^*(\mathfrak{g}(n)) = T_{t,n}^*,$$
$$J(\mathfrak{g}) = J, J^0(\mathfrak{g}) = J^0, J_t^0(\mathfrak{g}) = J_t^0, J(\mathfrak{g}(n)) = J_n, J^0(\mathfrak{g}(n)) = J_n^0, J_t^0(\mathfrak{g}(n)) = J_{t,n}^0.$$

The norm defined by Equation (2.2) will be denoted by $|\cdot|_t$ for $\mathfrak{g}$ and by $|\cdot|_{t,n}$ for $\mathfrak{g}_n$.

Lemma 7.2. *Suppose $f\mid_{SO_{HS}(n)} \in \mathcal{H}L^2(SO_{HS}(n), \mu_t^n)$ for all n. Then $\|f\|_{t,n} \leqslant \|f\|_{t,n+1}$ for any n.*

Proof. First of all, $\|f\|_{t,n}^2 = |(1-D)_{n,e}^{-1} f|_{t,n}^2$ by the Driver-Gross isomorphism, where $|(1-D_n)_e^{-1} f|_{t,n}^2 = |\sum_{k=0}^{\infty} (D_n^k f)(e)|_{t,n}^2$. Note that

$$|(D_n^k f)(e)|^2 = \sum_{1 \leqslant i_m \leqslant dim \mathfrak{g}_n} |\tilde{\xi}_{i_1} ... \tilde{\xi}_{i_k} f(e)|^2$$

for any orthonormal basis $\{\xi_l\}_{l=1}^{dim \mathfrak{g}_n}$ of $\mathfrak{g}_n$. Therefore $|(D_n^k f)(e)|^2 \leqslant |(D_{n+1}^k f)(e)|^2$, so the claim holds. ☐

Theorem 6.4 allows us to introduce the following definition.

Definition 7.3. $\mathcal{H}L^2(SO(\infty))$ is a space of continuous functions on $SO(\infty)$ such that their restrictions to $SO_{HS}(n)$ are holomorphic for every n and $\|f\|_{t,\infty} = \sup_n \{\|f\|_{t,n}\} = \lim_{n\to\infty} \|f\|_{t,n} < \infty$.

Theorem 7.4. *The embedding of $\mathcal{HP}$ into $\mathcal{H}L^2(SO(\infty))$ can be extended to an isometry from $\mathcal{H}L^2(SO_{HS}, \mu_t)$ into $\mathcal{H}L^2(SO(\infty))$.*

Proof. By Theorem 5.4 $\mathcal{HP} \subset L^2(SO_{HS}, \mu_t)$. In addition, by Corollary 5.8

$$\|p\|_{t,n} \xrightarrow[n\to\infty]{} \|p\|_t, \qquad p \in \mathcal{HP}.$$

Therefore $\|p\|_{t,\infty} = \|p\|_t$ and so the embedding is an isometry.

$\mathcal{H}L^2(SO_{HS}, \mu_t)$ is the closure of $\mathcal{HP}$, therefore the isometry extends to it from $\mathcal{HP}$. ☐

Theorem 7.5. $(1-D)_e^{-1}$ *is an isometry from $\mathcal{H}L^2(SO(\infty))$ into J_t^0.*

Proof. T_n is a subalgebra of T. Note that T_n' can be easily identified with a subspace of T'. Namely, for any $\alpha_n \in T_n'$ we can define α as follows:

$$\alpha = \begin{cases} \alpha_n & \text{on } T_n \\ 0 & \text{on } T_n^{\perp} \end{cases}$$

Therefore $T_n' = (T_n^{\perp})^0$. Define Π_n to be an orthogonal projection from T to T_n. Let Π_n' denote the following map from T' to T_n': $(\Pi_n' \alpha)(x) = \alpha(\Pi_n x), \alpha \in T', x \in T$. Then $\Pi_n' \circ (1-D)_e^{-1} : \mathcal{H}L^2(SO_{HS}, \mu_t) \longrightarrow T_n'$ is equal to $(1-D)_{n,e}^{-1}$. Indeed, note that if we choose an orthonormal basis of $\mathfrak{g}$ so that $\{\xi_m\}_{m=1}^{m=dim\mathfrak{g}_n}$ is an orthonormal basis of $\mathfrak{g}_n$, then Π_n can be described explicitly

$$\Pi_n(\xi_{k_1}\otimes\xi_{k_2}\otimes...\otimes\xi_{k_l}) = \begin{cases} 0 \ \text{ if } \ k_s > dim\ \mathfrak{g}_n \text{ for some } \ 1\leqslant s\leqslant l \\ \xi_{k_1}\otimes\xi_{k_2}\otimes...\otimes\xi_{k_l} \text{ otherwise}\end{cases}$$

