LÉVY PROCESSES AND THEIR SUBORDINATION IN MATRIX LIE GROUPS

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Abstract. Lévy processes in matrix Lie groups are studied. Subordination (random time change) is used to show that quasi-invariance of the Brownian motion in a Lie group induces absolute continuity of the laws of the corresponding pure jump processes. These results are applied to several examples which are discussed in detail.

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1. Introduction

There has been considerable interest in Lévy processes in matrix Lie groups over the last several years. In particular, M.Liao recently published a book [13] on Lévy processes in Lie groups. We will mention some directions in which this study has developed. Our primary goal is to study properties of pure jump processes in Lie groups.

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The main obstacle to understanding such processes can be easily seen by comparing a Brownian motion and a pure jump process in a Lie group G. For simplicity we assume that G is a connected Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$. Then a Brownian motion in G can be described as the result of rolling of a Brownian motion in the Lie algebra $\mathfrak{g}$ of G into the group G. In other words, linearization of a Brownian motion in G can be interpreted as a Brownian motion in $\mathfrak{g}$. This approach does not really work for pure jump processes since we would like to know the size of jumps after exponentiating of a jump process in $\mathfrak{g}$ to G, and possibly identify the process similarly to how it is done in the Euclidean case. For example, let us consider a compound Poisson process of exponential type

$$\Pi_t = \prod_{i=1}^{N_t} e^{x_i},$$

where $\{N(t) : t \geqslant 0\}$ is a Poisson process, x_i are i.i.d. random variables taking values in $\mathfrak{g}$, which are also independent of $\{N(t) : t \geqslant 0\}$. Then there is no clear connection with an easily identifiable stochastic process in the Lie algebra as we have in the case of a Brownian motion. Some of the classical results have been established for Lévy processes in Lie groups. We briefly discuss them later in the paper. First, Lévy processes in Lie groups were characterized by G. A. Hunt in [12] as Markov processes in G with a generator having the form (2.1). In [1] D. Applebaum and H. Kunita proved that Lévy processes give rise to solutions of a stochastic differential equation as described in Theorem 2.3. There were numerous works on an analogue of the classical Lévy -Khinchin formula. The original one is due to R. Gangolli [7]. In [2] D.Applebaum obtained a Lévy -Khinchin-type representation of a Lévy process by a stochastic differential equation. These formulae are restricted to the case of spherically symmetric processes. We give basic facts of these type of processes in Section 4. This restriction to spherically symmetric processes allows not only to use an analogue of the Fourier transform, but also describes the corresponding stochastic processes in symmetric spaces. Our aim is to study processes which are not necessarily spherically symmetric, therefore forcing us to consider the processes in the whole group rather than in a symmetric space.

One of the main methods we use in this paper is subordination, random time-change, of one stochastic process by another. This technique has been applied and studied extensively in $\mathbb{R}^n$. In the case of Lie groups the most closely related paper to our results is [4]. There the author studied spherically symmetric processes on semi-simple Lie groups. Then the Lévy -Khinchin representation and the characteristic functions of the subordinated process are calculated in terms of the characteristics of the original process and of the subordinator process. We study the subordination procedure in the case where the original process is not necessarily spherically symmetric. It is worth noting that a Brownian motion in a non-Abelian Lie group starting at the identity is not spherically symmetric.

The use of subordination allows us to control the times of the jumps, and sometimes enables to identify the resulting process in the group. Another advantage of subordination is used in Theorem 5.2 which shows that this procedure preserves the property of processes having absolutely continuous laws. In particular, using quasi-invariance of the Brownian motion in a Lie group we can explicitly write the Radon-Nikodym derivative of the corresponding subordinated processes. This formula involves the heat kernel for the corresponding Laplacian on G.

Our paper is structured as follows. Section 2 gives basic definitions and facts about Lévy processes in Lie groups. In Section 3 we prove quasi-invariance of the heat kernel measure in two cases: compact Lie groups and semi-simple Lie groups of non-compact type. For the original proof we refer to [19], though we follow the proof given in [8]. Section 4 gives basic facts of spherically symmetric processes and spherical transforms. In Section 5 we introduce the subordination procedure. In particular, Equation (5.1) gives a way of finding the Lévy measure for the subordinated process if we know the law of the original process and the Lévy measure of the subordinator. Theorem 5.2 is one of the main results of the paper, which is later used in several examples. Section 6 contains several examples of groups: $\mathbb{R}$, $\mathbb{R}_+$, $\mathrm{SL}(2,\mathbb{R})$, $\mathrm{GL}(n,\mathbb{R})_+$ and the Heisenberg group. For $G=\mathrm{SL}(2,\mathbb{R})$ and $G=\mathrm{GL}(n,\mathbb{R})_+$ we describe the components of the Iwasawa decomposition of a Brownian motion as solutions of a system of stochastic differential equations. This allows us to identify the process which a Brownian motion in $\mathrm{GL}(n,\mathbb{R})_+$ induces in the corresponding double cosets space $\mathrm{SO}(n,\mathbb{R})\backslash\mathrm{GL}(n,\mathbb{R})_+/\mathrm{SO}(n,\mathbb{R})$. In addition, we can identify the subordinated process by reducing this example to the Euclidean case. As one can see the distribution of the subordinated process depends very much on the structure of the group. In addition, Theorem 6.4 gives an explicit formula for the heat kernel in $\mathrm{SL}(2,\mathbb{R})$ which is of independent interest.

2. Lévy processes in matrix Lie groups over $\mathbb{R}$

Let G be a connected Lie subgroup of $\mathrm{GL}_n(\mathbb{R})$ with the identity $e=I$, the identity matrix in $\mathrm{GL}_n(\mathbb{R})$. We denote the dimension of G by d. Its Lie algebra $\mathfrak{g}$ will be identified with the left-invariant vector fields at the identity e. For any $X\in\mathfrak{g}$ we denote by X^l the corresponding left-invariant vector field. In the case of matrix Lie groups it is differentiation in the direction of X^l, that is, for a function $f: G\to\mathbb{R}$ we define

$$\frac{\partial f}{\partial X^l}(g)=\frac{d}{dt}\bigg|_{t=0} f(e^{tX}g),\ g\in G,$$

if such a derivative exists.

Throughout this paper we will work with a filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_t,P)$, where $(\Omega,\mathcal{F},P)$ is a probability space, $\mathcal{F}_t$ is a σ-field contained in $\mathcal{F}$. We assume that $\mathcal{F}=\lim_{t\uparrow\infty}\mathcal{F}_t$, and whenever necessary that all P-null sets are contained in $\mathcal{F}_t$ for any t in $(0,\infty)$, and that the filtration $\mathcal{F}_t$ is right-continuous $\mathcal{F}_t=\cap_{s>t}\mathcal{F}_s$.

Definition 2.1. Let g_t be a stochastic process with values in G, and let t be in $(0,\infty)$.

(1) $g_s^{-1}g_t$ is called **the right increment** of the process g_t, and $g_tg_s^{-1}$ is called **the left increment** of the process g_t for $s<t$;

(2) the process g_t has **independent right (left) increments** if for any $0<t_1<t_2<...<t_n$

$$g_0, g_0^{-1}g_{t_1}, g_{t_1}^{-1}g_{t_2}, ..., g_{t_{n-1}}^{-1}g_{t_n}\ (g_0, g_{t_1}g_0^{-1}, g_{t_2}g_{t_1}^{-1}, ..., g_{t_n}g_{t_{n-1}}^{-1})$$

are independent;

(3) the process g_t with independent right (left) increments has **stationary right (left) increments** if $g_s^{-1}g_t\overset{d}{=}g_0^{-1}g_{t-s}$ $(g_tg_s^{-1}\overset{d}{=}g_{t-s}g_0^{-1})$ for any $s<t$;

(4) the process g_t is **càdlàg** if almost all of its paths are right continuous on $[0,\infty)$ and have left limits on $(0,\infty)$;
(5) the process g_t is **a left (right) Lévy process** in G if it is a càdlàg process with independent and stationary right (left) increments.

Let g_t be a left Lévy process in G with $g_0 = e$. The transition probability semigroup of this process is given by $P_t f(g) = E[f(gg_t)]$ for any non-negative Borel function f on G. Then the distribution μ_t of g_t is a weakly continuous convolution semigroup of probability measures on G satisfying

$$P_t f(g) = \int_G f(gh) d\mu_t(h), \ t \in [0,\infty).$$

G.A.Hunt ([12]) gave a full characterization of left Lévy processes in G by describing their generators as

$$Lf(g) = \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j} X_i^l X_j^l f(g) + \sum_{i=1}^{d} c_i X_i^l f(g) + \int_G \left[f(gh) - f(g) - \sum_{i=1}^{d} x_i(h) X_i^l f(g) \right] d\Pi(h), \tag{2.1}$$

for any smooth function f on G with compact support. Here $\{X_i\}_{i=1}^d$ is a basis of $\mathfrak{g}$, x_i are real-valued functions on G such that for any $g \in G$ we have $g = \exp\left(\sum_{i=1}^d x_i(g) X_i\right)$, $A = \{a_{i,j}\}_{i,j=1}^d$ is a non-negative definite symmetric matrix, and Π is a measure on G satisfying $\Pi(\{e\}) = 0$, $\int_U \sum_{i=1}^d x_i^2 d\Pi < \infty$, $\Pi(G\backslash U) < \infty$ for some compact neighborhood U of e.

