TAYLOR MAP ON GROUPS ASSOCIATED WITH A II_1-FACTOR

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Abstract. A notion of the heat kernel measure is introduced for the L^2 completion of a hyperfinite II_1-factor with respect to the trace. Some properties of this measure are derived from the corresponding stochastic differential equation. Then the Taylor map is studied for a space of holomorphic functions square integrable with respect to the heat kernel measure. We also define a skeleton map from this space to a Hilbert space of holomorphic functions on a certain Cameron-Martin group. This group is a subgroup of the group of invertible elements of the II_1-factor.

Heat kernel measure, II_1-factor, infinite-dimensional group, stochastic differential equation

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1. Introduction

In this work we continue to study the type of heat kernel analysis on infinite-dimensional groups developed in [4], [5]. These papers dealt with so called Hilbert-Schmidt (infinite-dimensional) complex groups of operators on a Hilbert space. In the present article we consider similar problems for a hyperfinite II_1-factor $\mathcal{C}$ realized as the weak closure of a subalgebra of the CAR-algebra.

The structure of this paper is as follows. The exact description of the II_1-factor is given in Section 2. In this representation $\mathcal{C}$ is a subalgebra of the bounded operators on a Hilbert space of skew symmetric tensors, $\Lambda(H)$. The completion of this II_1-factor in the L^2 norm induced by a trace on $\mathcal{C}$ is a Hilbert space $L^2(\mathcal{C})$. Elements of the space $L^2(\mathcal{C})$ can be identified with possibly unbounded operators. We use the fact that $L^2(\mathcal{C})$ is a Hilbert space to build a heat kernel measure. This

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measure, μ_t, is the transition probability of a solution of a stochastic differential equation with state space $L^2(\mathcal{C})$. We will describe this stochastic differential equation and its solution X_t in Section 3. The Wiener process in $L^2(\mathcal{C})$ has a complex linear nonnegative trace class operator Q on $L^2(\mathcal{C})$ as its covariance operator. The operator Q also determines "new" stronger inner product on $L^2(\mathcal{C})$. Kolmogorov's backward equation for the process X_t is a heat equation in a sense. This fact justifies the name for the heat kernel measure μ_t defined in Section 4.

The main difference from the case of Hilbert-Schmidt groups studied in [4] and [5] is that the heat kernel measure lives in a space of unbounded operators, which makes the case of the II_1-factor more complicated. Some of the new features are addressed in Section 6. There, in addition to the stochastic differential equation determining the heat kernel measure, we consider a weak version of this equation. This "new" equation is a stochastic differential equation whose state space is not $L^2(\mathcal{C})$ but $\Lambda(H)$. We can view its solution at time t as an operator applied to the initial value. Then this operator is in general an unbounded linear operator although its mathematical expectation is a bounded operator. In [4], [5] we proved that the stochastic process analogous to X_t actually lives in the group of invertible elements of $B(H)$ for a Hilbert space H. In the case of II_1-factor the situation is more complicated, since X_t takes values in $L^2(\mathcal{C})$, not $\mathcal{C}$ itself. We will show that the elements $X_t(\omega)$ have right inverses in $L^2(\mathcal{C})$ defined by another stochastic differential equation.

In Section 7 we consider two spaces of holomorphic functions and discuss their properties. One of these spaces, $\mathcal{H}L^2(\mu_t)$, is the closure in $L^2(L^2(\mathcal{C}), \mu_t)$ of holomorphic polynomials on $L^2(\mathcal{C})$. In addition we consider a space of holomorphic functions on a "smaller space" with a direct limit type norm. One of our main results, Theorem 4, describes how these two spaces are related and the role of the Taylor map. Also in this section we define the Cameron-Martin group G_{CM}. One of the remarkable properties of this group is that we can define holomorphic versions, skeletons, for the elements of $\mathcal{H}L^2(\mu_t)$. Note that elements of $\mathcal{H}L^2(\mu_t)$ might not be even continuous functions on $\mathcal{C}$, still we can define holomorphic skeletons on a much smaller set, the Cameron-Martin group G_{CM}. Finally we explain why the trace class covariance operator Q is essential to obtain a nontrivial Taylor map.

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2. Description of a II_1-factor

We use a representation of a II_1-factor as the weak closure of a subalgebra of the CAR-algebra. In this description we follow, in particular, L. Gross [6] and I. E. Segal [10]. Let H be a complex Hilbert space with an inner product $(\cdot, \cdot)$. Denote by $\Lambda^n(H)$ the space of skew symmetric tensors of rank n over H. Put $\Lambda^0(H) = \mathbb{C}$ and write $\Lambda(H) = \oplus_{n=0}^{\infty} \Lambda^n(H)$. The "bare vacuum" $\Omega = 1$ is the element in $\Lambda^0(H) \subset \Lambda(H)$. For any x in H there exists a bounded operator C_x such that $C_x u = (n+1)^{\frac{1}{2}} x \wedge u, u \in \Lambda^n(H)$, where $x \wedge u$ denotes $P_a(x \otimes u)$ and P_a is the antisymmetric projection. If we denote $A_x = C_x^*$, then $C_x A_y + A_y C_x = (x, y)I$, which are the canonical anti-commutation relations.

We fix a conjugation $\mathcal{J}$ on H, that is, $\mathcal{J}$ is antilinear, antiunitary and idempotent. We will call an element x in H real with respect to $\mathcal{J}$ if $x = \mathcal{J}x$. Let us define

$B_x \stackrel{\text{def}}{=} C_x + A_{\mathcal{J}x}$ for any x in H. Note that $B_xB_y + B_yB_x = 2\langle x, y\rangle I$, where $\langle x, y\rangle = (x, \mathcal{J}y)$.

