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Quasi-invariance for the pinned Brownian motion on a Lie group

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Abstract

We give a new proof of the well-known fact that the pinned Wiener measure on a Lie group is quasi-invariant under right multiplication by finite energy paths. The main technique we use is the time reversal. This approach is different from what B. Driver used to prove quasi-invariance for the pinned Brownian motion on a compact Riemannian manifold.

Keywords: Pinned Brownian motion; Lie group; Quasi-invariance; Girsanov density

1. Introduction

The goal of this article is to prove that the pinned Wiener measure on a Lie group is quasi-invariant under right multiplication by finite energy paths. The result is not new (cf. Malliavin and Malliavin, 1990; Driver, 1994), but our proof is different and simpler. The main technique we use is the time reversal. This approach is different from what Driver used in Driver (1994), where he proved quasi-invariance for the pinned Brownian motion on a compact Riemannian manifold.

The paper is organized as follows. In Section 2 we introduce the notation and describe some well known properties of the Brownian motion on a Lie group G. In Section 3 we show that the unpinned Wiener measure on G is quasi-invariant under right multiplication by finite energy paths. It is certainly a well-known fact, see Shigekawa (1984). The main result of our paper is Theorem 13 which is proved in Section 4. Our proof is based on the quasi-invariance for the unpinned Wiener measure

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and the time reversal to deal with the endpoint singularity of the pinned Brownian motion. We also give a formula for the Girsanov density. Finally, Section 5 explains how the formula for the Girsanov density can be interpreted at the endpoint, though our proof of existence of the density does not use this interpretation. In addition, Section 5 shows that the pinned Brownian motion is a semi-martingale by a method which is simpler than the approach of J.-M. Bismut in Bismut (1984). He used the time reversal to prove pointwise estimates for the gradient of the heat kernel, though these estimates are not needed to deal with the endpoint singularity of the pinned Brownian motion. In conclusion we should mention that L. Gross in Gross (1991) addressed the endpoint singularity of the pinned Brownian motion.

2. Notation and basics

Let G be a connected n-dimensional (real) Lie group. Its Lie algebra $\mathfrak{g}$ will be identified with left-invariant vector fields at the identity e. We assume that there is an Ad_G-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$. The corresponding norm is denoted by $|\cdot|$. The existence of an Ad_G-invariant inner product implies that G is of compact type, that is, G is locally isomorphic to a compact Lie group (Hall, 1999; Driver, 1995). Without loss of generality the group G will be identified with a Lie subgroup of $GL(\mathbb{R}^d)$ for some d. By dg we will denote the (bi-invariant) Haar measure on G. We will use the following notation

(1) $W(G)=\{\omega\in C([0,T],G),\omega(0)=e\}$ is the space of all continuous paths in G beginning at the identity e,
(2) $W^e(G)=\{\omega\in W(G),\omega(0)=\omega(T)=e\}$ is the space of all continuous pinned paths in G,
(3) $H(G)=\{h\in W(G), h$ is absolutely continuous and the norm $\|h\|_H^2=\int_0^T|h(s)^{-1}h'(s)|^2\,\mathrm{d}s$ is finite$\}$ is the Cameron–Martin (finite energy) subset of $W(G)$.
(4) $H^e(G)=W^e(G)\cap H(G)$ is the Cameron–Martin (finite energy) subset of $W^e(G)$.

Let $\{\xi_i\}$ be an orthonormal basis of $\mathfrak{g}$ in $\langle\cdot,\cdot\rangle$. The Laplacian Δ is a left-invariant second order differential operator on $C^\infty(G)$ defined by

$$\Delta f(g)=\frac{1}{2}\sum_{i=1}^{n}\partial_i^2 f(g), \tag{1}$$

where $\partial_i f(g)=\tilde{\xi}_i f(g)=\mathrm{d}/\mathrm{d}t|_{t=0}f(g\exp(t\xi_i))$ and so $\partial_i=\tilde{\xi}_i$ is a left-invariant vector field on G corresponding to ξ_i. Denote by $p(t,x_0,y)$ the heat kernel on G, that is, $p(t,x)=p(t,x_0,x)$ is the fundamental solution to the heat equation

$$\frac{\partial p}{\partial t}=\Delta p,\quad t>0,$$

$$p(0,\cdot)=\delta_{x_0}(\cdot).$$

We define the Wiener measure μ on $W(G)$ and the pinned Wiener measure μ^e on $W^e(G)$ by their finite dimensional distributions.

Definition 1. Let the map $F(\vec{s}): W(G) \to G^n$ be defined by $F(\vec{s})(x) = (x(s_1), \ldots, x(s_n))$, where $\vec{s} = (s_1, \ldots, s_n)$, $x \in W(G)$.

- The Wiener measure μ is determined by the following finite dimensional distributions for a function $g: G^n \to \mathbb{R}$. Suppose $0 \leqslant s_1 < \cdots < s_n \leqslant T$, then the distribution is given by

$$E_\mu[g \circ F(\vec{s})] = \int_{W(G)} (g \circ F(\vec{s}))(x)\, \mathrm{d}\mu(x) = \int_{G^n} g(\vec{r})p(\vec{s}, e, \vec{r})\, \mathrm{d}\vec{r},$$

  where $\mathrm{d}\vec{r} = \mathrm{d}r_1 \ldots dr_n$, $\vec{r} = (r_1, \ldots, r_n)$ and the density $p(\vec{s}, e, \vec{r})$ is given by

$$p(\vec{s}, e, \vec{r}) = p(s_1, e, r_1)p(s_2 - s_1, r_1, r_2) \ldots p(s_n - s_{n-1}, r_{n-1}, r_n). \tag{2}$$