The Lévy measure Π can also be described as follows. Note that for any left Lévy process we can write $g_t = g_{t-} g_{t-}^{-1} g_t = g_t g_{t-}^{-1} g_{t-}$, so we will call $g_{t-}^{-1} g_t$ a right jump, and $g_t g_{t-}^{-1}$ a left jump. Denote by N the counting measure of the right jumps

$$N\left([0,t] \times B\right) = \#\{s \in (0,t] : g_{s-}^{-1} g_s \neq e, g_{s-}^{-1} g_s \in B\},$$

for any $B \in \mathfrak{B}(G)$. According to Proposition 1.4 of [13] N is a Poisson random measure on $\mathbb{R}_+ \times G$, and its characteristic measure is the Lévy measure Π of g_t.

The connection between the Lévy measure Π of the process g_t and its law $\nu_t(dg)$ can be described as follows. Namely, according to [11], p.308, the measure Π is uniquely determined by $\nu_t(dg)$ via

$$\int_{G\backslash\{e\}} f(g)\Pi(dg) = \lim_{t\to 0} \frac{1}{t} \int_G f(g)\nu_t(dg), \ f \in C^1(G\backslash\{e\}). \tag{2.2}$$

Remark 2.2. A left Lévy process g_t is continuous if and only if $\Pi = 0$. In this case g_t is a left-invariant diffusion process in G with generator

$$Lf(g) = \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j} X_i^l X_j^l f(g) + \sum_{i=1}^{d} c_i X_i^l f(g).$$

We will say that a left Lévy process is pure jump if its diffusion part in decomposition (2.1) is 0.

Theorem 2.3 describes a stochastic differential equation corresponding to a left Lévy process g_t. For more details on the result, based on works by D. Applebaum and H. Kunita, we refer to Theorem 1.2, p. 19 of [13].

Theorem 2.3. *Let g_t be a left Lévy process in G, then for any smooth real-valued function f on G*

$$f(g_t) = f(g_0) + M_t^f + \int_0^t Lf(g_s)ds,$$

where

$$M_t^f = \sum_{i=1}^d \int_0^t X_i^l f(g_{s-})dB_s^i + \int_0^t \int_G [f(g_{s-}h) - f(g_{s-})]\tilde{N}(dsdh)$$

is an L^2-martingale, B_t is a d-dimensional Brownian motion with the covariance matrix A, $\tilde{N}$ is the compensated random measure of N on $\mathbb{R}_+ \times G$, and N and B_t are independent. The integrals in the expansion of M_t^f are understood in the Itô sense. In addition, there is a one-to-one correspondence between left Lévy processes in G and the following triples: a Brownian motion B_t with the covariance matrix A, drift $\{c_i\}_{i=1}^d$ and measure N.

Remark 2.4. The Îto integrals are not intrinsic, but their use might be more convenient if one wants to exploit some standard facts on Lévy processes.

3. Quasi-invariance for the Brownian motion in G

In this section we show quasi-invariance of the Brownian motion in G in two cases: when G is of compact type, and when G is a semi-simple group of non-compact type. The proof follows the one published in [8] (given there for Lie groups of compact type), but we present the main ingredients of the exposition here to first show how it can be extended to semi-simple groups of non-compact type, and secondly to give the Radon-Nikodym formula to be used later. Shigekawa in [19] gave a proof of quasi-invariance of the Brownian motion in a general Lie group, but for our purposes it is enough to give a very explicit proof in the case of matrix Lie groups.

3.1. **The case of a Lie group of compact type.** Let G be as before. We assume that there is an Ad_G-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$. The corresponding norm is denoted by $|\cdot|$. The existence of an Ad_G-invariant inner product implies that G is of compact type, that is, G is locally isomorphic to a compact Lie group([10]). By dg we will denote the (bi-invariant) Haar measure on G. We will use the following notation

Notation 3.1.
(1) *$W(G) = \{\omega \in C([0, T], G), \omega(0) = e\}$ is the space of all continuous paths in G beginning at the identity e,*
(2) *$H(G) = \{h \in W(G), h$ is absolutely continuous and the norm $\|h\|_H^2 = \int_0^T |h(s)^{-1}h'(s)|^2 ds$ is finite$\}$ is the Cameron-Martin (finite energy) subset of $W(G)$.*

The Wiener measure on $W(G)$ will be denoted by μ. It is well known (e. g. [15]) that μ is the probability distribution for the Brownian motion g_t on G defined by the Itô stochastic differential equation

$$dg_t = g_t dW_t + \frac{1}{2} g_t \sum_1^d \xi_i^2 dt, \ g_0 = e, \tag{3.1}$$

where W_t is the Brownian motion on the Lie algebra $\mathfrak{g}$ with the identity operator as its covariance and $\{\xi_i\}_{i=1}^d$ is an orthonormal basis of the Lie algebra $\mathfrak{g}$. The process W_t can be described in terms of the basis $\{\xi_i\}_{i=1}^d$ as $W_t = \sum_{i=1}^d b_i^t \xi_i$ where b_i^t are real-valued Brownian motions mutually independent on a probability space $(\Omega, \mathcal{F}, P)$. Equivalently the Brownian motion g_t is the solution of the Stratonovich stochastic differential equation

$$\delta g_t = g_t \delta W_t, \ g_0 = e.$$

Suppose $\tilde{g}_t$ is the translation of g_t defined by $\tilde{g}_t = g_t h(t)$, where $h \in H(G)$. The translated Wiener measure μ_h is defined as the probability distribution of the translated process $g_t h(t)$ for $h \in H(G)$. Let

$$Y_t(x) = \int_0^t x(s)^{-1} \delta x(s) = \int_0^t x(s)^{-1} dx(s) - \frac{1}{2} \int_0^t \sum_{i=1}^d \xi_i^2 ds \tag{3.2}$$

for $0 \leqslant t \leqslant T$ and $x \in W(G)$.

The process $\tilde{g}_t$ satisfies the following stochastic differential equation

$$d\tilde{g}_t = g_t dW_t h(t) + \frac{1}{2} g_t \sum_1^d \xi_i^2 h(t) dt + g_t h'(t) dt =$$

$$\tilde{g}_t h(t)^{-1} dW_t h(t) + \frac{1}{2} \tilde{g}_t \sum_1^d (h(t)^{-1} \xi_i h(t))^2 dt + \tilde{g}_t h(t)^{-1} h'(t) dt, \quad \tilde{g}_0 = e.$$

Note that for any $f, k \in \mathfrak{g}$ according to Lévy 's criterion

$$\begin{aligned} E_\mu \langle \int_0^t h(s)^{-1} dB_s h(s), f \rangle \langle \int_0^t h(s)^{-1} dB_s h(s), k \rangle = \\ E_\mu \int_0^t \langle dB_s, h(s) f h(s)^{-1} \rangle \int_0^t \langle dB_s, h(s) k h(s)^{-1} \rangle = \\ \int_0^t \langle h(s) f h(s)^{-1}, h(s) k h(s)^{-1} \rangle ds = t \langle f, k \rangle \end{aligned} \tag{3.3}$$

since the inner product $\langle \cdot, \cdot \rangle$ is Ad-invariant.

This means that $d\tilde{W}_t = h(t)^{-1} dW_t h(t)$ is a Brownian motion with the same covariance as W_t. In addition, $\{h(t)^{-1} \xi_i h(t)\}_{i=1}^d$ is an orthonormal basis of $\mathfrak{g}$ since $\langle \cdot, \cdot \rangle$ is Ad-invariant. This means that we can rewrite the stochastic differential equation for $\tilde{g}_t$ as

$$d\tilde{g}_t = \tilde{g}_t d\tilde{W}_t + \frac{1}{2} \tilde{g}_t \sum_{i=1}^d \xi_i^2 dt + \tilde{g}_t h(t)^{-1} h'(t) dt, \ \tilde{g}_0 = e.$$

Then Girsanov's theorem implies that the law of $\tilde{g}_t$ is absolutely continuous with respect to the law of g_t, and an $L^1(d\mu)$-Radon-Nikodym derivative $D(h)$ can be written as follows in Theorem 3.2.

Theorem 3.2 ([8]). *Suppose h is in $H(G)$. Then the measure μ is equivalent to its translate μ_h, and the Radon-Nikodym density is given by the formula*

$$D(h)(x) = \frac{d\mu_h}{d\mu}(x) = \exp\Big(\int_0^T \langle h(s)^{-1}h'(s), dY_s(x)\rangle - \frac{1}{2}\|h\|_H^2\Big), \tag{3.4}$$

$x \in W(G)$.

Remark 3.3. We actually have shown that μ is quasi-invariant if and only if the inner product $\langle\cdot,\cdot\rangle$ is Ad_G-invariant. Indeed, the covariance of the translated Brownian motion $\tilde{W}_t$ is the same as of the original Brownian motion W_t if and only if the inner product is Ad_G-invariant as is shown by (3.3).