Let $\mathcal{C}$ be the smallest weakly closed algebra of operators on $\Lambda(H)$ containing all the operators $B_x, x \in H$. We consider the trace on $\mathcal{C}$ given by $tr(u) = (u\Omega, \Omega)$. This trace function is a positive linear functional on $\mathcal{C}$ such that $tr(AB) = tr(BA)$ for any $A, B \in \mathcal{C}$. By Corollary 3.4 of [10] the space $\mathcal{C}$ with this trace is a hyperfinite II_1-factor. The trace determines a Hermitian inner product on $\mathcal{C}$ by $(A, B)_{L^2(\mathcal{C})} = tr(B^*A)$. In addition we will use the real inner product $\langle A, B\rangle_{L^2(\mathcal{C})} = Re(A, B)_{L^2(\mathcal{C})}$. In a standard way we can define $L^2(\mathcal{C})$ as the completion of $\mathcal{C}$ in the norm $\|A\|_2 = (Tr(A^*A))^{\frac{1}{2}}$. Similarly we can define $L^p(\mathcal{C})$ as the completion of $\mathcal{C}$ in the norm $\|A\|_p = (Tr(|A|^p))^{\frac{1}{p}}$ for $1 \leqslant p < \infty$. The elements of $L^p(\mathcal{C})$ may be identified with (unbounded) operators on H. Finally $L^\infty(\mathcal{C})$ is simply the space $\mathcal{C}$ itself with the operator norm $\|\cdot\|$. These spaces $L^p(\mathcal{C})$ have properties similar to the ones of L^p-spaces in Lebesgue integration. For example, if $A, B \in L^2(\mathcal{C})$, then $AB \in L^1(\mathcal{C})$. We will also use the inequality from Corollary 12.2 of [11] $\|AB\|_{L^2(\mathcal{C})} \leqslant \|A\|\|B\|_{L^2(\mathcal{C})}$ and $\|BA\|_{L^2(\mathcal{C})} \leqslant \|A\|\|B\|_{L^2(\mathcal{C})}$ for any element A in $L^\infty(\mathcal{C})$ and $B \in L^2(\mathcal{C})$. By Theorem 5 of [6] the map $u \longmapsto u\Omega$ extends to a unitary map from $L^2(\mathcal{C})$ onto $\Lambda(H)$. In particular if we choose a real (with respect to $\mathcal{J}$) orthonormal basis $\{x_1, x_2, ..., x_n, ...\}$ in H, then $\{B_{x_{i_1}}B_{x_{i_2}}...B_{x_{i_n}}, i_1 < i_2 < ... < i_n, 0 \leqslant n\}$ is an orthonormal basis in $L^2(\mathcal{C})$.

3. Construction of the heat kernel measure

Let Q be a complex linear nonnegative trace class operator on $L^2(\mathcal{C})$. Denote by $\mathcal{C}_Q$ the subspace $Q^{1/2}L^2(\mathcal{C})$ and by $L_2^0 = L_2(\mathcal{C}_Q, L^2(\mathcal{C}))$ the space of the Hilbert-Schmidt operators from $\mathcal{C}_Q$ to $L^2(\mathcal{C})$ with the (Hilbert-Schmidt) norm $\|\cdot\|_{L_2^0}$. For the purpose of solving a stochastic differential equation in $L^2(\mathcal{C})$ we will view this space as a real space. Take an orthonormal basis $\{\xi_n\}_{n=1}^\infty$ of $\mathcal{C}_Q$ as a real space with the inner product $\langle A, B\rangle = \langle Q^{-1/2}A, Q^{-1/2}B\rangle_{L^2(\mathcal{C})}$. The corresponding norm will be denoted by $|\cdot|$. We can complete $\{Q^{-1/2}\xi_n\}_{n=1}^\infty$ to an orthonormal (in L^2-norm) basis $\{e_n\}_{n=1}^\infty$ of $L^2(\mathcal{C})$. We will assume throughout the paper that $\mathcal{C}_Q$ is a subspace of $\mathcal{C}$ and that the operator $\sum_{n=1}^\infty \xi_n^*\xi_n$ is an element of $\mathcal{C}$. In Section 6 we will also need that $\sum_{n=1}^\infty \xi_n\xi_n^*$ is an element of $\mathcal{C}$ to prove the estimate in Equation (6.3). Their operator norms are denoted by $\|\sum_{n=1}^\infty \xi_n\xi_n^*\| = \Xi_1$ and $\|\sum_{n=1}^\infty \xi_n^*\xi_n\| = \Xi_2$. The first constant will be used in the proofs of Theorem 1 and 2. Here the sum is assumed to converge in the operator norm $\|\cdot\|$. Lemma 3.1 proves that the operators $\sum_{n=1}^\infty \xi_n^*\xi_n$ and $\sum_{n=1}^\infty \xi_n\xi_n^*$ are independent of the choice of the basis $\{\xi_n\}_{n=1}^\infty$. This Lemma is essentially the same as Lemma 5.2 in [5]. The main goal of this section is to prove Theorem 1.

Theorem 1. *Let W_t be the Wiener process in $L^2(\mathcal{C})$ with the covariance operator Q.*

(1) *The stochastic differential equation*

$$dX_t = X_t dW_t, \quad X_0 = I \tag{3.1}$$

has a unique solution in $L^2(\mathcal{C})$, up to equivalence, among the processes satisfying

$$\boldsymbol{P}\left(\int_0^T \|X_s\|^2_{L^2(\mathcal{C})} ds < \infty\right) = 1.$$

(2) *For any $p \geqslant 2$ there exists a constant $C_{p,T} > 0$ such that*

$$\sup_{t\in[0,T]} \boldsymbol{E}\|X_t\|^p_{L^2(\mathcal{C})} \leqslant C_{p,T}.$$

Lemma 3.1. *The operators $\sum_{n=1}^{\infty} \xi_n^* \xi_n$, $\sum_{n=1}^{\infty} \xi_n \xi_n^*$ and $\sum_{n=1}^{\infty} \xi_n^2$ are independent of the choice of the basis $\{\xi_n\}_{n=1}^{\infty}$. Moreover,*

$$\sum_{n=1}^{\infty} \xi_n^2 = 0. \tag{3.2}$$

Proof. Define a bilinear real form on $L^2(\mathcal{C}) \times L^2(\mathcal{C})$ by $L(f,g) = \lambda(Q^{1/2} f Q^{1/2} g)$, where λ is a real bounded linear functional on $L^2(\mathcal{C})$. Then $f \mapsto L(f,g)$ is a bounded linear functional on $L^2(\mathcal{C})$ and so $L(f,g) = \langle f, \tilde{g}\rangle_{L^2(\mathcal{C})}$ for some $\tilde{g} \in L^2(\mathcal{C})$. There exists a linear operator $B : g \mapsto \tilde{g}$ on $L^2(\mathcal{C})$ such that $L(f,g) = \langle f, Bg\rangle_{L^2(\mathcal{C})}$. We can actually compute B. Indeed, there exists an element $h \in L^2(\mathcal{C})$ such that $\lambda(x) = \langle x, h\rangle_{L^2(\mathcal{C})}$. Then

$$L(f,g) = \langle Q^{1/2} f Q^{1/2} g, h\rangle_{L^2(\mathcal{C})}$$

and therefore

$$\begin{aligned} L(f,g) &= Tr\left(h^* Q^{1/2} f Q^{1/2} g\right) \\ &= \langle Q^{1/2} f, h(Q^{1/2} g)^*\rangle_{L^2(\mathcal{C})} = \langle f, Q^{1/2}(h(Q^{1/2} g)^*)\rangle_{L^2(\mathcal{C})}. \end{aligned}$$