- The pinned Wiener measure μ^e is defined by its finite dimensional distributions

$$E_{\mu^e}[g \circ F(\vec{s})] = \int_{W^0(G)} (g \circ F(\vec{s}))(x)\, \mathrm{d}\mu^e(x) = \int_{G^n} g(\vec{r})p^e(\vec{s}, \vec{r}, y)\, \mathrm{d}\vec{r},$$

  where the density $p^e(\vec{s}, \vec{r}, y)$ is given by

$$p^e(\vec{s}, \vec{r}, y) = p(s_1, e, r_1)p(s_2 - s_1, r_1, r_2) \ldots p(s_n - s_{n-1}, r_{n-1}, r_n) \times \frac{p(T - s_n, r_n, y)}{p(T, e, y)}. \tag{3}$$

  We will also use the following notation:

$$P_s f(x(s)) = E_\mu f(x(s)), \quad P_s^e f(x(s)) = E_{\mu^e} f(x(s)).$$

- The translated Wiener measure μ_h is defined as the probability distribution of the translated process $x(t)h(t)$ for $h \in H(G)$. Similarly we denote by μ_h^e the translated pinned Wiener measure for $h \in H^e(G)$.

It is well known (e.g. McKean, 1969) that μ is the probability distribution for the Brownian motion on G defined by the Itô stochastic differential equation

$$\mathrm{d}x(t) = x(t)\, \mathrm{d}B(t) + \frac{1}{2} x(t) \sum_1^n \xi_i^2\, \mathrm{d}t, \quad x(0) = e, \tag{4}$$

where $B(t)$ is the Brownian motion on the Lie algebra $\mathfrak{g}$ (with the identity operator as its covariance) and $\{\xi_i\}_1^n$ is an orthonormal basis of the Lie algebra $\mathfrak{g}$. The process $B(t)$ can be described in terms of the basis $\{\xi_i\}_1^n$ as $B(t) = \sum_1^n b_i^t \xi_i$ where b_i^t are real-valued Brownian motions mutually independent on a probability space $(\Omega, \mathscr{F}, P)$. Equivalently the Brownian motion $x(t)$ is the solution of the Stratonovich stochastic differential equation

$$\mathrm{d}x(t) = x(t) \circ \mathrm{d}B(t), \quad x(0) = e.$$

To see the connection between the Wiener measure μ and (4) we will show that Kolmogorov's backward equation for (4) is actually the heat equation. Recall that P_t is the transition probability of the process $x(t)$. Then for any smooth bounded function

$\varphi : G \to \mathbb{R}$, function $v(t,g) = P_t\varphi(g)$ satisfies the following equation (Kolmogorov's backward equation)

$$\frac{\partial}{\partial t} v(t,g) = Lv(t,g)$$

$$v(e,g) = \varphi(g), \quad t > 0, \; g \in G, \tag{5}$$

where the differential operator L is defined by

$$Lv \stackrel{\text{def}}{=} = \frac{1}{2}\sum_{i=1}^{n} [v''(g)(g\xi_i, g\xi_i) + v'(g)(g\xi_i^2)].$$

Our goal is to show that L is the Laplacian on G

$$(\Delta v)(g) = \frac{1}{2}\sum_{i=1}^{n} \partial_i^2 v(g). \tag{6}$$

Let us calculate derivatives of $v : G \to \mathbb{R}$ in the direction of ξ_i

$$(\partial_i v)(g) = v'(g) \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (g\exp(t\xi_i)) = v'(g)(g\xi_i)$$

and therefore

$$(\partial_i^2 v)(g) = v''(g)(g\xi_i, g\xi_i) + v'(g)(g\xi_i^2).$$

3. Quasi-invariance for the Wiener measure on *G*

The goal of this section is to prove quasi-invariance of the Wiener measure μ. But first we show the process $x(t)$ lives in $GL(\mathbb{R}^d)$.

Proposition 2. *The Brownian motion $x(t)$ lives in $GL(\mathbb{R}^d)$ with probability one and the inverse to $x(t)$ satisfies the following Itô stochastic differential equation*:

$$\mathrm{d}z(t) = -\mathrm{d}B(t)z(t) + \frac{1}{2}\sum_{i=1}^{n} \xi_i^2 z(t)\,\mathrm{d}t, \; z(0) = e.$$

Proof. One can show that $x(t)z(t)$ is equal to e with probability one by applying the Itô formula to $\Lambda(x(t)z(t))$ for any linear functional Λ (for example, Λ can be taken to be a matrix entry of $x(t)z(t)$). □

Let $\tilde{x}(t)$ be the translation of $x(t)$ defined by $\tilde{x}(t) = x(t)h(t)$, where $h \in H(G)$. Recall that μ_h the distribution of the translated process $\tilde{x}(t)$. Let

$$Y_t(x) = \int_0^t x(s)^{-1} \circ \mathrm{d}x(s) = \int_0^t x(s)^{-1}\,\mathrm{d}x(s) - \frac{1}{2}\int_0^t \sum_{i=1}^{n} \xi_i^2\,\mathrm{d}s \tag{7}$$

for $0 \leqslant t \leqslant T$ and $x \in W(G)$.

Theorem 3. *Suppose h is in $H(G)$. Then the measure μ is equivalent to μ_h and the Radon–Nikodym density is given by the formula*

$$D(h)(x) = \frac{\mathrm{d}\mu_h}{\mathrm{d}\mu}(x) = \exp\left(\int_0^T \langle h(s)^{-1}h'(s), \mathrm{d}Y_s(x)\rangle - \frac{1}{2}\|h\|_H^2\right), \tag{8}$$

$x \in W(G)$.