Corollary 3.4. *Let $h(t) = e^{\xi t}$ for some $\xi \in \mathfrak{g}$ with $t \in [0,T]$. Then the Radon-Nikodym derivative $D(h)$ is given by*

$$D(h)(x) = \exp\left(Y_T^\xi(x) - \frac{|\xi|^2 T}{2}\right), \tag{3.5}$$

$x \in W(G)$, where $Y_t^\xi(x) = \langle \xi, Y_t\rangle$.

3.2. **The case of a semi-simple Lie group of non-compact type.** Let G be a semi-simple matrix group, and $\mathfrak{g}$ its Lie algebra. Cartan's criterion implies then that the Killing form $B(X,Y)$ on $\mathfrak{g}$ is non-degenerate. Let $\Theta : G \to G$ be a Cartan involution, that is, a non-identity map such that $\Theta^2 g = g$ for any $g \in G$. For example, $\Theta g = (g^T)^{-1}$ is such an involution. Then its differential is a Cartan involution on $\mathfrak{g}$. For $\Theta g = (g^T)^{-1}$, the corresponding Lie algebra isomorphism is $\theta X = -X^T$. The Lie algebra $\mathfrak{g}$ can be decomposed into a direct sum of the eigenspaces $\mathfrak{k}$ and $\mathfrak{p}$ corresponding to the only eigenvalues of θ, 1 and -1. This is called the Cartan decomposition of the Lie algebra $\mathfrak{g}$. If the Killing form is negative definite on $\mathfrak{g}$, then G is of compact type. This is the case which has been considered in the previous subsection, and in particular, it implies that there is an Ad_G-invariant inner product on $\mathfrak{g}$. Now let us consider the case where G is of non-compact type. The basic example of a semi-simple Lie group of non-compact type is $\mathrm{SL}(n,\mathbb{R})$.

Notation 3.5. (1) *The inner product on $\mathfrak{g}$ is given by*

$$\langle X, Y\rangle = -B(X, \theta(Y)),$$

where $B(\cdot,\cdot)$ is the Killing form on $\mathfrak{g}$;

(2) *let $K = \{g \in G : \Theta(g) = g\}$;*

(3) *let $\mathfrak{a}$ be a maximal Abelian subspace of $\mathfrak{p}$, and A be the (Abelian) Lie subgroup of G generated by $\mathfrak{a}$.*

Note that the inner product $\langle\cdot,\cdot\rangle$ is Ad_K-invariant, but not Ad_G-invariant. It is also clear that $\mathfrak{k}$ is the Lie algebra of the Lie subalgebra K.

The next theorem states that the Wiener measure on a connected semi-simple Lie group G of non-compact type is quasi-invariant, only with respect to a smaller Cameron-Martin space, namely, $H(K)$ as defined in Notation 3.1.

Theorem 3.6. *Suppose k is in $H(K)$. Then the measure μ is equivalent to μ_k and the Radon-Nikodym density is given by the formula*

$$D(k)(x) = \frac{d\mu_k}{d\mu}(x) = \exp\Big(\int_0^T \langle k(s)^{-1}k'(s), dY_s(x)\rangle - \frac{1}{2}\|k\|_H^2 \Big), \tag{3.6}$$

$x \in W(G)$.

Proof. The proof goes along the lines of the proof of Theorem 3.2, and uses the fact that the inner product $\langle \cdot, \cdot \rangle$ is Ad_K-invariant. □

Corollary 3.7. *Let $k(t) = e^{\xi t}$ for some $\xi \in \mathfrak{k}$. Then the Radon-Nikodym derivative $D(k)$ is given by*

$$D(k)(x) = \exp\left(Y_T^{\xi}(x) - \frac{|\xi|^2 T}{2} \right),$$

$x \in W(G)$*, where* $Y_t^{\xi}(x) = \langle \xi, Y_t \rangle$.

In what follows we use the direct sum Iwasawa decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$, where $\mathfrak{n}$ is a nilpotent Lie subalgebra of $\mathfrak{g}$. The subalgebra $\mathfrak{n}$ can be described as the projection to $\mathfrak{p}$ of the space spanned by the root spaces of either positive or negative roots. The corresponding Lie group is denoted by N. For the Cartan decomposition of G we also need to fix a Weyl chamber $\mathfrak{a}_+$ with the corresponding Lie group $A_+ = \exp(\mathfrak{a}_+)$. We recall that the exponential map is a diffeomorphism from $\mathfrak{a}$ onto A, and therefore the closure $\overline{A_+}$ of A_+ satisfies $\overline{A_+} = \exp(\overline{\mathfrak{a}_+})$. The following theorem is a compilation of several results in [10]: Theorem 1.1 in Chapter VI, Theorem 1.1 and Theorem 1.3 in Chapter IX.

Theorem 3.8 (Structure theorem for semi-simple Lie groups of non-compact type)**.** *Let G be a connected semi-simple Lie group, and let K be defined as in Notation 3.5. Then*

1. *K is a connected closed subgroup of G containing the center Z of G;*
2. *K is compact if and only if the center Z of G is finite. Then K is a maximal compact subgroup of G;*
3. *the inner product $\langle \cdot, \cdot \rangle$ and the space $\mathfrak{p}$ are Ad_K-invariant;*
4. *the map $(k, X) \mapsto ke^X$ is a diffeomorphism from $K \times \mathfrak{p}$ onto G;*
5. ***Cartan decomposition** $K\overline{A_+}K$: any $g \in G$ can be written as $g = k_1 a k_2$, where $k_1, k_2 \in K$, and a is a unique element in $\overline{A_+}$;*
6. ***Iwasawa decomposition** NAK : the map $(n, a, k) \mapsto kan$ is a diffeomorphism of $N \times A \times K$ onto G.*

The following result appears in Liao's book, though the original application of the Iwasawa decomposition to a Brownian motion in Lie groups was done by M.-P. Malliavin and P. Malliavin in [14].

Proposition 3.9. *[p. 130 in [13]] Let G be a semi-simple group of non-compact type. Suppose g_t is a Brownian motion in the group G. Choose an orthonormal basis $\{\xi_i\}_{i=1}^d$ of $\mathfrak{g}$ in such a way that $\{\xi_i\}_{i=1}^k$ is an orthonormal basis of $\mathfrak{k}$. Let $g_t = n_t a_t k_t$ be the Iwasawa decomposition of g_t, and $g_t = k_t^{(1)} a_t k_t^{(2)}$ the Cartan decomposition of g_t. Then the processes n_t, a_t, k_t and a_t^+ satisfy the following Stratonovich stochastic differential equations*

$$
\begin{aligned}
\delta k_t &= \sum_{i=1}^{d} [\mathrm{Ad}_{k_t}\xi_i]_{\mathfrak{k}}\, k_t \circ \delta W_t^i, \\
\delta a_t &= \sum_{i=k+1}^{d} a_t\, [\mathrm{Ad}_{k_t}\xi_i]_{\mathfrak{a}}\, \delta W_t^i, \\
\delta n_t &= \sum_{i=k+1}^{d} n_t \mathrm{Ad}_{a_t}\, [\mathrm{Ad}_{k_t}\xi_i]_{\mathfrak{n}}\, \delta W_t^i, \qquad (3.7)\\
\delta a_t^+ &= \sum_{i=k+1}^{d} a_t^+ \left[\mathrm{Ad}_{k_t^{(2)}}\xi_i\right]_{\mathfrak{a}} \delta W_t^i,
\end{aligned}
$$

where $\dim \mathfrak{k} = \mathrm{k}$, $\dim \mathfrak{g} = \mathrm{d}$.

Proof. By the Itô formula applied to $g_t = n_t a_t k_t$ we have

$$\delta g_t = \delta n_t a_t k_t + n_t \delta a_t k_t + n_t a_t \delta k_t = g_t \delta W_t.$$

Then we can multiply this equation on the left by $a_t^{-1} n_t^{-1}$ and on the right by k_t^{-1} to see that

$$a_t^{-1} n_t^{-1} \delta n_t a_t + a_t^{-1} \delta a_t + \delta k_t k_t^{-1} = k_t \delta W_t k_t^{-1}. \qquad (3.8)$$

The invariance of the space $\mathfrak{n}$ under the adjoint action by elements from K implies the result. The equation for a_t^+ can be derived similarly. □

Corollary 3.10. *We can write the processes a_t and a_t^+ as $a_t = e^{H_t}$ and $a_t^+ = e^{H_t^+}$, where H_t and H_t^+ are the solutions to the following stochastic differential equations*

$$\delta H_t = \sum_{i=k+1}^{d} [\mathrm{Ad}_{k_t}\xi_i]_{\mathfrak{a}}\, \delta W_t^i; \qquad (3.9)$$

$$\delta H_t^+ = \sum_{i=k+1}^{d} \left[\mathrm{Ad}_{k_t^{(2)}}\xi_i\right]_{\mathfrak{a}} \delta W_t^i. \qquad (3.10)$$

4. Spherically symmetric processes

This section describes a Harish-Chandra transform, i.e. a spherical Fourier transform, with respect to a spherical function on G for a Brownian motion in G.