Thus $Bg = Q^{1/2}(h(Q^{1/2}g)^*)$ for some $h \in L^2(\mathcal{C})$. Therefore B is trace class and since the trace is independent of a basis, the trace of B over $L^2(\mathcal{C})$ is

$$TrB = \sum_{n=1}^{\infty} \langle e_n, Be_n\rangle_{L^2(\mathcal{C})} = \sum_{n: e_n \in \mathcal{C}_Q} \lambda(Q^{1/2} e_n Q^{1/2} e_n) = \lambda\left(\sum_{n=1}^{\infty} \xi_n^2\right)$$

does not depend on the choice of $\{\xi_n\}_{n=1}^{\infty}$ for any λ. Use the bilinear forms $M_1(f,g) = \lambda((Q^{1/2}f)^* Q^{1/2} g)$ and $M_2(f,g) = \lambda(Q^{1/2} f (Q^{1/2} g)^*)$ to verify that $\sum_{n=1}^{\infty} \xi_n^* \xi_n$ and $\sum_{n=1}^{\infty} \xi_n \xi_n^*$ are independent of the choice of the basis.

Finally, choose $\{\xi_n\}_{n=1}^{\infty}$ so that $\xi_{2n} = i\xi_{2n-1}$ for $n = 1, 2, ...$, where $i = \sqrt{-1}$. Then $(\xi_{2n-1})^2 + (\xi_{2n})^2 = 0$.

□

Lemma 3.1 explains why stochastic differential equation (3.1) determining the heat kernel measure has no drift term. The stochastic differential equation would have a drift term if the real inner product $\langle \cdot, \cdot \rangle_{L^2(\mathcal{C})}$ were not the real part of a Hermitian inner product.

Proof of Theorem 1. To prove this theorem we will use Theorem 7.4 from the book by G.DaPrato and J.Zabczyk [1], p.186. We will check that the hypotheses of that theorem are satisfied. Let $B : L^2(\mathcal{C}) \longrightarrow L_2^0$ be defined by $B(X)U = XU$ for $U \in \mathcal{C}_Q$. The hypotheses are

(1) $B(X)$ is a measurable mapping from $L^2(\mathcal{C})$ to L_2^0;
(2) $\|B(X_1) - B(X_2)\|_{L_2^0} \leqslant C\|X_1 - X_2\|_{L^2(\mathcal{C})}$ for any $X_1, X_2 \in L^2(\mathcal{C})$;
(3) $\|B(X)\|_{L_2^0}^2 \leqslant K(1 + \|X\|_{L^2(\mathcal{C})}^2)$ for any $X \in L^2(\mathcal{C})$.

We want to check that $B(X)$ is in L_2^0 for any X from $L^2(\mathcal{C})$. First of all, $B(X)U \in L^2(\mathcal{C})$ for any $U \in \mathcal{C}_Q$. Indeed, $B(X)U = XU \in L^2(\mathcal{C})$, since $U \in \mathcal{C}_Q \subseteq \mathcal{C}$ and $X \in L^2(\mathcal{C})$.

Now let us verify that $B(X)$ is in L_2^0. Compute the Hilbert-Schmidt norm of B as an operator from $\mathcal{C}_Q$ to $L^2(\mathcal{C})$.

$$\|B(X)\|_{L_2^0}^2 = \sum_{n=1}^{\infty} \langle B(X)\xi_n, B(X)\xi_n \rangle_{L^2(\mathcal{C})} = \sum_{n=1}^{\infty} \langle X\xi_n, X\xi_n \rangle_{L^2(\mathcal{C})}$$
$$= \sum_{n=1}^{\infty} tr(X^*X\xi_n\xi_n^*) = tr(X^*X \sum_{n=1}^{\infty} \xi_n\xi_n^*) \leqslant \Xi_1 \|X\|_{L^2(\mathcal{C})}^2 < \infty,$$

since $\sum_{n=1}^{\infty} tr(AB_n) = tr(A \sum_{n=1}^{\infty} B_n)$ for $A \in L^1(\mathcal{C})$, $B_n \in \mathcal{C}$ such that $\sum_{n=1}^{\infty} B_n$ converges to a bounded operator in the operator norm. Now one can use this estimate to verify conditions 2 and 3. □

4. Definition of the heat kernel measure

We define a Borel measure μ_t by

$$\int_{L^2(\mathcal{C})} f(X)\mu_t(dX) = Ef(X_t(I)) = P_{t,0}f(I)$$

for any bounded Borel function f on $L^2(\mathcal{C})$.

Definition 4.1. μ_t is called *the heat kernel measure on* $L^2(\mathcal{C})$. The space of all square integrable functions on $L^2(\mathcal{C})$ is denoted by $L^2(L^2(\mathcal{C}), \mu_t)$ and the corresponding norm by $\|f\|_{L^2(L^2(\mathcal{C}),\mu_t)} = \|f\|_t$.

The name for this measure can be explained by interpreting Kolmogorov's backward equation for the process X_t as the heat equation in a sense. First of all, the coefficient B depends only on $X \in L^2(\mathcal{C})$; therefore the transition probability is stationary $P_{s,t}f(X) = Ef(X(t,s;X)) = P_{t-s}f(X)$. According to Theorem 9.16 in [1] for any $\varphi \in C_b^2(L^2(\mathcal{C}))$ and $X \in L^2(\mathcal{C})$ the function $v(t,X) = P_t\varphi(X)$ is a unique strict solution in $C_b^{1,2}(L^2(\mathcal{C}))$ for the parabolic type equation (Kolmogorov's backward equation)

$$\begin{aligned} &\frac{\partial}{\partial t}v(t,X) = \frac{1}{2}Tr[v_{XX}(t,X)(B(X)Q^{1/2})(B(X)Q^{1/2})^*], \quad t > 0, X \in L^2(\mathcal{C}) \\ &v(0,X) = \varphi(X). \end{aligned} \tag{4.1}$$

Here $C_b^n(L^2(\mathcal{C}))$ denotes the space of all functions from $L^2(\mathcal{C})$ to $\mathbb{R}$ that are n-times continuously Frechet differentiable with all derivatives up to order n bounded and

$C_b^{k,n}(L^2(\mathcal{C}))$ denotes the space of all functions from $[0,T]\times L^2(\mathcal{C})$ to $\mathbb{R}$ that are k-times continuously Frechet differentiable with respect to t and n-times continuously Frechet differentiable with respect to X with all partial derivatives continuous in $[0,T]\times L^2(\mathcal{C})$ and bounded.