Proof. The process $\tilde{x}(t)$ satisfies the following stochastic differential equation

$$\begin{aligned}\mathrm{d}\tilde{x}(t) &= x(t)\,\mathrm{d}B(t)h(t) + \frac{1}{2}x(t)\sum_1^n \xi_i^2 h(t)\,\mathrm{d}t + x(t)h'(t)\,\mathrm{d}t \\ &= \tilde{x}(t)h(t)^{-1}\,\mathrm{d}B(t)h(t) + \frac{1}{2}\tilde{x}(t)\sum_1^n (h(t)^{-1}\xi_i h(t))^2\,\mathrm{d}t \\ &\quad + \tilde{x}(t)h(t)^{-1}h'(t)\,\mathrm{d}t, \quad \tilde{x}(0) = e.\end{aligned}$$

Note that for $f, k \in \mathfrak{g}$

$$\begin{aligned}&E_\mu \left\langle \int_0^t h(s)^{-1}\,\mathrm{d}B(s)h(s), f\right\rangle \left\langle \int_0^t h(s)^{-1}\,\mathrm{d}B(s)h(s), k\right\rangle \\ &= E_\mu \int_0^t \langle \mathrm{d}B(s), h(s)fh(s)^{-1}\rangle \int_0^t \langle \mathrm{d}B(s), h(s)kh(s)^{-1}\rangle \\ &= \int_0^t \langle h(s)fh(s)^{-1}, h(s)kh(s)^{-1}\rangle \mathrm{d}s = t\langle f, k\rangle \end{aligned} \tag{9}$$

since $\langle\cdot,\cdot\rangle$ is Ad-invariant.

This means that $\mathrm{d}\tilde{B}_t = h(t)^{-1}\,\mathrm{d}B(t)h(t)$ is a Brownian motion with the same covariance as $B(t)$. In addition, $\{h(t)^{-1}\xi_i h(t)\}_{i=1}^n$ is an orthonormal basis of $\mathfrak{g}$ since $\langle\cdot,\cdot\rangle$ is Ad-invariant. This means that we can rewrite the stochastic differential equation for $\tilde{x}(t)$ as

$$\mathrm{d}\tilde{x}(t) = \tilde{x}(t)\,\mathrm{d}\tilde{B}_t + \frac{1}{2}\tilde{x}(t)\sum_{i=1}^n \xi_i^2\,\mathrm{d}t + \tilde{x}(t)h(t)^{-1}h'(t)\,\mathrm{d}t,\ \tilde{x}(0) = e.$$

Therefore by Girsanov's theorem (2) the law of $\tilde{x}(t)$ is absolutely continuous with respect to the law of $x(t)$. Moreover, there is an $L^1(\mathrm{d}\mu)$-Radon–Nikodym derivative $D(h)$. □

Remark 4. We actually have shown that μ is quasi-invariant if and only if the inner product $\langle\cdot,\cdot\rangle$ is Ad_G-invariant. Indeed, the covariance of the translated Brownian motion $\tilde{B}_t$ is the same as of the original Brownian motion $B(t)$ if and only if the inner product is Ad_G-invariant as is shown by (9).

4. Quasi-invariance for the pinned Wiener measure

Let $H^e_\varepsilon(G)=\{h\in H(G): h(s)=e \text{ for } s\in[T-\varepsilon,T]\}$. Note that $H^e_\varepsilon(G)\subset H^e(G)$. As before $Y_t(x)=\int_0^t x(s)^{-1}\circ \mathrm{d}x(s)$ for $0\leqslant t\leqslant T$ and $x\in W(G)$. Note that Y_t is a μ-Brownian motion on $\mathfrak{g}$ by (4). The Girsanov density $D(h)$ is well-defined on $H^e_\varepsilon(G)$ μ^e-a.s. by Lemma 7.

Our proof consists of two parts. First we prove the quasi-invariance of μ^e with respect to h in $H^e_\varepsilon(G)$ and with the Radon–Nikodym derivative given by (8). The second part of the proof is to show that the Girsanov density $D(h)$ has a continuous (in h) extension to a map from $H^e(G)$ to $L^1(\mathrm{d}\mu^e)$. This extension to $H^e(G)$ will be also denoted by $D(h)$.

Note that one cannot use (8) to define $D(h)$ for $h\in H^e(G)$ but not in $H^e_\varepsilon(G)$. The reason for that is that the pinned Brownian motion has a singularity at the endpoint and the integral in (8) needs an interpretation if $x\in W^e(G)$. Our proof of quasi-invariance for the pinned Wiener measure does not use the interpretation for $x\in W^e(G)$, though we will discuss in Section 5 how to define Y_T μ^e-a.s.

Theorem 5. *For any $h\in H^e_\varepsilon(G)$ and a bounded measurable function f*

$$E_{\mu^e_h} f = E_{\mu^e} f D(h).$$

Our proof is based on the following lemma.