Definition 4.1. Suppose G is a connected semi-simple Lie group of non-compact type, and K is a maximal compact subgroup of G.

(1) A function φ on G is called **spherical** if $\varphi(k_1 g k_2) = \varphi(g)$ for any $g \in G$, $k_1, k_2 \in K$;
(2) a spherical function φ is **an elementary spherical function** if in addition it satisfies

$$\int_K \varphi(gkh)dk = \varphi(g)\varphi(h), \; g, h \in G,$$

where dk is the normalized Haar measure on K and $\varphi(e) = 1$.

Definition 4.2. A stochastic process g_t in G is **spherically symmetric** (or spherical) if

$$P\left(g_t \in k_1 A k_2\right) = P\left(g_t \in A\right)$$

for any $k_1, k_2 \in K$, $A \in \mathfrak{B}(G)$, $t > 0$.

Suppose G is a matrix Lie group which is either compact or a semi-simple group of non-compact type. Let K be either the group G itself or a maximal compact subgroup of G, and $\mathfrak{k}$ its Lie algebra. We assume that $\mathfrak{k}$ is equipped with an Ad_K-invariant inner product $\langle\cdot,\cdot\rangle$. By g_t we denote the Brownian motion in G defined by stochastic differential equation (3.1). Then the following holds.

Proposition 4.3. *Suppose G is a matrix Lie group which is either compact or a semi-simple group of non-compact type. Let g_t be a solution to the Stratonovich stochastic differential equation*

$$\delta g_t = g_t \delta W_t, \tag{4.1}$$

where W_t is the Brownian motion in $\mathfrak{g}$ described above, and g_0 is assumed to have a spherically symmetric distribution. Then g_t is spherically symmetric for any t.

Proof. Let $\tilde{g}_t = k_1 g_t k_2$ for any $k_1, k_2 \in K$. Then $\tilde{g}_t$ satisfies the following stochastic differential equation

$$\delta \tilde{g}_t = k_1 \delta g_t k_2 = \sum_{i=1}^{k} k_1 g_t \delta b_i^t \xi_i k_2 = \sum_{i=1}^{k} \tilde{g}_t \delta b_i^t k_2^{-1} \xi_i k_2.$$

Note that since the inner product on $\mathfrak{g}$ is Ad_K-invariant, $\{k_2^{-1}\xi_i k_2\}_{i=1}^d$ is an orthonormal basis of δ. Therefore g_t and $\tilde{g}_t$ have the same laws for any $t > 0$, and g_0 and $\tilde{g}_0$ have the same distributions. □

Remark 4.4. Note that $g_0 = e$ is not a spherically symmetric random variable.

Definition 4.5. The spherical transform of a Lévy process g_t is defined as

$$\hat{g}_\varphi(t) = E\left(\varphi(g_t)\right),$$

where φ is a spherical function on G.

Remark 4.6. By Propositions 3.9 and 4.3 for a spherically symmetric process g_t in Equation (4.1) we have

$$\hat{g}_\varphi(t) = E\left(\varphi(a_t^+)\right),$$

where a_t^+ is the Abelian component of the Cartan decomposition of g_t.

5. Subordination

Definition 5.1. Suppose T_t is an increasing $\mathbb{R}_+$-valued Lévy process, and X_t is a Markov process with values in G. Then X_{T_t} is called **a subordinated process with the subordinator T_t.**

We always assume that X_t and T_t are independent. Note that Remark 4.6 shows that for spherically symmetric processes the subordination procedure is reduced to subordination of the Abelian component of the process.

It is well-known that in the case where X_t is a continuous Lévy process with the law μ_t, and T_t is a pure jump (that is, it has a zero continuous part) subordinator with the Lévy measure ν and the law ν_t, then the subordinated process Y_t is pure jump with the Lévy measure given by

$$\Pi^Y(B) = \int_0^\infty \mu_s(B)\nu(ds), \ B \in \mathcal{B}(G \setminus \{e\}). \tag{5.1}$$

Indeed, by (2.2)

$$\begin{aligned}
&\int_{G\setminus\{e\}} \int_0^\infty f(g)\mu_s(dg)\nu(ds) = \lim_{t\to 0} \frac{1}{t} \int_0^\infty \int_G f(g)\mu_s(dg)\nu_t(ds) \\
&= \lim_{t\to 0} \frac{1}{t} \int_G \int_0^\infty f(g)\mu_s(dg)\nu_t(ds) = \lim_{t\to 0} \frac{1}{t} \int_G f(g) \int_0^\infty \mu_s(dg)\nu_t(ds),
\end{aligned}$$

which yields the desired result. We will apply Equation (5.1) to the case where X_t is a Brownian motion in G in Section 6.

The next theorem shows that subordination preserves two Markov processes having laws which are absolutely continuous with respect to each other. We also give a formula for the Radon-Nikodym density of the subordinated processes in the case of the initial processes being a Brownian motion in a Lie group and its shifted version.

Theorem 5.2. *Let $t \in [0, \infty)$.*

(1) Suppose that $X_1(t)$ and $X_2(t)$ are two Markov processes with the laws in the path spaces μ_1 and μ_2 respectively. If $\mu_1 << \mu_2$, then so are the laws of $Y_1(t) = X_1(T_t)$ and $Y_2(t) = X_2(T_t)$;

(2) Let $X_1(t) = g_t$ be the Brownian motion in G as defined by (3.1), and $X_2(t) = g_t e^{\xi t}$ for some $\xi \in \mathfrak{k}$. Denote by $p_t(g)$ the heat kernel corresponding to g_t and by ν_t the distribution of T_t. Then the Radon-Nikodym density of the laws of $Y_1(t) = X_1(T_t)$ and $Y_2(t) = X_2(T_t)$ is

$$D(t, g_{T_t}) = \frac{e^{\varphi(g_{T_t})} \int\limits_0^\infty e^{-\frac{|\xi|^2 x}{2}} p_x(g_{T_t}) d\nu_t(x)}{\int\limits_0^\infty p_x(g_{T_t}) d\nu_t(x)}, \tag{5.2}$$

where $\varphi(x) = Y_t^\xi(x) = \langle \xi, Y_t(x) \rangle$, $x \in W(G)$, where $Y_t(x)$ is defined in Equation (3.5).

Proof. (1) Suppose first that T_t is non-random, then the statement holds. Now let ν_t be the distribution of T_t in $\mathbb{R}_+$, then by a general theory of conditional expectations (e.g. [5]) for any $A \in \mathcal{F}$

$$P(Y_i(\cdot) \in A) = \int_0^\infty P(Y_i(s) \in A | T_t = s) d\nu_t(s), \ i = 1, 2.$$

Suppose $P(Y_i(s) \in A) = 0$, then $P(Y_i(s) \in A | T_t = s) = 0$ P-a.s., which reduces the question to the case where T_t is non-random. In particular, if $\mu_1 << \mu_2$, then $P(Y_2(s) \in A | T_t = s) = 0$ implies $P(Y_1(s) \in A | T_t = s) = 0$.

(2) Recall that by Corollary 3.7 if $k(t) = e^{\xi t}$ for some $\xi \in \mathfrak{k}$, the Radon-Nikodym derivative $D(k)$ is given by

$$D(k)(x) = \exp\left(Y_T^{\xi}(x) - \frac{|\xi|^2 T}{2}\right),$$

$x \in W(G)$, where $Y_t^{\xi}(x) = \langle \xi, Y_t(x) \rangle$. Set $\varphi(x) = Y_t^{\xi}(x)$, then by Fubini's theorem

$$Ef(g_{T_t}) = \int_0^{\infty}\int_G f(g)p_x(g)dgd\nu_t(x) =$$

$$\int_G f(g)\int_0^{\infty} p_x(g)d\nu_t(x)dg;$$

$$Ef(g_{T_t}e^{\xi T_t}) = \int_0^{\infty}\int_G f(g)p_x(g)e^{\varphi(g)}e^{-\frac{|\xi|^2 x}{2}}dgd\nu_t(x) =$$

$$\int_G f(g)e^{\varphi(g)}\int_0^{\infty} p_x(g)e^{-\frac{|\xi|^2 x}{2}}d\nu_t(x)dg,$$

which gives the density

$$D(t,g) = \frac{e^{\varphi(g)}\int_0^{\infty} e^{-\frac{|\xi|^2 x}{2}}p_x(g)d\nu_t(x)}{\int_0^{\infty} p_x(g)d\nu_t(x)}.$$

□

One of the problems in dealing with subordination for stochastic processes in Lie groups is that there is no appropriate Fourier transform for processes which are not spherically symmetric. At the same time, a spherically symmetric process g_t in G as defined in Proposition 4.3 and the corresponding shifted process $\tilde{g}_t = g_t k(t)$, $k(t) \in H(K)$ have the same spherical transforms, making them indistinguishable. One of the possible approaches to avoid this impasse will be presented in the case $G = \mathrm{SL}(2, \mathbb{R})$.