One can rewrite Equation ((4.1)) as the heat equation. First

$$\begin{aligned}
&Tr[v_{XX}(t,X)(B(X)Q^{1/2})(B(X)Q^{1/2})^*] \\
&\quad = \sum_{n=1}^{\infty} v_{XX}(t,X)(B(X)Q^{1/2}e_n \otimes B(X)Q^{1/2}e_n) \\
&\quad = \sum_{n=1}^{\infty} v_{XX}(t,X)(XQ^{1/2}e_n \otimes XQ^{1/2}e_n) \\
&\quad = \sum_{n=1}^{\infty} v_{XX}(t,X)(X\xi_n \otimes X\xi_n),
\end{aligned}$$

where $v_{XX}(t,X)$ is viewed as a functional on $L^2(\mathcal{C})\otimes L^2(\mathcal{C})$. This means that for any smooth bounded function $\varphi(X): L^2(\mathcal{C})\to\mathbb{R}$, the function $v(t,X)=P_t\varphi(X)$ satisfies this equation, which can be considered as the heat equation

$$\begin{aligned}
&\frac{\partial}{\partial t}v(t,X) = Lv(t,X) \\
&v(0,X)=\varphi(X), t>0, X\in L^2(\mathcal{C}),
\end{aligned} \tag{4.2}$$

where the differential operator L on the space $C_b^{1,2}(L^2(\mathcal{C}))$ is defined by

$$Lv \stackrel{\text{def}}{=} \frac{1}{2}\sum_{n=1}^{\infty} v_{XX}(t,X)((X)\xi_n \otimes (X)\xi_n).$$

Our goal is to show that L is a Laplacian in a sense. More precisely, L is a half of the sum of second derivatives in the directions of an orthonormal basis of $\mathcal{C}_Q$. Define the Laplacian by

$$(\Delta v)(X) = \frac{1}{2}\sum_{n=1}^{\infty}(\tilde{\xi_n}\tilde{\xi_n}v)(X), \tag{4.3}$$

where the left-invariant vector field $\tilde{\xi_n}$ corresponding to ξ_n is defined by $(\tilde{\xi_n}v)(X) = \frac{d}{dt}\mid_{t=0} v(Xe^{t\xi_n})$ for a function $v: L^2(\mathcal{C})\to\mathbb{R}$.

Let us calculate derivatives of $v: L^2(\mathcal{C})\to\mathbb{R}$ in the direction of ξ_n

$$(\tilde{\xi_n}v)(X) = v_X(X)\frac{d}{dt}\mid_{t=0}(Xe^{t\xi_n}) = v_X(X)(X\xi_n)$$

and therefore

$$(\tilde{\xi_n}\tilde{\xi_n}v)(X) = v_{XX}(X)(X\xi_n\otimes X\xi_n) + v_X(X\xi_n^2).$$

Thus by Lemma 3.1 the Laplacian is equal to

$$(\Delta v)(X) = \frac{1}{2}\sum_{n=1}^{\infty}[v_{XX}(X)(X\xi_n\otimes X\xi_n) + v_X(X\xi_n^2)] = \frac{1}{2}\sum_{n=1}^{\infty} v_{XX}(X)(X\xi_n\otimes X\xi_n).$$

Since $\tilde{\xi_n}$ is a left-invariant vector field, the Laplacian Δ is a left-invariant differential operator such that $Lv=\Delta v$ for any $v\in C_b^2(L^2(\mathcal{C}))$.

5. Approximation of the process

Let a real (with respect to $\mathcal{J}$) orthonormal basis $\{x_m\}_{m=1}^{\infty}$ of the Hilbert space H be fixed. Denote by H_n the complex subspace of H with the basis $\{x_m\}_{m=1}^{n}$, and by $\mathcal{C}_n$ the corresponding II_1-factor. The subspace H_n is invariant under the conjugation $\mathcal{J}$. Note that since H_n is a finite-dimensional space, we have $\mathcal{C}_n = L^2(\mathcal{C}_n)$ as linear spaces though with different norms. We will denote by G_n the group of invertible elements of $\mathcal{C}_n$. Then G_n is a complex connected Lie group with the Lie algebra $\mathcal{C}_n$.

In what follows we assume that $\mathcal{C}_n$ are invariant subspaces of Q. This condition ensures that the heat kernel measures on G_n approximate the heat kernel measure μ_t (defined in Section 4) on $L^2(\mathcal{C})$. The approximation is essential in proving of Theorem 4. Let P_n be the projection onto $L^2(\mathcal{C}_n)$. By the Q-invariance of $\mathcal{C}_n$ the projection P_n from $\mathcal{C}$ onto $\mathcal{C}_n$ (defined in terms of the norm $|\cdot|$) is the restriction of the projection from $L^2(\mathcal{C})$ onto $L^2(\mathcal{C}_n)$ (defined in terms of the norm $|\cdot|_{L^2(\mathcal{C})}$).

Choose a real orthonormal basis $\{\xi_m\}_{m=1}^{\infty}$ of $\mathcal{C}$ in the same way as it was done in Section 3. In addition, we will assume that the first $d_n = \dim L^2(\mathcal{C}_n) = 2^n$ basis elements form an orthonormal basis of $L^2(\mathcal{C}_n)$. Consider the equation

$$dX_{n,t} = B_n(X_{n,t})dW_t, \qquad X_{n,t}(0) = I,$$

where

$$B_n(X)U = X(P_nU).$$

This equation has a unique solution by the same arguments as in Section 3. Denote $Q_n = P_nQP_n$, where P_n is the projection onto $L^2(\mathcal{C}_n)$.

Lemma 5.1. *$X_{n,t}$ is a solution of the equation*

$$\begin{aligned} dX_{n,t} &= B_n(X_{n,t})dW_{n,t}, \\ X_{n,t}(0) &= I, \end{aligned} \tag{5.1}$$

where $W_{n,t} = P_nW_t$.