Lemma 6. *For any $h\in H^e_\varepsilon(G)$ and a bounded measurable function f*

$$E_{\mu^e_h}\{f|\mathscr{F}_{T-\varepsilon}\} = E_{\mu^e}\{f|\mathscr{F}_{T-\varepsilon}\}.$$

Proof. For any $\vec{s}=(s_1,\ldots,s_n)$ we denote $\vec{s}_\varepsilon=(s_1,\ldots,s_{k_\varepsilon})$, where k_ε is such that $s_{k_\varepsilon}\leqslant T-\varepsilon<s_{k_\varepsilon+1}$. Note that for a cylindrical function $g\circ F(\vec{s})$ conditional expectations for μ^e and μ conditioned by $\mathscr{F}_{T-\varepsilon}$ are

$$\begin{aligned} E_{\mu^e}\{g\circ F(\vec{s})|\mathscr{F}_{T-\varepsilon}\} = \int_{G^{n-k_\varepsilon}} & g(x(s_1),\ldots,x(s_{k_\varepsilon}),r_{k_\varepsilon+1},\ldots,r_n) \\ & \times p(s_{k_\varepsilon+1}-s_{k_\varepsilon},x(s_{k_\varepsilon}),r_{k_\varepsilon+1})\ldots p(s_n-s_{n-1},r_{n-1},r_n) \\ & \times \frac{p(T-s_n,r_n,e)}{p(T-s_{k_\varepsilon},x(s_{k_\varepsilon}),e)}\,\mathrm{d}r_{k_\varepsilon+1}\ldots\mathrm{d}r_n. \end{aligned} \quad (10)$$

Then the statement of Lemma 6 follows from (10) since $h(s)=e$ for $T-\varepsilon\leqslant s\leqslant T$ and $h\in H^e_\varepsilon(G)$. □

Proof of Theorem 5. By Lemma 6

$$E_{\mu^e_h} f = E_{\mu^e_h} E_{\mu^e_h}\{f|\mathscr{F}_{T-\varepsilon}\} = E_{\mu^e_h} E_{\mu^e}\{f|\mathscr{F}_{T-\varepsilon}\}.$$

Denote $g(x)=E_{\mu^e}\{f|\mathscr{F}_{T-\varepsilon}\}$. Note that by (10) function g does not depend on $x(s)$ for $T-\varepsilon\leqslant s\leqslant T$. Let us denote the path $x(s)h(s)$ by $xh(s)$. Then

$$E_{\mu_h^e}g(x)=E_{\mu^e}g(xh)=E_{\mu}g(xh)\,\frac{p(T-(T-\varepsilon)),xh(T-\varepsilon),e)}{p(T,e,e)}$$
$$=E_{\mu}g(xh)\,\frac{p(\varepsilon,x(T-\varepsilon),e)}{p(T,e,e)}=E_{\mu_h}g(x)\,\frac{p(\varepsilon,x(T-\varepsilon),e)}{p(T,e,e)}$$

since $h\in H_\varepsilon^e(G)$. Thus

$$E_{\mu_h^e}f=E_{\mu_h}E_{\mu^e}\{f|\mathscr{F}_{T-\varepsilon}\}\,\frac{p(\varepsilon,x(T-\varepsilon),e)}{p(T,e,e)}$$
$$=E_{\mu}E_{\mu^e}\{f|\mathscr{F}_{T-\varepsilon}\}\,\frac{p(\varepsilon,x(T-\varepsilon),e)}{p(T,e,e)}D(h)$$
$$=E_{\mu}E_{\mu^e}\{fD(h)|\mathscr{F}_{T-\varepsilon}\}\,\frac{p(\varepsilon,x(T-\varepsilon),e)}{p(T,e,e)}$$
$$=E_{\mu^e}E_{\mu^e}\{fD(h)|\mathscr{F}_{T-\varepsilon}\}=E_{\mu^e}fD(h). \qquad \square$$

Let $\mathscr{F}_t=\sigma\{x(s),\ x\in W(G),\ 0\leqslant s\leqslant t\}$, $\mathscr{F}$ is the Borel σ-field. Note that the measures μ and μ^e are mutually absolutely continuous on $\mathscr{F}_t$, $0\leqslant t<T$ as the following lemma asserts. For compact connected manifolds it is a part of Theorem 2.3 in (3).

Lemma 7. *Suppose* $0\leqslant t<T$. *Then on* $\mathscr{F}_t$

$$\frac{\mathrm{d}\mu^e}{\mathrm{d}\mu}(x)=\frac{p(T-t,x,e)}{p(T,e,e)}.$$

Proof. Suppose $f\in L^1(\mathscr{F}_t,\mu)$ for $0\leqslant t\leqslant T$. Then we need to show that

$$E_{\mu^e}f(x)=E_{\mu}\left[f(x)\,\frac{p(T-t,x,e)}{p(T,e,e)}\right]. \tag{11}$$

It is enough to check 11 for a smooth cylindrical function f. Suppose that a cylindrical function $f(x(s))$ depends on $x(s)$ only for $0\leqslant s\leqslant t$, that is, f is a cylindrical function $f(x)=g\circ F(\vec{s})(x)$ where $s_n=t$ and g is a smooth function on G^n. Then

$$E_{\mu}g\circ F(\vec{s},t)(x)\,\frac{p(T-t,x,e)}{p(T,e,e)}$$
$$=\int_{G^n}g(\vec{r})\,\frac{p(T-t,r_n,e)}{p(T,e,e)}\,p(\vec{s},e,\vec{r})\,\mathrm{d}\vec{r}=E_{\mu^e}g(x(s)). \qquad \square$$

Theorem 8. *$D(h)$ on $H_\varepsilon^e(G)$ can be extended to a continuous map from $H^e(G)$ to $L^1(\mathrm{d}\mu^e)$.*

Our proof will be based on several lemmas. For $0 \leqslant t \leqslant T$ denote

$$D(h)(t)(x) = \exp\left(\int_0^t \langle h(s)^{-1}h'(s), \mathrm{d}Y_s(x)\rangle - \frac{1}{2}\int_0^t |h(s)^{-1}h'(s)|^2\right), \qquad (12)$$

where $x \in W^e(G)$ and $h \in H_\varepsilon^e(G)$.