And $\langle \Pi_n'\circ(1-D)_e^{-1}f, \xi_{k_1}\otimes\xi_{k_2}\otimes...\otimes\xi_{k_l}\rangle = \langle (1-D)_e^{-1}f, \Pi_n(\xi_{k_1}\otimes\xi_{k_2}\otimes...\otimes\xi_{k_l})\rangle =$

$$\begin{cases} 0 \ \text{ if } \ k_s > dim\ \mathfrak{g}_n \text{ for some } \ 1\leqslant s\leqslant l \\ \langle (1-D)_e^{-1}f, \xi_{k_1}\otimes\xi_{k_2}\otimes...\otimes\xi_{k_l}\rangle \text{ otherwise}\end{cases}$$

So $\Pi_n'\circ(1-D)_e^{-1} = (1-D)_{n,e}^{-1}$. B. Driver and L. Gross proved in [7] that $\Pi_n'\circ(1-D)_e^{-1}$ is an isometry from $\mathcal{H}L^2(SO_{HS}(n),\mu_t^n)$ into $J_{t,n}^0$. Let us define a restriction map R_n: $\mathcal{H}L^2(SO(\infty)) \longrightarrow \mathcal{H}L^2(SO_{HS}(n),\mu_t^n)$ by $f\mapsto f|_{SO_{HS}(n)}$. Thus we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{H}L^2(SO(\infty)) & \xrightarrow{(1-D)_e^{-1}} & J_t^0 \\ {\scriptstyle R_n}\Big\downarrow & & \Big\downarrow{\scriptstyle \Pi_n'} \\ \mathcal{H}L^2(SO_{HS}(n),\mu_t^n) & \xrightarrow[\text{Driver-Gross}]{(1-D)_{n,e}^{-1}} & J_{t,n}^0 \end{array}$$

Now we can prove that $(1-D)_e^{-1}$ is an isometry.

$$\|f\|_{t,\infty} = \lim_{n\to\infty}\|f\|_{t,n} = \lim_{n\to\infty}\|R_nf\|_{t,n}$$

by Proposition 7.2. It is clear that $|\Pi_n'\alpha|_t = |\Pi_n'\alpha|_{t,n} \xrightarrow[n\to\infty]{} |\alpha|_t$ for any $\alpha\in T'$. In particular, $|\Pi_n'\circ(1-D)_e^{-1}f|_{t,n} \xrightarrow[n\to\infty]{} |(1-D)_e^{-1}f|_t$. At the same time $|\Pi_n'\circ(1-D)_e^{-1}f|_{t,n} = |(1-D)_{e,n}^{-1}\circ R_nf|_{t,n} = \|R_nf\|_{t,n} \xrightarrow[n\to\infty]{} \|f\|_{t,\infty}$ by the Driver-Gross isomorphism. □

Theorem 7.6. *$\mathcal{H}L^2(SO(\infty))$ is a Hilbert space.*

Proof. It is clear that $\|\cdot\|_{t,\infty}$ is a seminorm. Suppose that $\|f\|_{t,\infty}=0$. Then $\|f\|_{t,n}=0$ for any $n,t>0$. We know that $\mathcal{H}L^2(SO_{HS}(n),\mu_t^n)$ is a Hilbert space, therefore $f|_{SO_{HS}(n)}=0$ for all n. By Lemma 6.4 we have that $f|_{SO_{HS}(\infty)}=0$. Thus $\|\cdot\|_{t,\infty}$ is a norm.

Let us now show that $\mathcal{H}L^2(SO(\infty))$ is a complete space. Suppose $\{f_m\}_{m=1}^\infty$ is a Cauchy sequence in $\mathcal{H}L^2(SO(\infty))$. Then $\{f_m\,|_{SO_{HS}(n)}\}_{m=1}^\infty$ is a Cauchy sequence in $\mathcal{H}L^2(SO_{HS}(n),\mu_t^n)$ for all n. Therefore there exists $g_n\in\mathcal{H}L^2(SO_{HS}(n))$ such that $f_m\,|_{SO_{HS}(n)} \xrightarrow[m\to\infty]{} g_n$. Note that $g_{n+m}|_{SO_{HS}(n)}=g_n$. In addition,

$$\|g_n\|_{t,n} \leqslant \|f_m|_{SO_{HS}(n)}\|_{t,n} + \|f_m|_{SO_{HS}(n)}-g_n\|_{t,n} \leqslant \|f_m\|_{t,\infty} + \|f_m|_{SO_{HS}(n)}-g_n\|_{t,n}. \tag{7.1}$$

Note that $\{\|f_m\|_{t,\infty}\}_{m=1}^\infty$ is again a Cauchy sequence, so it has a (finite) limit as $m\to\infty$. Taking a limit in (7.1) as $m\to\infty$ we get that $\{\|g_n\|_{t,n}\}_{n=1}^\infty$ are uniformly bounded.

By Lemma 6.4 there exists a continuous function g on $SO_{HS}(\infty)$ such that $g\,|_{SO_{HS}(n)}=g_n$. Thus $g\in\mathcal{H}L^2(SO(\infty))$. Now we need to prove that $f_m \xrightarrow[m\to\infty]{} g$

in $\mathcal{H}L^2(SO(\infty))$. Let $\alpha = (1-D)_e^{-1}g$, $\alpha_m = (1-D)_e^{-1}f_m$. By Theorem 7.5 $(1-D)_e^{-1}$ is an isometry, so α_m is a Cauchy sequence in J_t^0. Thus there exists $\alpha' \in J_t^0$ such that $\alpha_m \xrightarrow[m\to\infty]{} \alpha'$ (in J_t^0). The question is whether $\alpha = \alpha'$.