Proof. First we check that P_nQP_n is the covariance operator of $W_{n,t}$. By the definition of a covariance operator we have that $\langle f, Qg\rangle_{L^2(\mathcal{C})} = E\langle f, W_1\rangle_{L^2(\mathcal{C})}\langle g, W_1\rangle_{L^2(\mathcal{C})}$, and therefore

$$\begin{aligned} \langle f, Q_ng\rangle_{L^2(\mathcal{C})} = \langle P_nf, QP_ng\rangle_{L^2(\mathcal{C})} &= E\langle P_nf, W_1\rangle_{L^2(\mathcal{C})}\langle P_ng, W_1\rangle_{L^2(\mathcal{C})} \\ &= E\langle f, P_nW_1\rangle_{L^2(\mathcal{C})}\langle g, P_nW_1\rangle_{L^2(\mathcal{C})}. \end{aligned}$$

Thus Equation (5.1) is actually the same as Equation (3.1) with the Wiener process that has the covariance Q_n instead of Q. □

Theorem 2. *Denote by $\mathcal{H}_2$ the space of equivalence classes of $L^2(\mathcal{C})$-valued predictable processes with the norm*

$$|||X|||_2 = (\sup_{t\in[0,T]} E\|X_t\|^2_{L^2(\mathcal{C})})^{1/2}.$$

Then

$$|||X_{n,t} - X_t|||_2 \xrightarrow[n\to\infty]{} 0.$$

Proof of Theorem 2. We want to apply the local inversion theorem (see, for example, Lemma 9.2, from the book by DaPrato and Zabczyk [1], p. 244) to $K(x, X) = x + \int_0^t B(X)dW_t$, where x is the initial value of X and $X = X(x, t)$ is an $L^2(\mathcal{C})$-valued predictable process. Analogously we define $K_n(x, X) = x + \int_0^t B_n(X)dW_t$. To apply this lemma we need to check that K and K_n satisfy the following conditions

(1) For any $X_1(t)$ and $X_2(t)$ from $\mathcal{H}_2$
$$\sup_{t\in[0,T]} E\|K(x, X_1) - K(x, X_2)\|^2_{L^2(\mathcal{C})} \leqslant \alpha \sup_{t\in[0,T]} E\|X_1(t) - X_2(t)\|^2_{L^2(\mathcal{C})},$$
where $0 \leqslant \alpha < 1$

(2) For any $X_1(t)$ and $X_2(t)$ from $\mathcal{H}_2$
$$\sup_{t\in[0,T]} E\|K_n(x, X_1) - K_n(x, X_2)\|^2_{L^2(\mathcal{C})} \leqslant \alpha \sup_{t\in[0,T]} E\|X_1(t) - X_2(t)\|^2_{L^2(\mathcal{C})},$$
where $0 \leqslant \alpha < 1$

(3) $\lim_{n\to\infty} K_n(x, X) = K(x, X)$ in $\mathcal{H}_2$.

Proof of 1. To estimate the part with the stochastic differential we will use Lemma 7.2 from [1], p.182: for any $r \geqslant 1$ and for arbitrary L_2^0-valued predictable process $\Phi(t)$,

$$E(\sup_{s\in[0,t]} \| \int_0^s \Phi(u)dW(u)\|^{2r}_{L^2(\mathcal{C})}) \leqslant C_r E(\int_0^t \|\Phi(s)\|^2_{L_2^0} ds)^r, t \in [0, T], \tag{5.2}$$

where $C_r = (r(2r - 1))^r (\frac{2r}{2r-1})^{2r^2}$. Thus

$$\begin{aligned} E\|K(x, X_1) - K(x, X_2)\|^2_{L^2(\mathcal{C})} &= E\| \int_0^t B(X_1) - B(X_2)dW_s\|^2_{L^2(\mathcal{C})} \\ &\leqslant 4E \int_0^t \|B(X_1) - B(X_2)\|^2_{L_2^0} ds \leqslant 4\Xi_1 E \int_0^t \|X_1 - X_2\|^2_{L^2(\mathcal{C})} ds \\ &\leqslant 4t\Xi_1 \sup_{t\in[0,T]} E\|X_1 - X_2\|^2_{L^2(\mathcal{C})}. \end{aligned}$$

Note that for small t we can make $4t\Xi_1$ as small as we wish, therefore 1 holds.

Proof of 2. Similarly to the proof of Theorem 1 we have that

$$\begin{aligned} \|B_n(X_1) - B_n(X_2)\|^2_{L_0^2} &= \|(X_1 - X_2)P_n(\cdot)\|^2_{\mathcal{C}_Q\to L^2(\mathcal{C})} \\ &\leqslant \Xi_1\|X_1 - X_2\|^2_{L^2(\mathcal{C})}. \end{aligned}$$

Now use the same estimates as in 1 to see that 2 holds.

Proof of 3. Here again we will use ((5.2)) to estimate the part with the stochastic differential

$$|\!|\!|K_n(x, X) - K(x, X)|\!|\!|^2_2 = |\!|\!| \int_0^t (B_n(X) - B(X))dW_t|\!|\!|^2_2$$

$$= \sup_{t\in[0,T]} E\| \int_0^t (B_n(X) - B(X))dW_s\|^2_{L^2(\mathcal{C})} \leqslant 4E \int_0^t \|B_n(X) - B(X)\|^2_{L_2^0} ds.$$

Now let us estimate $\|B(X) - B_n(X)\|^2_{L_2^0}$

$$\|B(X)-B_n(X)\|^2_{L^0_2} = \sum_{m=1}^{\infty} \|(B(X)-B_n(X))\xi_m\|^2_{L^2(\mathcal{C})}$$
$$= \sum_{m=1}^{\infty} \|X(I-P_n)\xi_m\|^2_{L^2(\mathcal{C})} = \sum_{m=d_n}^{\infty} \|X\xi_m\|^2_{L^2(\mathcal{C})}$$
$$\leqslant \| \sum_{m=d_n}^{\infty} \xi_m\xi_m^* \| \|X\|^2_{L^2(\mathcal{C})} \xrightarrow[n\to\infty]{} 0$$

by the assumptions on $\sum\limits_{m=1}^{\infty} \xi_m\xi_m^*$.

We know that there are unique elements X, X_n in the space $\mathcal{H}_2$ such that $X = K(x, X)$, $X_n = K_n(x, X_n)$ and therefore $\lim\limits_{n\to\infty} X_n = X$ for any x by the local inversion lemma. □

Denote $P^n_t f(X) = Ef(X_n(t, X))$, $X \in \mathcal{C}_n$ and $v^n(t, X) = P^n_t f(X)$. Then similarly to (4.1) the following holds for $v^n(t, X)$. For any $f \in C^2_b(\mathcal{C}_n)$ the function $v^n(t, X)$ is a unique strict solution from $C^{1,2}_b(\mathcal{C}_n)$ for the parabolic type equation

$$\frac{\partial}{\partial t} v^n(t, X) = \frac{1}{2} Tr[v^n_{XX}(B_n(X)Q^{1/2})(B_n(X)Q^{1/2})^*]$$
$$v^n(0, X) = f(X), t > 0, X \in \mathcal{C}_n.$$

We want to show that P^n_t corresponds to the heat kernel measure defined on G_n as on a Lie group (i. e. as in the finite-dimensional case).