Lemma 9. *$h \mapsto D(h)(T/2)$ is a continuous map from $H^e(G)$ to $L^2(\mathrm{d}\mu^e)$.*

Proof. By Lemma 7

$$E_{\mu^e}\left|D(h_n)\left(\frac{T}{2}\right)(x) - D(h)\left(\frac{T}{2}\right)(x)\right|^2$$

$$= E_\mu\left|D(h_n)\left(\frac{T}{2}\right)(x) - D(h)\left(\frac{T}{2}\right)(x)\right|^2 \frac{p(T-\frac{T}{2}, x_{T/2}, e)}{p(T,e,e)}$$

$$\leqslant CE_\mu\left|D(h_n)\left(\frac{T}{2}\right)(x) - D(h)\left(\frac{T}{2}\right)(x)\right|^2$$

since $p(T/2, r, e)$ is bounded in r. Now we can use the fact that for $0 \leqslant s \leqslant T/2$ the process Y_s is a μ-Brownian motion on $\mathfrak{g}$, therefore for an L^2-function $f : [0,T] \to \mathfrak{g}$

$$E_\mu \exp\left(\int_0^{T/2} \langle f(s), \mathrm{d}Y_s(x)\rangle - \frac{1}{2}\int_0^{T/2} |f(s)|^2\right) = 1.$$

Therefore

$$E_\mu\left|D(h_n)\left(\frac{T}{2}\right)(x) - D(h)\left(\frac{T}{2}\right)(x)\right|^2$$

$$= \exp\left(\int_0^{T/2} |h^{-1}h'|^2\,\mathrm{d}s\right) - 2\exp\left(\int_0^{T/2} \langle h^{-1}h', h_n^{-1}h_n'\rangle\,\mathrm{d}s\right)$$

$$+ \exp\left(\int_0^{T/2} |h_n^{-1}h_n'|^2\,\mathrm{d}s\right).$$

Note that $\|h_n - h\|_H \underset{n\to\infty}{\longrightarrow} 0$ implies $\int_0^{T/2} |h^{-1}h' - h_n^{-1}h_n'|^2\,\mathrm{d}s \underset{n\to\infty}{\longrightarrow} 0$. Therefore

$$E_{\mu^e}\left|D(h_n)\left(\frac{T}{2}\right)(x) - D(h)\left(\frac{T}{2}\right)(x)\right|^2 \underset{n\to\infty}{\longrightarrow} 0. \qquad \square$$

Denote the time reversal map on $W^e(G)$ by $Rx(s) = x(T-s)$.

Lemma 10. *$h \mapsto D(Rh)(T/2)(Rx)$ is a continuous map from $H^e(G)$ to $L^2(\mathrm{d}\mu^e)$.*

Proof. Let g be a function on $H(G)$. If $g(x(s)) = g(x_{s \leqslant T/2})$ depends on $x(s)$ only for $T/2 \leqslant s \leqslant T$, then $g(Rx(s))$ depends on $x(s)$ only for $0 \leqslant s \leqslant T/2$. Note that finite dimensional μ^e-distributions for x and Rx are the same by (2) and therefore

$$E_{\mu^e} g(x) = E_{\mu^e} g(Rx) = E_{\mu} g(Rx) \, \frac{p(\frac{T}{2}, x_{T/2}, e)}{p(T, e, e)}$$

according to (11). Then the result follows similarly to the proof of Lemma 9. □

Lemma 11. *For any $\varepsilon > 0$ and $h \in H_\varepsilon^e(G)$*

$$D(h)(T)(x) = D(h)\left(\frac{T}{2}\right)(x) D(Rh)\left(\frac{T}{2}\right)(Rx), \ \ \mu^e\text{-a.s. } x \in W(G).$$

Proof. The statement of Lemma 11 follows from the following fact. Suppose that a deterministic function $f : [0, T] \to \mathfrak{g}$ is in $C([0, T], \mathfrak{g})$ and $f(t) = 0$ for $t \in [T - \varepsilon, T]$. Then

$$\int_{T/2}^{T} \langle f(s), \mathrm{d}Y_s(x) \rangle = -\int_0^{T/2} \langle f(T - s), \mathrm{d}Y_s(Rx) \rangle. \tag{13}$$

First we show that for any $\Delta > 0$

$$Y_{t+\Delta}(Rx) - Y_t(Rx) = -(Y_{T-t}(x) - Y_{T-t-\Delta}(x)), \ \ \mu^e\text{-a.s.}$$

In what follows all the identities must be understood μ^e-a.s. By the definition

$$Y_t(x) = \lim_{\max_i |s_{i+1} - s_i| \to 0} \sum_{i=0}^{n} \frac{x(s_{i+1})^{-1} + x(s_i)^{-1}}{2} (x(s_{i+1}) - x(s_i)),$$

where $0 = s_0 < s_1 < \cdots < s_n < s_{n+1} = t$. Then for $\Delta_0 = 0 < \Delta_1 < \cdots < \Delta_n < \Delta_{n+1} = \Delta$

$$\begin{aligned} &Y_{t+\Delta}(Rx) - Y_t(Rx) \\ &\quad = \lim_{\max_i |\Delta_{i+1} - \Delta_i| \to 0} \sum_{i=0}^{n} \frac{x(T - t - \Delta_{i+1})^{-1} + x(T - t - \Delta_i)^{-1}}{2} \\ &\qquad \times (x(T - t - \Delta_{i+1}) - x(T - t - \Delta_i)). \end{aligned}$$