6. Examples

6.1. $\mathbb{R}$**.** Let B_t be an $\mathbb{R}$-valued standard Brownian motion, and $\gamma_t(a,b)$ a Gamma-process with mean a and variance equal to b. The density of this process with respect to the Lebesgue measure on $\mathbb{R}_+$ is

$$g_t(x) = \left(\frac{a}{b}\right)^{\frac{a^2 t}{b}} \frac{x^{\frac{a^2 t}{b}-1}\exp\left(-\frac{ax}{b}\right)}{\Gamma\left(\frac{a^2 t}{b}\right)}, \ x > 0,$$

and the characteristic function is

$$\varphi_{\gamma_t}(u) = \left(\frac{1}{1 - iu\frac{b}{a}}\right)^{\frac{a^2 t}{b}}.$$

Proposition 6.1. *Let* $X_1(t) = B(t)$, $X_2(t) = B(t) + ct$, $c \in \mathbb{R}$. *Then the law of* $Y_1(t) = B(T_t)$ *is absolutely continuous with respect to the law of* $Y_2(t) = B(T_t) + cT_t$, *where* $T_t = \gamma_t(1,1)$, *a standard Gamma-process with the mean and variance equal to* 1. *The Radon-Nikodym density for the Lévy measures of* $Y_1(t)$ *and* $Y_2(t)$ *is*

$$p_c(x) = e^{cx} e^{\left(\sqrt{2} - \sqrt{c^2+2}\right)|x|}.$$

Proof. In this case we not only calculate the characteristic functions for the subordinated processes, but actually identify the subordinated processes. We give two proofs: one is a direct calculation using the characteristic functions, and another is an application of Equation (5.1).

The characteristic function of the subordinated process $Y_t = B_{\gamma_t(1,1)}$ is

$$\varphi_{Y_t}(u) = E\left[e^{iuB_{\gamma_t(1,1)}}\right] = E\left[e^{-\frac{u^2}{2}\gamma_t(1,1)}\right] = \int_0^\infty \frac{x^{t-1}\exp(-x)}{\Gamma(t)} e^{-\frac{u^2}{2}x} dx = \int_0^\infty \frac{x^{t-1}}{\Gamma(t)} e^{-(\frac{u^2}{2}+1)x} dx = \left(\frac{u^2}{2} + 1\right)^{-t} = \left(1 + \frac{iu}{\sqrt{2}}\right)^{-t} \left(1 - \frac{iu}{\sqrt{2}}\right)^{-t},$$

so $Y_t = \gamma_t^1(\frac{1}{\sqrt{2}}, \frac{1}{2}) - \gamma_t^2(\frac{1}{\sqrt{2}}, \frac{1}{2})$. Now let $X_t = B_t + ct$ for some constant c, and set $Z_t = X_{\gamma_t(1,1)}$. Then

$$\varphi_{Z_t}(u) = E\left[e^{iuB_{\gamma_t(1,1)} + iuc\gamma_t(1,1)}\right] = E\left[e^{\left(iuc - \frac{u^2}{2}\right)\gamma_t(1,1)}\right] = \int_0^\infty \frac{x^{t-1}\exp(-x)}{\Gamma(t)} e^{-\left(\frac{u^2}{2} - iuc\right)x} dx = \int_0^\infty \frac{x^{t-1}}{\Gamma(t)} e^{-(\frac{u^2}{2} - iuc + 1)x} dx = \left(\frac{u^2}{2} - iuc + 1\right)^{-t} = \left(1 - iu\frac{c + \sqrt{c^2+2}}{2}\right)^{-t} \left(1 + iu\frac{\sqrt{c^2+2} - c}{2}\right)^{-t},$$

so $Z_t = \gamma_t^1\left(\frac{c+\sqrt{c^2+2}}{2}, \left(\frac{c+\sqrt{c^2+2}}{2}\right)^2\right) - \gamma_t^2\left(\frac{\sqrt{c^2+2}-c}{2}, \left(\frac{\sqrt{c^2+2}-c}{2}\right)^2\right)$. The Lévy measure for Z_t is then

$$g_c(x)dx = \begin{cases} \frac{\exp\left(-\frac{2|x|}{\sqrt{c^2+2}-c}\right)}{|x|} dx, & x < 0 \\ \frac{\exp\left(-\frac{2x}{\sqrt{c^2+2}+c}\right)}{x} dx, & x > 0 \end{cases} = e^{cx} \frac{e^{-\sqrt{c^2+2}|x|}}{|x|} dx,$$

and for Y_t

$$g_0(x)dx = \frac{e^{-\sqrt{2}|x|}}{|x|} dx.$$

Thus $g_c(x)dx = p_c(x)g_0(x)dx$, where

$$p_c(x) = e^{cx} e^{\left(\sqrt{2}-\sqrt{c^2+2}\right)|x|}.$$

Note that Equation (5.1) gives the same answer. The Lévy measure for a standard Gamma process is $\frac{e^{-s}}{s} ds$, $s > 0$, and so the Lévy measure for the Brownian motion subordinated by $\gamma(1,1)$ is

$$\int_0^\infty \frac{e^{-\frac{(x-cs)^2}{2s}}}{\sqrt{2\pi s}} \frac{e^{-s}}{s} ds dx = e^{cx} \int_0^\infty \frac{e^{-\frac{x^2}{2s}}}{\sqrt{2\pi s}} \frac{e^{-(\frac{c^2+2}{2})s}}{s} ds dx = e^{cx} \frac{e^{-\sqrt{c^2+2}|x|}}{|x|} dx.$$

In the latter equation we used the following identity

$$\int_0^\infty \frac{e^{-\frac{a^2}{s}-b^2 s}}{s^{3/2}} ds = \frac{e^{-2ab}}{a} \int_{-\infty}^\infty e^{-t^2} \frac{\sqrt{t^2+4ab}-t}{\sqrt{t^2+4ab}} dt = \frac{e^{-2ab}}{a}\sqrt{\pi}, \tag{6.1}$$

where $t = b\sqrt{s} - \frac{a}{\sqrt{s}}$. □

6.2. $\mathbb{R}_+ \setminus \{0\}$**.** Suppose $G = \mathbb{R}_+ \setminus \{0\}$, then $\mathfrak{g} = \mathbb{R}$. The Brownian motion on G satisfies the stochastic differential equation

$$dX_t = X_t d\beta_t + \frac{1}{2} X_t dt,$$

where β_t is a real-valued Brownian motion. The solution to this stochastic differential equation is $X_1(t) = e^{\beta_t}$, the geometric Brownian motion. Now consider the subordinated process $M_t = X_1(T_t) = e^{\beta_{T_t}} = e^{N_t}$, where $N_t = \beta_{T_t}$. We assume that the process T_t is of the pure jump type, and therefore so is N_t.

Note that if $T_t = \gamma_t(1,1)$ is a standard Gamma-process with the mean and variance 1, and $X_2(t) = e^{\beta_t + ct}$, then this example reduces to the previous one since the characteristic function of an $\mathbb{R}_+$-valued process A_t is

$$\varphi_{A_t}(u) = E\left[e^{iu \ln A_t}\right].$$

More generally, Theorem 3.2 says that the law of $X_1(t)$ is equivalent to the law of $X_t^s = e^{\beta_t + f(t)}$, where f is a real-valued measurable function such that $f' \in L^2([0,T])$, where f' is understood in the distributional sense. This comes from the observation that the Cameron-Martin space in this case is

$$H(G) = \{h(t) : [0,T] \to \mathbb{R}_+, \int_0^T \left|\frac{h'(t)}{h(t)}\right|^2 dt < \infty\} = \{h(t) = e^{f(t)}, f' \in L^2([0,T])\}.$$

6.3. $\mathrm{SL}(2,\mathbb{R})$**.** Let $G = \mathrm{SL}(2,\mathbb{R})$. In this case we can describe the Cartan and Iwasawa decompositions of a Brownian motion in G more explicitly that for a general matrix Lie group, or even for $\mathrm{SL}(n,\mathbb{R})$ if $n > 2$. Let us choose an orthonormal (with respect to the inner product $\langle X, Y\rangle = Tr(XY^T)$ for $X, Y \in \mathrm{SL}(2,\mathbb{R})$ induced by the Killing form) basis of $\mathfrak{g}$ to be

$$\xi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \xi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \xi_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

An Iwasawa decomposition for $\mathrm{SL}(2,\mathbb{R})$ is

$$
\begin{aligned}
A =& \{\begin{pmatrix} e^{H} & 0 \\ 0 & e^{-H} \end{pmatrix}, H \in \mathbb{R}\}; \\
N_{-} =& \{\begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix}, N \in \mathbb{R}\}; \\
K =& \{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \theta \in \mathbb{R}\},
\end{aligned} \tag{6.2}
$$

and in the Cartan decomposition the Abelian component can be chosen as

$$
A_{+} = \{\begin{pmatrix} e^{H} & 0 \\ 0 & e^{-H} \end{pmatrix}, H \geqslant 0\};
$$