Note that for any $X \in \mathcal{C}_n$

$$Tr[v^n_{XX}(t, X)(B_n(X)Q^{1/2})(B_n(X)Q^{1/2})^*] = \sum_{m=1}^{2d_n} v^n_{XX}(t, X)(X\xi_m \otimes X\xi_m).$$

Thus $Lv = \frac{1}{2} Tr[v^n_{XX}(t, X)(B_n(X)Q^{1/2})(B_n(X)Q^{1/2})^*]$ is equal to the Laplacian

$$\Delta_n v = \frac{1}{2}\sum_{k=1}^{2d_n} (\tilde{\xi_k}\tilde{\xi_k} v)(X),$$

where the left-invariant vector field $\tilde{\xi_n}$ corresponding to ξ_n is defined by $(\tilde{\xi_n} v)(X) = \frac{d}{dt}\mid_{t=0} v(Xe^{t\xi_n})$ for a function $v : L^2(\mathcal{C}_n) \to \mathbb{R}$. Thus, the transition probability P^n_t is equal to $\mu^n_t(dX)$, where the latter is the heat kernel measure on G_n defined as a Haar measure on G_n with the heat kernel as the density.

6. More properties of the stochastic process determining the heat kernel measure

In addition to Equation (3.1) we consider a stochastic differential equation on $\Lambda(H)$ whose solution can be viewed as a weak solution to Equation 3.1 in a sense. Let us explain the motivation informally. The main difficulty with the solution to Equation (3.1) is that the process X_t is in $L^2(\mathcal{C})$, whose elements in general are unbounded operators. Thus for fixed t, ω the operator $X_t(\omega)$ might not be defined for a fixed element of $\Lambda(H)$. Still there is a subspace $\mathcal{D}$ of $\Lambda(H)$ which is contained in the domains of all elements of $L^2(\mathcal{C})$. In particular, we can take the subspace $\mathcal{D}$

to be equal to $\bigcup_n \Lambda(H_n)$. Let us disregard this problem of the domain of the process X_t for a moment. Then we can take the adjoint of both sides of Equation (3.1) and apply it to an element in $\Lambda(H)$.

$$dX_t^* x = dW_t^* X_t^* x, \quad x \in \Lambda(H). \tag{6.1}$$

Denote $X_t^* x$ by x_t. Then Equation (6.1) becomes

$$dx_t = dW_t^* x_t, x_0 = x. \tag{6.2}$$

This equation has a unique solution in $\Lambda(H)$. Indeed, denote $D : \Lambda(H) \to L_0^2(\mathcal{C}_Q, \Lambda(H))$, $D(x)U = Ux, U \in \mathcal{C}_Q$. Then D satisfies conditions 1, 2 and 3 of Theorem 1. We have the following estimate of the L_0^2-norm

$$\|D(x)\|_{L_0^2}^2 = \sum_{n=1}^{\infty} \langle D(x)\xi_n^*, D(x)\xi_n^* \rangle_{\Lambda(H)} = \sum_{n=1}^{\infty} \langle \xi_n^* x, \xi_n^* x \rangle_{\Lambda(H)}$$
$$= \langle \sum_{n=1}^{\infty} \xi_n \xi_n^* x, x \rangle_{\Lambda(H)} \leqslant \Xi_1 \|x\|_{\Lambda(H)}.$$

Here we used the assumption that $\sum_{n=1}^{\infty} \xi_n \xi_n^*$ is a bounded operator on $\Lambda(H)$. As in proof of Theorem 1 this estimate implies conditions 1, 2 and 3.

Let us define a random operator $\tilde{X}_t$ as an operator on $\Lambda(H)$ by $\tilde{X}_t x = x_t$. From [12] (p. 179) we know that $E\tilde{X}_t^* \tilde{X}_t$ is a bounded operator on $\Lambda(H)$ as an operator on the initial value $x_0 = x \in \Lambda(H)$. Skorokhod's result is much more general than what we actually need. We can show that $E\tilde{X}_t^* \tilde{X}_t$ is a bounded operator directly. First, we can apply Itô's formula to $E\langle x_t, x_t \rangle_{\Lambda(H)}$ to get

$$E\langle x_t, x_t \rangle_{\Lambda(H)} = \langle x_0, x_0 \rangle_{\Lambda(H)} + \int_0^t E\langle \sum_n \xi_n^* \xi_n x_s, x_s \rangle_{\Lambda(H)} ds.$$

If we assume that $\sum_n \xi_n^* \xi_n$ is a bounded operator on $\Lambda(H)$, then

$$E\langle x_t, x_t \rangle_{\Lambda(H)} \leqslant \langle x_0, x_0 \rangle_{\Lambda(H)} + \Xi_2 \int_0^t E\langle x_s, x_s \rangle_{\Lambda(H)} ds.$$

By Gronwall's lemma $E\langle x_t, x_t \rangle_{\Lambda(H)} \leqslant e^{t\Xi_2} \langle x_0, x_0 \rangle_{\Lambda(H)} = c\langle x, x \rangle_{\Lambda(H)}$ and therefore $E\langle \tilde{X}_t x, \tilde{X}_t x \rangle_{\Lambda(H)} \leqslant c\langle x, x \rangle_{\Lambda(H)}$. Thus

$$\langle E\tilde{X}_t^* \tilde{X}_t x, x \rangle_{\Lambda(H)} \leqslant c\langle x, x \rangle_{\Lambda(H)} \tag{6.3}$$

and so $E\tilde{X}_t^* \tilde{X}_t$ is a bounded operator on $\Lambda(H)$.

The question is whether $\tilde{X}_t x = x_t = X_t^* x$ for all $x \in \Lambda(H)$, and if not then for which x it is so. Note that if we follow the same argument for finite-dimensional approximations $X_{n,t}$ to the process X_t, then $\tilde{X}_{n,t} x = x_{n,t} = X_{n,t}^* x$ with probability one. Indeed, in this case $X_{n,t}$ is a bounded operator for fixed x, ω and $X_{n,t} = \tilde{X}_{n,t}^*$. Let us see now what we can say about X_t^*.