Denote $t_i = T - t - \Delta_i$ for $i = 0, \ldots, n+1$, then $T - t = t_0 > t_1 > \cdots > t_n > t_{n+1} = T - t - \Delta$ and

$$\begin{aligned} &Y_{t+\Delta}(Rx) - Y_t(Rx) \\ &\quad = -\lim_{\max_i |t_{i+1} - t_i| \to 0} \sum_{i=0}^{n} \frac{x(t_{i+1})^{-1} + x(t_i)^{-1}}{2} (x(t_i) - x(t_{i+1})) \\ &\quad = -(Y_{T-t}(x) - Y_{T-t-\Delta}(x)). \end{aligned}$$

Now let $T/2 = s_0 < s_1 < \cdots s_n < s_{n+1} = T$, $\sigma = \max_i (s_{i+1} - s_i)$. Then

$$\int_{T/2}^{T} \langle f(s), \mathrm{d}Y_s(x) \rangle = \lim_{\sigma \to 0} \sum_{i=0}^{n} \left\langle \frac{f(s_i) + f(s_{i+1})}{2}, Y_{s_{i+1}}(x) - Y_{s_i}(x) \right\rangle.$$

Let $t_i = T - s_i$, so that $t_{n+1} = 0 < t_n < \cdots < t_1 < t_0 = T/2$. Then by the first part of the proof

$$\begin{aligned}
&\int_{T/2}^{T} \langle f(s), \mathrm{d}Y_s(x)\rangle \\
&\quad = \lim_{\sigma\to 0} \sum_{i=0}^{n} \left\langle \frac{f(T-t_i)+f(T-t_{i+1})}{2}, Y_{T-t_{i+1}}(x) - Y_{T-t_i}(x) \right\rangle \\
&\quad = \lim_{\sigma\to 0} \sum_{i=0}^{n} \left\langle \frac{f(T-t_i)+f(T-t_{i+1})}{2}, Y_{t_{i+1}}(Rx) - Y_{t_i}(Rx) \right\rangle \\
&\qquad - \int_{0}^{T/2} \langle f(T-s), \mathrm{d}Y_s(Rx)\rangle. \qquad \square
\end{aligned}$$

Remark 12. Note that since f is a deterministic function, we can use the Stratonovich integral definition to approximate the Itô stochastic integral. If one uses the Itô integral instead of the Stratonovich one in (13), the proof involves a quadratic covariation term which turns out to be 0.

Proof of Theorem 8. By Lemmas 9–11 $D(h)(T,x)$ is a continuous map from $H^e_\varepsilon(G)$ to $L^1(d\mu^e)$. $\square$

Theorem 13. *For any $h \in H^e(G)$ and a bounded measurable function f*

$$E_{\mu^e_h} f = E_{\mu^e} f D(h).$$

Proof. Now we can use Theorems 8 and 5, and the fact that any $h \in H(G)$ can be approximated by smooth functions from H^e_ε. $\square$

5. Y_t is a semimartingale

The goal of this section is to prove that the process Y_t (defined by (7)) is a μ^e-semimartingale for $0 \leqslant t \leqslant T$. The proof of quasi-invariance of the pinned Brownian motion in Section 4 does not use the following results. Our exposition follows (5). Another proof for a Riemannian manifold can be found in (1). Note that Y_t is a μ-Brownian motion in $\mathfrak{g}$ for $0 \leqslant t \leqslant T$.

Proposition 14. *$p(T-t, x, e)$ is a μ-martingale.*

Proof. Let $\{\xi_i\}_{i=1}^n$ be an orthonormal basis of $\mathfrak{g}$. Then we can write $\xi = \sum_{i=1}^n \alpha_i \xi_i$. Denote $Y^i_t = Y^{\xi_i}_t = \langle Y_t, \xi_i\rangle$. Note that $\mathrm{d}x(t) = x(t) \circ \mathrm{d}Y_t$ and therefore for a smooth function f

$$\begin{aligned} \mathrm{d}f(x(t)) &= \sum_{i=1}^{n} \tilde{\xi}_i f(x(t)) \circ \mathrm{d}Y_t^i \\ &= \sum_{i=1}^{n} \tilde{\xi}_i f(x(t))\, \mathrm{d}Y_t^i + \frac{1}{2} \sum_{i=1}^{n} \mathrm{d}(\tilde{\xi}_i f(x(t)))\, \mathrm{d}Y_t^i, \end{aligned} \tag{14}$$

where $\tilde{\xi}_i f$ is defined as in Section 2. Apply this formula to $\tilde{\xi}_i f(x(t))$

$$\begin{aligned} \mathrm{d}\tilde{\xi}_i f(x(t)) &= \sum_{j=1}^{n} \tilde{\xi}_j \tilde{\xi}_i f(x(t)) \circ \mathrm{d}Y_t^j \\ &= \sum_{j=1}^{n} \tilde{\xi}_j \tilde{\xi}_i f(x(t))\, \mathrm{d}Y_t^j + \frac{1}{2} \sum_{j=1}^{n} \mathrm{d}(\tilde{\xi}_j \tilde{\xi}_i f(x(t)))\, \mathrm{d}Y_t^j. \end{aligned}$$

Thus we can compute the covariance

$$\begin{aligned} &\mathrm{d}(\tilde{\xi}_i f(x(t)))\, \mathrm{d}Y_t^i \\ &\quad = \left(\sum_{j=1}^{n} \tilde{\xi}_j \tilde{\xi}_i f(x(t))\, \mathrm{d}Y_t^j + \frac{1}{2} \sum_{j=1}^{n} \mathrm{d}(\tilde{\xi}_j \tilde{\xi}_i f(x(t)))\, \mathrm{d}Y_t^j \right) \mathrm{d}Y_t^i = \tilde{\xi}_i^2 f(x(t))\, \mathrm{d}t. \end{aligned}$$