Suppose g_t is a Brownian motion in $G = \mathrm{SL}(2,\mathbb{R})$. Let $W_t = b_t^{(1)}\xi_1 + b_t^{(2)}\xi_2 + b_t^{(3)}\xi_3$, where $b_t^{(1)}, b_t^{(2)}, b_t^{(3)}$ are independent real-valued Brownian motions. Set

$$
\begin{aligned}
k_t &= \begin{pmatrix} \cos\theta_t & \sin\theta_t \\ -\sin\theta_t & \cos\theta_t \end{pmatrix}, \\
a_t &= \begin{pmatrix} e^{H_t} & 0 \\ 0 & e^{-H_t} \end{pmatrix}, \\
n_t &= \begin{pmatrix} 1 & 0 \\ N_t & 1 \end{pmatrix}.
\end{aligned}
$$

Proposition 6.2. *The processes θ_t, H_t and N_t satisfy the following system of stochastic differential equations*

$$
\begin{aligned}
\delta\theta_t =& \frac{1}{\sqrt{2}}\left(\delta b_t^{(1)} + \sin 2\theta_t \delta b_t^{(2)} + \cos 2\theta_t \delta b_t^{(3)}\right), \\
\delta H_t =& \frac{1}{\sqrt{2}}\left(\cos 2\theta_t \delta b_t^{(2)} - \sin 2\theta_t \delta b_t^{(3)}\right), \\
\delta N_t =& \sqrt{2} e^{-2H_t}\left(\sin 2\theta_t \delta b_t^{(2)} + \cos 2\theta_t \delta b_t^{(3)}\right).
\end{aligned} \tag{6.3}
$$

Proof. These stochastic differential equations can be derived in two ways. First, we can write Equation (3.7) in terms of the basis $\{\xi_1, \xi_2, \xi_3\}$. An alternative is to write Equation (3.8) for these particular matrices n_t, a_t, k_t. We will follow the second path.

$$
a_t^{-1} n_t^{-1} \delta n_t a_t + a_t^{-1}\delta a_t + \delta k_t k_t^{-1} = k_t \delta W_t k_t^{-1}.
$$

Let us first find the adjoint action of K on the basis $\{\xi_1, \xi_2, \xi_3\}$. For any $k = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in K$ we have

$$
\begin{aligned}
\mathrm{Ad}_k \xi_1 =& \xi_1, \\
\mathrm{Ad}_k \xi_2 =& \frac{1}{\sqrt{2}}\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}, \\
\mathrm{Ad}_k \xi_3 =& \frac{1}{\sqrt{2}}\begin{pmatrix} -\sin 2\theta & \cos 2\theta \\ \cos 2\theta & \sin 2\theta \end{pmatrix}.
\end{aligned} \tag{6.4}
$$

Now let us find all terms on the left hand side of Equation (3.8)

$$a_t^{-1}n_t^{-1}\delta n_t a_t = \begin{pmatrix} 0 & 0 \\ e^{2H_t}\delta N_t & 0 \end{pmatrix},$$
$$a_t^{-1}\delta a_t = \begin{pmatrix} \delta H_t & 0 \\ 0 & -\delta H_t \end{pmatrix}.$$
$$\delta k_t k_t^{-1} = \begin{pmatrix} 0 & \delta\theta_t \\ -\delta\theta_t & 0 \end{pmatrix},$$

For the right hand side of Equation (3.8) we use Equation (6.4) with $k = k_t$

$$k_t\delta W_t k_t^{-1} = \frac{1}{\sqrt{2}}\times$$
$$\begin{pmatrix} \cos 2\theta_t\delta b_t^{(2)} - \sin 2\theta_t\delta b_t^{(3)} & \delta b_t^{(1)} + \sin 2\theta_t\delta b_t^{(2)} + \cos 2\theta_t\delta b_t^{(3)} \\ -\delta b_t^{(1)} + \sin 2\theta_t\delta b_t^{(2)} + \cos 2\theta_t\delta b_t^{(3)} & -\cos 2\theta_t\delta b_t^{(2)} + \sin 2\theta_t\delta b_t^{(3)} \end{pmatrix}.$$

Adding these four matrices gives Equation (6.3). □

Proposition 6.3. *The processes θ_t and $\sqrt{2}H_t$ are independent real-valued Brownian motions. The process $N_t \stackrel{\mathfrak{L}}{=} \sqrt{2}\sinh(-2B_t)$, where B_t is a standard $\mathbb{R}$-valued Brownian motion.*

Proof. The first part follows from the Lévy criterion once the Stratonovich integrals in Equation (6.3) are rewritten in the Itô form. For the second part we follow [16]. This is a version of Bougerol's identity. Define $Y_t = 2e^{-2B_t}\int_0^t e^{2B_s}dW_s$, where B_s and W_s are standard independent $\mathbb{R}$-valued Brownian motions. Then $\sqrt{2}N_t \stackrel{\mathfrak{L}}{=} Y_t$ due to the invariance of the law of a Brownian motion under time reversal at a fixed time. Bougerol's identity states that $\sinh(-2B_t) \stackrel{\mathfrak{L}}{=} Y_t$. Indeed, by the Itô formula

$$\begin{aligned} d\sinh(-2B_t) = &- 2\cosh(-2B_t)dB_t + 2\sinh(-2B_t)dt = \\ &- 2\sqrt{1 + (\sinh(-2B_t))^2}dB_t + 2\sinh(-2B_t)dt, \end{aligned}$$

so $\sinh(-2B_t)$ has the following generator

$$\mathfrak{L}f(x) = 2(1+x^2)\frac{d^2 f}{dx^2} + 2x\frac{df}{dx}.$$

The process Y_t has the same generator as can be seen from the following stochastic differential equation

$$dY_t = -2Y_t dB_t + 2dW_t + 2Y_t dt \stackrel{\mathfrak{L}}{=} \sqrt{4Y_t^2 + 4}dB_t + 2Y_t dt.$$

□

In a similar manner we can derive a stochastic differential equation for the Abelian component in the Cartan decomposition of g_t, a Brownian motion in $G = \mathrm{SL}(2,\mathbb{R})$. Namely, let $g_t = k_t^{(1)}a_t^+k_t^{(2)}$, where $k_t^{(1)}, k_t^{(2)} \in K$, $a_t^+ \in A_+$, then by the first part of this proof

$$\delta H_t^+ = \frac{1}{\sqrt{2}}\left(\cos 2\theta_t\delta b_t^2 - \sin 2\theta_t\delta b_t^3\right). \tag{6.5}$$

Let $k(t)=\begin{pmatrix}\cos ct & \sin ct\\ -\sin ct & \cos ct\end{pmatrix}$, and consider the shifted Brownian motion $\tilde{g}_t = g_t h(t) = n_t a_t \tilde{k}_t$, where $\tilde{k}_t = k_t k(t)$ or $\tilde{k}_t = \begin{pmatrix}\cos(\theta_t+ct) & \sin(\theta_t+ct)\\ -\sin(\theta_t+ct) & \cos(\theta_t+ct)\end{pmatrix}$.

Note that the characters of $G=\mathrm{SL}(2,\mathbb{R})$ are determined by their values on the elements of the one-dimensional subgroups A, N and K. Thus for any $g = nak, a\in A, n\in N, k\in K$

$$\chi(g)=\chi(n)\chi(a)\chi(k)=e^{i\lambda_n N}e^{i\lambda_a H}e^{i\lambda_k\theta},\ \lambda_n,\lambda_a,\lambda_k\in\mathbb{R},$$

where $N,H,\theta\in\mathbb{R}$ are defined as in Equation (6.2). Then the characteristic functions for g_t and $\tilde{g}_t$ can be found as follows

$$\begin{aligned}\widehat{g_t}(\chi)=E_\mu\chi(g_t)(\lambda_a,\lambda_n,\lambda_k) =&E_\mu e^{i\lambda_n N_t}e^{i\lambda_a H_t}e^{i\lambda_k\theta_t},\\ E_\mu\chi(\tilde{g}_t)(\lambda_a,\lambda_n,\lambda_k) =&E_\mu e^{i\lambda_n N_t}e^{i\lambda_a H_t}e^{i\lambda_k(\theta_t+ct)},\end{aligned}$$

and by Equation (5.2)

$$\widehat{\tilde{g}_t}(\lambda_a,\lambda_n,\lambda_k)=e^{i\lambda_k ct}\widehat{g_t}(\lambda_a,\lambda_n,\lambda_k)=e^{\frac{-c^2t}{2}}E_\mu\left[\chi(g_t)(\lambda_a,\lambda_n,\lambda_k)e^{cb_t^{(1)}}\right].$$

Now we can change time by a Gamma process for the components a_t, n_t and k_t, and use the previous examples.

For this example we can also write the heat kernel corresponding to the Brownian motion g_t, though this expansion does not allow us to use (5.1) to find the Lévy measure for the subordinated process.