Lemma 6.1. *For any $x \in \mathcal{D}$*

$$\tilde{X}_t x = X_t^* x \quad \textit{with probability one.}$$

Proof. Any x in $\mathcal{D}$ is in the domain of any operator Y from $L^2(\mathcal{C})$. By Corollary 12.12 and Corollary 16.1 of [11] we have that

$$\|Yx\|_{\Lambda(H)} \leqslant \|Y\|_{L^2(\mathcal{C})}\|x\|_\infty \tag{6.4}$$

where $\|x\|_\infty = \|\sum \alpha_{j_1...j_m} B_{j_1}...B_{j_m}\|$ if $x = \sum \alpha_{j_1...j_m} x_{j_1} \wedge ... \wedge x_{j_m}$. By Theorem 2 we have $E\|X_t - X_{n,t}\|^2_{L^2(\mathcal{C})} \to 0$. Then by inequality (6.4) and usual properties of mathematical expectation there is a subsequence such that $\|X_t x - X_{n_k,t}x\|_{\Lambda(H)} \to 0$ a.s. Similarly $E\|\tilde{X}_t x - \tilde{X}_{n,t}x\|^2_{\Lambda(H)} \to 0$. In addition $X^*_{n,t}x = \tilde{X}_{n,t}x$ for large enough n. □

Proposition 6.2. *For any x in $\Lambda(H)$ we have that x belongs to the domain of X_t^* and $X_t^* x = \tilde{X}_t x$ with probability one.*

Proof. There is a sequence $\{x_n\}$ in $\mathcal{D}$ such that $x_n \to x$ in $\Lambda(H)$. By Equation (6.3) there is a constant c such that

$$E\|\tilde{X}_t y\|^2_{\Lambda(H)} \leqslant c\|y\|^2_{\Lambda(H)}$$

for any $y \in \mathcal{D}$. Hence by Lemma 6.1 there is a subsequence x_{n_k} such that $X_t^* x_{n_k} \to \tilde{X}_t x$ a.s. This implies the result since any element of $L^2(\mathcal{C})$ is a closed operator (see Section 2 of [11]). □

The fact that the process X_t is in general an unbounded operator makes the task of finding its inverse complicated. In terms of the heat kernel measure μ_t, it means that we can not view μ_t as a heat kernel measure on a group. However the support of this measure enjoys some of the properties of a group. For example, Theorem 3 shows that for μ_t-almost all elements of $L^2(\mathcal{C})$ there exists a right inverse in $L^2(\mathcal{C})$. This right inverse is defined as a solution of the stochastic differential equation (6.5). At the same time we can not claim that there is a left inverse in $L^2(\mathcal{C})$. Note that if X, X^{-1} are elements of $L^2(\mathcal{C})$, then XX^{-1} is a well defined element of $L^1(\mathcal{C})$, and the equality $XX^{-1} = I$ can be understood in $L^1(\mathcal{C})$-terms.

Theorem 3. *Consider the stochastic differential equation in $L^2(\mathcal{C})$*

$$dZ_t = -dW_t Z_t, \quad Z_0 = I. \tag{6.5}$$

It has a unique solution in $L^2(\mathcal{C})$ and its solution satisfies $X_t Z_t = I$ with probability one.

Proof. This can be shown by applying Itô's formula to $\lambda(X_t Z_t)$ where λ is an arbitrary bounded linear functional on $L^1(\mathcal{C})$.

$$\lambda(X_tZ_t) = \lambda(X_0Z_0) + \int_0^t \lambda(X_s dW_s Z_s - X_s dW_s Z_s) + \frac{1}{2}\int_0^t \lambda(-\sum_n X_s\xi_n^2 Z_s - \sum_n X_s \xi_n^2 Z_s)ds = \lambda(I) - \int_0^t \lambda(X_s \sum_n \xi_n^2 Z_s)ds = \lambda(I)$$

by Lemma 3.1. □

Taking an inverse is not the only problem with group properties of the support of the heat kernel measure μ_t. For any T in $L^2(\mathcal{C})$ we can informally define a product TX in $L^2(\mathcal{C})$ for μ_t-almost all X in $L^2(\mathcal{C})$. This product can be defined as a solution of the stochastic differential equation $dY_t = Y_t dW_t$ with the initial condition $Y_0 = T$. However we can not define a product XT in a similar way and the product TX lacks associativity.

Proposition 6.3 gives an estimate of the L^p norm of the solution of the stochastic differential equation in Theorem 1.

Proposition 6.3. *For any $p \geqslant 2, t > 0$*

$$E\|X_t\|^p_{L^2(\mathcal{C})} < \frac{1}{C_{p,t}}(e^t C_{p,t} - 1),$$

where $C_{p,t} = C_{\frac{p}{2}} 2^{\frac{p}{2}-1}\Xi_1^{\frac{p}{2}} t^{\frac{p}{2}-1}, C_p = (p(2p-1))^p(\frac{2p}{2p-1})^{2p^2}$.

Proof. First of all, let us estimate $E\|\int_0^t B(X_s)dW_s\|^p_{L^2(\mathcal{C})}$. From part 3 of the proof of Theorem 1 we know that $\|B(X)\|^2_{L^0_2} \leqslant \Xi_1(\|X\|^2_{L^2(\mathcal{C})} + 1)$. In addition we will use Lemma 7.2 from the book by DaPrato and Zabczyk [1], p.182: for any $r \geqslant 1$ and for an arbitrary L^0_2-valued predictable process $\Phi(t)$,

$$E(\sup_{s\in[0,t]} \|\int_0^s \Phi(u)dW(u)\|^{2r}_{L^2(\mathcal{C})}) \leqslant C_r E(\int_0^t \|\Phi(s)\|^2_{L^0_2} ds)^r, t \in [0,T], \tag{6.6}$$

where $C_r = (r(2r-1))^r(\frac{2r}{2r-1})^{2r^2}$. Thus

$$\begin{aligned} E\|\int_0^t B(X_s)dW_s\|^p_{L^2(\mathcal{C})} &\leqslant C_{\frac{p}{2}} E(\int_0^t \|B(X_s)\|^2_{L^0_2} ds)^{\frac{p}{2}} \\ &\leqslant C_{\frac{p}{2}}\Xi_1^{\frac{p}{2}} E(\int_0^t ((\|X\|^2_{L^2(\mathcal{C})} + 1)ds)^{\frac{p}{2}} \\ &\leqslant C_{\frac{p}{2}}\Xi_1^{\frac{p}{2}} t^{\frac{p}{2}-1} E\int_0^t (\|X\|^2_{L^2(\mathcal{C})} + 1)^{\frac{p}{2}} ds \end{aligned} \tag{6.7}$$

Now we can use inequality $(x+1)^q \leqslant 2^{q-1}(x^q+1)$ for any $x \geqslant 0$ for the estimate ((6.7))

$$E\|\int_0^t B(X_s)dW_s\|^p_{L^2(\mathcal{C})} \leqslant C_{\frac{p}{2}}\Xi_1^{\frac{p}{2}} t^{\frac{p}{2}-1} 2^{\frac{p}{2}-1} E\int_0^t (1 + \|X_s\|^p_{L^2(\mathcal{C})})ds = C_{\frac{p}{2}} 2^{\frac{p}{2}-1}\Xi_1^{\frac{p}{2}} t^{\frac{p}{2}-1}(t + E\int_0^t \|X\|^p_{L^2(\mathcal{C})} ds).$$

Finally,

$$E\|X_t\|^p_{L^2(\mathcal{C})} \leqslant E\|\int_0^t B(X_s)dW_s\|^p_{L^2(\mathcal{C})} \leqslant C_{p,t}(t + E\int_0^t \|X_s\|^p_{L^2(\mathcal{C})} ds),$$

where $C_{p,t} = C_{\frac{p}{2}} 2^{\frac{p}{2}-1} \Xi_1^{\frac{p}{2}} t^{\frac{p}{2}-1}$.