This means that

$$\mathrm{d}f(x(t)) = \sum_{i=1}^{n} \tilde{\xi}_i f(x(t))\, \mathrm{d}Y_t^i + \Delta f(x(t))\, \mathrm{d}t.$$

Therefore

$$\begin{aligned} \mathrm{d}p(T-t, x(t), e) &= \frac{\partial p}{\partial t}(T-t, x(t), e) + \sum_{i=1}^{n} \tilde{\xi}_i p(T-t, x(t), e) \circ \mathrm{d}Y_t^i \\ &= -\Delta p(T-t, x(t), e) + \sum_{i=1}^{n} \tilde{\xi}_i p(T-t, x(t), e) \circ \mathrm{d}Y_t^i \\ &= \sum_{i=1}^{n} \tilde{\xi}_i p(T-t, x(t), e)\, \mathrm{d}Y_t^i. \qquad \square \end{aligned} \tag{15}$$

Proposition 15. *Define $b(t)$ as a solution to the stochastic differential equation*

$$\mathrm{d}b(t) = \mathrm{d}Y_t - \frac{1}{2} \nabla \log p(T-t, x(t), e)\, \mathrm{d}t.$$

Then $b(t)$ is a $\mathfrak{g}$-valued μ^e-Brownian motion for $0 \leqslant t < T$.

Proof. We will denote $C = 1/p(T,e,e)$. Suppose $0 \leqslant s < t < T$. By the Itô formula (with respect to the measure μ) we have

$$d[(Y_t^i - Y_s^i)p(T-t,x(t),e)]$$
$$= p(T-t,x(t),e)d(Y_t^i - Y_s^i) + (Y_t^i - Y_s^i)\mathrm{d}p(T-t,(t),e)$$
$$+\frac{1}{2}d(Y_t^i - Y_s^i)\,\mathrm{d}p(T-t,x(t),e).$$

The first two terms are μ-martingales, we will denote their sum by M_t. By (15)

$$\mathrm{d}p(T-t,x(t),e) = \sum_j \tilde{\xi}_j p(T-t,x(t),e)\,\mathrm{d}Y_t^j.$$

Now we can compute the covariance

$$\begin{aligned} d(Y_t^i - Y_s^i)\,\mathrm{d}p(T-t,x(t),e) &= \mathrm{d}Y_t^i\,\mathrm{d}p(T-t,x(t),e) \\ &= \mathrm{d}Y_t^i \sum_j \tilde{\xi}_j p(T-t,x(t),e)\,\mathrm{d}Y_t^j \\ &= \tilde{\xi}_i p(T-t,x(t),e)\,\mathrm{d}t. \end{aligned}$$

Therefore

$$(Y_t^i - Y_s^i)p(T-t,x(t),e) = 0 + \int_s^t \mathrm{d}M_\tau + \frac{1}{2}\int_s^t \tilde{\xi}_i p(T-\tau,x(\tau))\,\mathrm{d}\tau.$$

By Lemma 7 for any bounded $\mathscr{F}_s$-measurable function f

$$\begin{aligned} &E_{\mu^e}(Y_t^i - Y_s^i)f = CE_\mu(Y_t^i - Y_s^i)f\,p(T-t,x(t),e) \\ &= \frac{1}{2}CE_\mu f\int_s^t \tilde{\xi}_i p(T-\tau,x_\tau,e)\,\mathrm{d}\tau = \frac{1}{2}C\int_s^t E_\mu f(\tilde{\xi}_i p(T-\tau,x_\tau,e))\,\mathrm{d}\tau \\ &= \frac{1}{2}C\int_s^t E_\mu f(\tilde{\xi}_i \log(p(T-\tau,x_\tau,e))p(T-\tau,x_\tau,e))\,\mathrm{d}\tau \\ &= \frac{1}{2}\int_s^t E_{\mu^e} f(\tilde{\xi}_i \log(p(T-\tau,x_\tau)))\,\mathrm{d}\tau \\ &= E_{\mu^e} f\,\frac{1}{2}\int_s^t (\tilde{\xi}_i \log(p(T-\tau,x_\tau)))\,\mathrm{d}\tau. \end{aligned}$$

Then $b(t)^i = \int_0^t \mathrm{d}Y_\tau^i - 1/2\int_0^t \tilde{\xi}_i p(T-\tau,x_\tau)\,\mathrm{d}\tau$ is a μ^e-martingale for any $i = 1,\dots,n$. Define $b(t) = \sum_{i=1}^n b(t)^i \xi_i$. Note that

$$\mathrm{d}b(t)^i b(t)^j = \mathrm{d}Y_t^i\,\mathrm{d}Y_t^j = \delta_{i,j}\,\mathrm{d}t.$$

Therefore by Lévy's Theorem $b(t)^i$ is a μ^e-Brownian motion for any $i = 1,\dots,n$. □

Proposition 16. *Let* $f \in L^\infty(G, dg)$ *and* $\xi \in \mathfrak{g}$. *Then*

$$E_\mu[f(x(t))(\tilde{\xi} \log p(t, x(t)))] = -\frac{1}{t} E_\mu[f(x(t)) Y_t^\xi], \quad x(t) \in W(G),$$

where $Y_t^\xi = \langle Y_t, \xi \rangle$.