Theorem 6.4. (1) *The Laplace-Beltrami operator on* $\mathrm{SL}(2,\mathbb{R})$ *in the coordinates* (N,H,θ) *is*

$$\Delta=\frac{1}{2}\frac{\partial^2}{\partial H^2}+\frac{\partial^2}{\partial\theta^2}+2e^{-4H}\frac{\partial^2}{\partial N^2}+2e^{-2H}\frac{\partial^2}{\partial\theta\partial N}+\frac{\partial}{\partial H}; \tag{6.6}$$

(2) *the heat kernel on* $\mathrm{SL}(2,\mathbb{R})$ *is given by*

$$\begin{aligned}p_t(N,H,\theta)=&\\ &\sum_{n\in\mathbb{Z}}e^{i\theta n}\int_{-\infty}^{\infty}e^{t\lambda_n^p}\varphi_n^p(N,e^{-2H})dp+\\ &e^{i2\theta}e^{-2t}\varphi_2^0(N,e^{-2H})+e^{-i2\theta}e^{-2t}\varphi_{-2}^0(N,e^{-2H})+\\ &\sum_{n=3}^{\infty}e^{i\theta n}\sum_{l=0}^{\left[\frac{n}{2}\right]-1}e^{t\lambda_n^l}\varphi_n^l(N,e^{-2H})+\sum_{n=3}^{\infty}e^{-i\theta n}\sum_{l=0}^{\left[\frac{n}{2}\right]-1}e^{t\lambda_n^l}\varphi_n^l(N,e^{-2H}),\end{aligned}$$

where

$$\begin{cases}\lambda_n^p = -2(p^2+\frac{1}{4}+n^2), & p\in\mathbb{R},\\ \lambda_n^l = 2\left((\frac{|n|-1}{2}-l)^2-\frac{1}{4}-n^2\right), & 0\leqslant l<\frac{|n|-1}{2}, l\in\mathbb{Z},\end{cases}$$

$\varphi_n^p(x,y)$ *and* $\varphi_n^l(x,y)$ *are generalized eigenfunctions for the Maass Laplacian (see* [17], [18]*), and* $[k]$ *is the largest integer smaller than* k.

Proof. First we would like to compute the differential operator

$$(\tilde{\xi} f)(g) = \frac{d}{dt}\Big|_{t=0} f(ge^{t\xi})$$

in terms of (N, H, θ), where N, H, θ are the parameters of the Iwasawa decomposition described in Equation (6.2). For any $g \in \mathrm{SL}(2, \mathbb{R})$ we have

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} e^H \cos\theta & e^H \sin\theta \\ Ne^H \cos\theta - e^{-H} \sin\theta & Ne^H \sin\theta + e^{-H} \cos\theta \end{pmatrix}.$$

Then for any ξ in the Lie algebra of $\mathrm{SL}(2, \mathbb{R})$ we denote by $N(t), H(t), \theta(t)$ the parameters of the Iwasawa decomposition for $ge^{t\xi}$. Then

$$(\tilde{\xi} f)(g) = \frac{\partial f}{\partial N}(g) N'(0) + \frac{\partial f}{\partial H}(g) H'(0) + \frac{\partial f}{\partial \theta}(g) \theta'(0).$$

Let ξ_1, ξ_2, ξ_3 be the same basis of the Lie algebra of $\mathrm{SL}(2, \mathbb{R})$ as before. Then for ξ_1 we have $N_1(t) = N_1(0) = N$, $H_1(t) = H_1(0) = H$, $\theta_1(t) = \theta + \frac{t}{\sqrt{2}}$, so

$$L_1 f = (\tilde{\xi_1} f) = \frac{1}{\sqrt{2}} \frac{\partial f}{\partial \theta}.$$

For ξ_2 we have $e^{t\xi_2} = \begin{pmatrix} e^{t/\sqrt{2}} & 0 \\ 0 & e^{-t/\sqrt{2}} \end{pmatrix}$, therefore the corresponding parameters $N_2(t), H_2(t), \theta_2(t)$ satisfy

$$\begin{aligned} e^{H_2(t)} \cos\theta_2(t) =& ae^{t/\sqrt{2}}, \\ e^{H_2(t)} \sin\theta_2(t) =& be^{-t/\sqrt{2}}, \\ N_2(t) e^{H_2(t)} \cos\theta_2(t) - e^{-H_2(t)} \sin\theta_2(t) =& ce^{t/\sqrt{2}}, \\ N_2(t) e^{H(t)} \sin\theta_2 + e^{-H_2(t)} \cos\theta_2(t) =& de^{-t/\sqrt{2}}. \end{aligned}$$

Differentiating these equations at $t = 0$ we see that

$$\begin{aligned} N_2'(0) =& -\frac{2\sqrt{2}ab}{(a^2+b^2)^2} = -\sqrt{2} e^{-2H} \sin 2\theta, \\ H_2'(0) =& \frac{a^2 - b^2}{\sqrt{2}(a^2+b^2)} = \frac{\cos 2\theta}{\sqrt{2}}, \\ \theta_2'(0) =& -\frac{\sqrt{2}ab}{(a^2+b^2)} = -\frac{\sin 2\theta}{\sqrt{2}}, \end{aligned}$$

and therefore

$$L_2 f = (\tilde{\xi_2} f) = -\sqrt{2} e^{-2H} \sin 2\theta \frac{\partial f}{\partial N} + \frac{\cos 2\theta}{\sqrt{2}} \frac{\partial f}{\partial H} - \frac{\sin 2\theta}{\sqrt{2}} \frac{\partial f}{\partial \theta}.$$

Similarly

$$L_3 f = (\tilde{\xi_3} f) = \sqrt{2} e^{-2H} \cos 2\theta \frac{\partial f}{\partial N} + \frac{\sin 2\theta}{\sqrt{2}} \frac{\partial f}{\partial H} + \frac{\cos 2\theta}{\sqrt{2}} \frac{\partial f}{\partial \theta}.$$

From these expressions for $L_i, i = 1, 2, 3$ we can derive Equation (6.6) for $\Delta = L_1^2 + L_2^2 + L_3^2$.

To find the heat kernel for this Laplacian we first look at the eigenfunctions of Δ of the form $f_n(N,H,\theta)=e^{in\theta}g(N,H)$, $n\in\mathbb{Z}$

$$\Delta f_n=\left(\frac{1}{2}\frac{\partial^2 g}{\partial H^2}-n^2g+2e^{-4H}\frac{\partial^2 g}{\partial N^2}+2ine^{-2H}\frac{\partial g}{\partial N}+\frac{\partial g}{\partial H}\right)e^{in\theta}=\lambda_n f_n.$$

Let $x=N, y=e^{-2H}$, then equivalently we would like to solve for $g(x,y)$

$$D_n g=y^2(g_{xx}+g_{yy})+inyg_x=(\frac{\lambda_n}{2}+n^2)g,$$

where D_n is the Maass Laplacian. The spectrum of this operator has been studied for a long time, we follow the exposition found in two papers by K. Oshima [17] and [18]. For the original treatment of the subject one can look at [6]. Note that usually the Maass Laplacian is labeled by "half-integer weights", namely, $D_k g=y^2(g_{xx}+g_{yy})-2ikyg_x$. The operator D_n has both discrete and continuous spectra. The elements in the spectrum are usually parameterized as $s(s-1)$, where

$$s=\begin{cases}\frac{1}{2}+ip, & p\in\mathbb{R},\\ \frac{|n|}{2}-l, & 0\leqslant l<\frac{|n|}{2}-\frac{1}{2}, l\in\mathbb{Z}.\end{cases}$$

Explicit formulae for the corresponding generalized eigenfunctions $\varphi_n^p(x,y)$ and $\varphi_n^l(x,y)$ can be found in the above mentioned papers. In particular, in these papers it is shown that these eigenfunctions give the full spectral decomposition. Then

$$\begin{cases}\lambda_n^p &= -2(p^2+\frac{1}{4}+n^2), & p\in\mathbb{R},\\ \lambda_n^l &= 2\left((\frac{|n|-1}{2}-l)^2-\frac{1}{4}-n^2\right), & 0\leqslant l<\frac{|n|-1}{2}, l\in\mathbb{Z}.\end{cases}$$

Note that for $n=-1,0,1$ the operator D_n has only continuous spectrum, and $\lambda_n<0$ for any $n\in\mathbb{Z}$. Denote by $[k]$ the largest integer smaller than k. Thus the heat kernel on $\mathrm{SL}(2,\mathbb{R})$ is

$$\begin{aligned}p_t(N,H,\theta)=&\\ &\sum_{n\in\mathbb{Z}}e^{i\theta n}\int_{-\infty}^{\infty}e^{t\lambda_n^p}\varphi_n^p(N,e^{-2H})dp+\\ &e^{i2\theta}e^{-2t}\varphi_2^0(N,e^{-2H})+e^{-i2\theta}e^{-2t}\varphi_{-2}^0(N,e^{-2H})+\\ &\sum_{n=3}^{\infty}e^{i\theta n}\sum_{l=0}^{[\frac{n}{2}]-1}e^{t\lambda_n^l}\varphi_n^l(N,e^{-2H})+\sum_{n=3}^{\infty}e^{-i\theta n}\sum_{l=0}^{[\frac{n}{2}]-1}e^{t\lambda_n^l}\varphi_n^l(N,e^{-2H}).\end{aligned}$$

□

6.4. $\mathrm{GL}(n,\mathbb{R})_+$. Let $G=\mathrm{GL}(n,\mathbb{R})_+$, the group of invertible matrices having positive determinant. This is not a semi-simple group, but one has $\mathrm{GL}(n,\mathbb{R})_+=\mathbb{R}_+\times\mathrm{SL}(n,\mathbb{R})$, and therefore $\mathrm{GL}(n,\mathbb{R})_+$ has the same Iwasawa decomposition as $\mathrm{SL}(n,\mathbb{R})$ with $K=\mathrm{SO}(n,\mathbb{R})$, A consisting of diagonal matrices with positive diagonal entries, and N is the subgroup of lower triangular matrices with 1s on the diagonal. Choose the following basis for the Lie algebra of G

$$X_{ij}=\frac{1}{\sqrt{2}}(e_{ij}-e_{ji}), i<j, \quad Y_i=e_{ii}, \quad Z_{ij}=\frac{1}{\sqrt{2}}(e_{ij}+e_{ji}), i<j,$$

where e_{ij} is the matrix with 1 at the ijth place, and 0s at all other places.