Using the same notation as in the proof of Theorem 7.5 we see that

$$\Pi'_n \alpha = (1-D)_{n,e}^{-1} g \qquad \text{and} \qquad \Pi'_n \alpha_m = (1-D)_{n,e}^{-1} f_m.$$

We know that $\mathcal{H}L^2(SO_{HS}(n), \mu_t^n)$ is a Hilbert space, therefore $\Pi'_n\alpha = \Pi'_n\alpha'$ for any n. Thus $\alpha = \alpha'$, which completes the proof. □

Remark 7.7. Note that $SO_{HS}(n)$ are not simply connected and therefore the map $(1-D)_e^{-1}$ may not be surjective. It is known that if a connected (finite dimensional) Lie group G is simply connected, then the map $(1-D)_e^{-1}$ is surjective. At the same time if G is not simply connected, then this map is not onto. Our proof of Theorem 7.5 did not rely on any specific properties of groups $SO_{HS}(n)$, therefore one can prove an analog of this theorem in a more abstract situation. In particular, if we consider a sequence of finite dimensional connected simply connected Lie groups $\{G_n\}_{n=1}^{\infty}$, the surjectivity of the map $(1-D)_e^{-1}$ can be proved.

8. Triviality of J_t^0 for an invariant inner product

If we have an isometry described above then the size of J_t^0 gives us information about how many square-integrable holomorphic functions might exist. Indeed, if J_t^0 is isomorphic to $\mathbb{C}$, then the only such functions can be constants. This is exactly the situation that happens in the case of an Ad-invariant inner product on $Lie(SO_{HS})$. To see this we will use the following theorem.

Theorem 8.1. *Suppose $\mathfrak{g}$ is a Lie algebra with an inner product $\langle\cdot,\cdot\rangle$. Assume that there is an orthonormal basis $\{\xi_k\}_{k=1}^{\infty}$ of $\mathfrak{g}$ such that for any k there are nonzero $\alpha_k \in \mathbb{C}$ and an infinite set of distinct pairs (i_m, j_m) satisfying $\xi_k = \alpha_k[\xi_{i_m}, \xi_{j_m}]$. Then J_t^0 is isomorphic to $\mathbb{C}$.*

Proof of Theorem 8.1. First we prove that ξ_k lies in the completion of J in the norm defined by (2.1) for any k, $t > 0$.

Indeed, denote $\eta_m = \xi_{i_m} \otimes \xi_{j_m} - \xi_{j_m} \otimes \xi_{i_m}$.

Then $\eta_m \perp \eta_l$ for any $m \neq l$ and $\|\eta_m\|_t = \|\eta_l\|_t$ for any m, l. From the assumptions on $\mathfrak{g}$ we know that $\eta_m - \frac{1}{\alpha_k}\xi_k \in J$ for all m. If we define $\mu_n = \frac{1}{n}\sum_{m=1}^n \eta_m$, then $\mu_n + \xi_k = \frac{1}{n}\sum_{m=1}^n (\eta_m + \xi_k)$ lies in J. Note that

$$\|\mu_n\|_t = \frac{1}{n}\sqrt{\sum_{m=1}^n \|\eta_m\|_t^2} = \frac{1}{\sqrt{n}}\|\eta_1\|_t \xrightarrow[n\to\infty]{} 0.$$

This shows that $\mu_n + \xi_k \xrightarrow[n\to\infty]{} \xi_k$ and therefore ξ_k lies in $\bar{J}$.

Denote by $T_t^{1<}$ a subspace of T_t containing tensors of order greater than 1, then $T_t = T_t^{1<} \oplus \mathbb{C}$. By the above J is dense in $T_t^{1<}$ for any $t > 0$ and $J_t^0 = T_t^* \cap J^0$ is isomorphic to $\mathbb{C}$ for such $\mathfrak{g}$. □

Corollary 8.2. *J_t^0 is trivial for $\mathfrak{g} = so_{HS}$ with $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{HS}$.*

Proof. Take an orthonormal basis $\{f_m\}_{m=1}^{\infty}$ in H_r. Then $b_{ij}, i < j$ defined by $b_{ij}f_i = -f_j, b_{ij}f_j = f_i, b_{ij}f_m = 0$ for $m \neq i, j$ is an orthogonal basis of so_{HS}. One can think of b_{ij} as the following infinite matrices:

$$b_{ij} = \begin{array}{cc} & \begin{array}{cccccc} \quad & i & & & j & \quad \end{array} \\ \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} & \begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & 0 & -1 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \end{array}$$

Operators b_{ij} satisfy the following identity: $[b_{ik}, b_{jk}] = -b_{ij}$ for $k \neq i, k \neq j$. Therefore J_t^0 is trivial by Theorem 8.1. $\square$

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