Proof. First assume that $f \in C_0^\infty(G)$. Then

$$E_\mu[t\tilde{\xi} f(x(t))] = E_\mu[f(x(t)) Y_t^\xi].$$

This integration by parts formula is a simple consequence of quasi-invariance of μ shown in Theorem 3. Indeed, Theorem 3 says that for $h(t) = e^{\xi t}$

$$E_\mu f(x(t)e^{\xi t}) = E_\mu \left[f(x(t)) \exp\left(\int_0^t \mathrm{d}Y_s^\xi - \frac{1}{2} t |\xi|^2 \right) \right]$$

$$= E_\mu \left[f(x(t)) \exp\left(Y_t^\xi - \frac{1}{2} t |\xi|^2 \right) \right].$$

Therefore

$$E_\mu \left[\frac{f(x(t)e^{\varepsilon \xi t}) - f(x(t))}{\varepsilon} \right]$$

$$= E_\mu \left[f(x(t)) \left(\frac{\exp(\varepsilon Y_t^\xi - \frac{1}{2}\varepsilon^2 t |\xi|^2) - 1}{\varepsilon} \right) \right] \to E_\mu[f(x(t)) Y_t^\xi]$$

as $\varepsilon \to 0$.

By Lemma 7 and the usual integration by parts formula for a smooth function f

$$E_\mu \tilde{\xi} f(x(t)) = \int_G \tilde{\xi} f(r) p(t, e, r) \, \mathrm{d}r = -\int_G f(r) \tilde{\xi} p(t, e, r) \, \mathrm{d}r$$

$$= -\int_G f(r) \tilde{\xi}(\log p(t, e, r)) p(t, e, r) \, \mathrm{d}r = -E_\mu[f(x(t))(\tilde{\xi} \log p(t, x(t)))]$$

Combining these two calculations we have that for a smooth function f

$$E_\mu(\tilde{\xi} f(x(t))) = -E_\mu[f(x(t))(\tilde{\xi} \log p(t, x(t)))] = \frac{1}{t} E_\mu[f(x(t)) Y_t^\xi].$$

Thus we have proved the formula for a $C_0^\infty(G)$-function and therefore it holds for any measurable bounded function. □

Corollary 17. *For* $0 \leqslant t < T$, $x(t) \in W(G)$

(1) $E_\mu[Y_t^\xi | \mathscr{F}_{x(t)}](x(t)) = -t\tilde{\xi} \log p(t, x(t))$,

(2) $E_\mu |\tilde{\xi} \log p(t, x(t))|^p \leqslant 1/t^p E_\mu |Y_t^\xi|^p < \infty$, $p \geqslant 1$.

Proposition 18. Y_t *is a* μ^e*-semimartingale for* $0 \leqslant t \leqslant T$.

Proof. All we need to check now is that

$$E_{\mu^e} \int_0^T |\tilde{\xi} \log p(T-t, x(t), e)| \, dt < \infty$$

for any $\xi \in \mathfrak{g}$. First of all,

$$E_{\mu^e} \int_0^T |\tilde{\xi} \log p(T-t, x(t), e)| \, dt$$

$$= E_{\mu^e} \int_0^{T/2} |\tilde{\xi} \log p(T-t, x(t), e)| \, dt + E_{\mu^e} \int_{T/2}^T |\tilde{\xi} \log p(T-t, x(t), e)| \, dt$$

First let us estimate the second term. Note that

$$cE_{\mu^e} \left[\int_{T/2}^T |\tilde{\xi} \log p(T-t, x(t), e)| \, dt \right] = E_{\mu^e} \left[\int_0^{T/2} |\tilde{\xi} \log p(t, x(t))| \, dt \right]$$

$$= E_{\mu} \left[\int_0^{T/2} |\tilde{\xi} \log p(t, x(t))| \, dt \, \frac{p(\frac{T}{2}, x(\frac{T}{2}))}{p(T, e, e)} \right]$$

since $x(t)$ and x_{T-t} have the same finite dimensional distributions. The function $p(\frac{T}{2}, x(\frac{T}{2}))$ is bounded in x therefore it is enough to check that

$$E_{\mu} \left[\int_0^{T/2} |\tilde{\xi} \log p(t, x(t))| \, dt \right] < \infty.$$

Now use Proposition 16 for $f(x(t)) = \text{sign}(\tilde{\xi} \log p(t, x(t)))$ and apply the fact that Y_t is a μ-Brownian motion

$$E_{\mu} \int_0^{T/2} |\tilde{\xi} \log p(t, x(t))| \, dt = -\int_0^{T/2} \frac{1}{t} E_{\mu}[(\text{sign} \log p(t, x(t))) Y_t^{\xi}] \, dt$$

$$\leqslant \int_0^{T/2} \frac{1}{t} E_{\mu} |Y_t^{\xi}| \, dt = \text{const} \int_0^{T/2} \frac{1}{\sqrt{t}} \, dt < \infty.$$

Let us now estimate the first term. Note that by Lemma 7

$$E_{\mu^e} |\tilde{\xi} \log p(T-t, x(t), e)| = E_{\mu^e} \frac{|\tilde{\xi} p(T-t, x(t), e)|}{p(T-t, x(t), e)} = E_{\mu} \frac{|\tilde{\xi} p(T-t, x(t), e)|}{p(T, e, e)}$$

which is uniformly bounded for $0 \leqslant t \leqslant T/2$. This follows, for example, from the heat kernel decomposition in the proof of Theorem 1.2 in (7). □

For further reading

The following references are also of interest to the reader: DaPrato and Zabczyk, 1992; Driver and Thalmaier, 2001; Lyons and Zheng, 1990.

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