Theorem 6.5. (1) *Let H_t^l be the lth entry of the solution to* (3.9). *Then $\{H_t^l\}_{l=1}^n$ are independent Brownian motions.*

(2) *Let H_t^+ be the solution of* (3.10). *Then the law of H_t^+ is given by*

$$P\{H_t^+ \in A \in \mathcal{B}(\mathbb{R}^n)\} = \int\limits_{f_n^{-1}(A)} \prod_{i=1}^n p_t(x_i) dx_1 ... dx_n, \tag{6.7}$$

where $f_n(x_1, ... x_n) = (y_1, ... y_n)$, $y_j = \max\limits_{(i_1,...,i_j)} \{\min\{e^x_{i_1}, ..., e^x_{i_j}\}\}$, $i_1, ..., i_j$ are distinct, and $p_t(x)$ is the standard Gaussian density.

(3) *The Brownian motion g_t in $\mathrm{GL}(n,\mathbb{R})_+$ induces a stochastic process in the double cosets space $\mathrm{SO}(n,\mathbb{R})\backslash\mathrm{GL}(n,\mathbb{R})_+/\mathrm{SO}(n,\mathbb{R})$ with the law given by* (6.7). *The corresponding subordinated process has the following Lévy measure*

$$\int\limits_{f_n^{-1}(A)} \int_0^\infty \prod_{i=1}^n p_s(x_i) \nu(ds) dx_1 ... dx_n, \ A \in \mathcal{B}(\mathbb{R}^n \backslash \{0\}),$$

where ν is the Lévy measure of a pure jump subordinator T_t .

Proof. (1) For any $k = \{k_{lm}\}_{l,m=1}^n \in SO(n,\mathbb{R})$ we have

$$\begin{aligned}
\mathrm{Ad}_k X_{ij} =& \sum_{l<m} (k_{il}k_{jm} - k_{jl}k_{im}) X_{lm}, \\
\mathrm{Ad}_k Y_i =& \sum_{l,m} k_{il}k_{im} e_{lm} = \sum_l k_{il}^2 e_{ll} + \sum_{l<m} k_{il}k_{im}(e_{lm} + e_{ml}) = \\
& \sum_l k_{il}^2 Y_l + \sqrt{2} \sum_{l<m} k_{il}k_{im} X_{lm} + 2\sqrt{2} \sum_{l<m} k_{il}k_{im} e_{ml}, \\
\mathrm{Ad}_k Z_{ij} =& \frac{1}{\sqrt{2}} \sum_{l,m} (k_{il}k_{jm} + k_{jl}k_{im}) e_{lm} = \\
& \sqrt{2} \sum_l k_{il}k_{jl} Y_l + \sum_{l<m} (k_{il}k_{jm} + k_{im}k_{jl}) X_{lm} + \sqrt{2} \sum_{l<m} (k_{il}k_{jm} + k_{im}k_{jl}) e_{lm}.
\end{aligned}$$

Then

$$\sum_{i=1}^n ([\mathrm{Ad}_k Y_i]_{\mathfrak{a}})^2 = \sum_{i=1}^n \left(\sum_{l=1}^n k_{il}^2 Y_l \right)^2 = \sum_{i=1}^n \sum_{l=1}^n k_{il}^4 Y_l,$$

$$\sum_{i<j}^n \left([\mathrm{Ad}_{k_t} Z_{ij}]_{\mathfrak{a}} \right)^2 = \sum_{i<j}^n \left(\sqrt{2} \sum_l k_{il}k_{jl} Y_l \right)^2 = 2 \sum_{i<j}^n \sum_l k_{il}^2 k_{jl}^2 Y_l.$$

In particular, for H_t^l, the lth entry of the solution to (3.9) we have

$$d\left\langle H_s^l, H_s^m\right\rangle_t = \left(\sum_{i=1}^n k_{il}^2 k_{im}^2 + 2\sum_{i<j}^n k_{il}k_{jl}k_{im}k_{jm}\right) dt = \left(\sum_{i,j=1}^n k_{il}k_{jl}k_{im}k_{jm}\right) dt = \left(\sum_{i=1}^n k_{il}k_{im}\right)^2 dt = \delta_{km} dt,$$

where δ_{km} is Kronecker's symbol. Therefore $\{H_t^l\}_{l=1}^n$ are independent Brownian motions.

(2) It is easy to see that for any j we have $y_j \geqslant y_{j+1}$. H_t^+ is a permutation of entries of H_t, so if we interpret both H_t and H_t^+ as random vectors in $\mathbb{R}^n$, we can see that

$$a_t^+ = f_n(a_t)$$

which gives the desired result.

(3) This part follows directly from part (2) and (5.1). □

Remark 6.6. Note that if we use the last part of Theorem 6.5 with $T_t = \gamma(1,1)$, then we can find explicitly the Lévy measure of the subordinated process in the double cosets space $\mathrm{SO}(n,\mathbb{R})\backslash\mathrm{GL}(n,\mathbb{R})_+/\mathrm{SO}(n,\mathbb{R})$ similarly to Example 6.2.

6.5. **Heisenberg group.** In order to fit this example into the case of matrix Lie groups we give a description of the Heisenberg group as a matrix group as it is done in [9]. Denote

$$[a,b,c] = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \ (a,b,c) = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}.$$

Then the Heisenberg group, $\mathbf{H}$, can be described as $\mathbf{H} = \{[a,b,c], a,b,c \in \mathbb{R}\}$ with the group multiplication

$$[a_1,b_1,c_1]\cdot[a_2,b_2,c_2] = [a_1+a_2, b_1+b_2, c_1+c_2+a_1b_2],$$

and the Heisenberg Lie algebra, $\mathfrak{h}$, can be described as $\mathfrak{h} = \{(a,b,c), a,b,c \in \mathbb{R}\}$ with the Lie bracket

$$[(a_1,b_1,c_1),(a_2,b_2,c_2)] = (0,0,a_1b_2 - a_2b_1),$$

and the inner product

$$\langle (a_1,b_1,c_1),(a_2,b_2,c_2)\rangle = Tr\left((a_2,b_2,c_2)^T(a_1,b_1,c_1)\right) = a_1a_2 + b_1b_2 + c_1c_2.$$

Then $X = (1,0,0)$, $Y = (0,1,0)$ and $Z = (0,0,1)$ is an orthonormal basis of $\mathfrak{h}$. Note that we have as usual

$$e^{(a,b,c)} = [a,b,c+\frac{ab}{2}].$$

The inner product on $\mathfrak{h}$ is Ad_g-invariant only if $g = [0,0,c]$, which means that in this case the Cameron-Martin subspace $H(\mathbf{H})$ is the space of curves $[0,0,k(t)]$, where $k(t) \in H(\mathbb{R})$.

First let us consider the Brownian motion corresponding to the full Laplacian $\Delta = \frac{1}{2}(\tilde{X}^2 + \tilde{Y}^2 + \tilde{Z}^2)$. Suppose W_t^X, W_t^Y and W_t^Z are independent real-valued Brownian motions. Then

Proposition 6.7. *The process* $g_t = [W_t^X, W_t^Y, W_t^Z + \int_0^t W_s^X \delta W_s^Y]$ *is a Brownian motion in* $\mathbf{H}$*, that is,* g_t *is the solution to stochastic differential equation* (3.1)*.*

Proof. This can be verified by a direct calculation

$$\delta g_t = (\delta W_t^X, \delta W_t^Y, \delta W_t^Z + W_t^X \delta W_t^Y),$$

and this is the same matrix as

$$g_t \delta W_t = \begin{pmatrix} 1 & W_t^X & W_t^Z + \int_0^t W_s^X \delta W_s^Y \\ 0 & 1 & W_t^Y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \delta W_t^X & \delta W_t^Z \\ 0 & 0 & \delta W_t^Y \\ 0 & 0 & 0 \end{pmatrix}.$$

□

Now we can apply (5.1) and (5.2). The heat kernel in this case can be written explicitly, though it is of limited use for our purpose.

In conclusion we can add that similarly we can consider the subelliptic Laplacian $\Delta_H = \frac{1}{2}(\tilde{X}^2 + \tilde{Y}^2)$ and the corresponding Brownian motion $g_t^H = [W_t^X, W_t^Y, \int_0^t W_s^X \delta W_s^Y]$. In this case the function φ in (5.2) $\varphi \equiv 0$, and therefore the Radon-Nikodym derivative has a simpler form than the one for the full Laplacian.

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