Thus, $E\|X_t\|^p_{L^2(\mathcal{C})} < \frac{1}{C_{p,t}}(e^t C_{p,t} - 1)$ by Gronwall's lemma. □

7. The Cameron-Martin group, holomorphic functions and Taylor map

Let $\mathfrak{g} = \bigcup\limits_{n=1}^{\infty} \mathcal{C}_n$. The operator Q is a bounded operator and therefore there is a constant c such that $|x| \geqslant c\|x\|$ for any $x \in \mathcal{C}_Q$, where $\|\cdot\|$ is the operator norm on $B(\Lambda(H))$. Note that the closure of $\mathfrak{g}$ in the norm $|\cdot|$ is $\mathcal{C}_Q$. Let us denote by G the connected component of I of invertible elements of $\mathcal{C}$ (e.g. [8]).

Let d denote the Riemannian metric $d(y,z) = \inf\limits_h \{\int\limits_0^1 \left|h^{-1}\dot{h}\right| ds\}$, where $h : [0,1] \to G$ is a piecewise differentiable continuous path, $h(0) = y, h(1) = z$, $\dot{h} = \frac{dh}{ds}$, $h' = h^{-1}\dot{h} \in \mathcal{C}_Q$. Let us define now the Cameron-Martin group G_{CM}.

Definition 7.1. $G_{CM} = \{x \in \mathcal{C} : d(x, I) < \infty\}$ is called *the Cameron-Martin group.*

Note that G_{CM} is a group and $G_n \subset G_{CM}$ for any n. We will denote $\cup_n G_n$ by G_∞. Let us now introduce holomorphic polynomials on $\mathcal{C}$.

Definition 7.2. A function $p : \mathcal{C} \longrightarrow \mathbb{C}$ is called *a holomorphic polynomial*, if p is a complex linear combination of monomials of the form $p(X) = \prod\limits_{m=1}^{k} \langle X, F_m\rangle_{L^2(\mathcal{C})}$, where $\{F_m\}_{m=1}^k$ are arbitrary elements of $L^2(\mathcal{C})$. We will denote the space of all such polynomials by $\mathcal{HP}$.

Note that $\langle X, F\rangle_{L^2(\mathcal{C})} = \langle X\Omega, F\Omega\rangle_{\Lambda(H)}$ and therefore we could consider polynomials of $\langle Xf, g\rangle_{\Lambda(H)}$ instead. For any $p \in \mathcal{HP}$, $X \in \mathcal{C}$, $\beta \in \mathcal{C}_Q$ the derivative of p_k in the direction of ξ is $(Dp_k)(X)(\xi) = \sum\limits_{i=1}^{k} \langle X, F_1\rangle_{L^2(\mathcal{C})} ... \langle X\xi, F_i\rangle_{L^2(\mathcal{C})} ... \langle X, F_m\rangle_{L^2(\mathcal{C})}$. Thus the derivative $(Dp_k)(X)(\xi)$ is complex linear in ξ. By Proposition 6.3 the holomorphic polynomials are square integrable with respect to the heat kernel measure. We will denote the closure of all holomorphic polynomials in $L^2(L^2(\mathcal{C}), \mu_t)$ by $\mathcal{H}L^2(\mu_t)$.

One of the important properties of the holomorphic polynomials is that they can be recovered on the Cameron-Martin group from their Taylor coefficients at the identity. This formula was first introduced for finite-dimensional Lie groups by B. Driver in [2].

Lemma 7.3. *(B. Driver's formula).*

Let g be a smooth path in G_{CM} such that $g(0) = I$ and g is a piecewise differentiable path from $[0,1]$ to G such that $g' = g^{-1}\dot{g}$ is in $\mathcal{C}_Q$.

Then for any $p \in \mathcal{HP}$

$$p(g(s)) = \sum_{n=0}^{\infty} \int\limits_{\Delta_n(s)} (D^n p)(I)(c(s_1) \otimes ... \otimes c(s_n)) d\vec{s},$$

where

$$\Delta_n(s) = \{(s_1, ..., s_n) \in \mathbb{R}^n : 0 \leqslant s_1 \leqslant s_2 ... \leqslant s_n \leqslant s\}, c(s) = g^{-1}(s)\frac{dg}{ds}.$$

Proof. Note that our result is stated for paths g which are smooth in G_{CM}, but we actually use weaker assumptions in our proof. In fact the formula holds for a smooth path in G.

It is enough to prove this formula for monomials. Take a monomial $p(X) = \prod_{m=1}^{k} \langle X, F_m\rangle_{L^2(\mathcal{C})}$. Similarly to Lemma 5.2 of [2]

$$p(g(s)) = \sum_{n=0}^{N-1} \int_{\Delta_n(s)} (D^n p)(I)(c(s_1) \otimes ... \otimes c(s_n)) d\vec{s} + \int_{\Delta_N(s)} (D^N p)(g(s_1))(c(s_1) \otimes ... \otimes c(s_N)) d\vec{s}.$$

We want to estimate the remainder

$$\Big| \int_{\Delta_N(s)} (D^N p)(g(s_1))(c(s_1) \otimes ... \otimes c(s_N)) d\vec{s} \Big| \leqslant \int_{\Delta_N(s)} \left|(D^N p)(g(s_1))(c(s_1) \otimes ... \otimes c(s_N))\right| d\vec{s}.$$

Note that the Nth derivative of the monomial is

$$(D^N p)(X)(\xi_1 \otimes ... \otimes \xi_N) = \sum \langle X\vec{\xi}^{\alpha_1}, F_1\rangle_{L^2(\mathcal{C})} ... \langle X\vec{\xi}^{\alpha_i}, F_i\rangle_{L^2(\mathcal{C})} ... \langle X\vec{\xi}^{\alpha_k}, F_k\rangle_{L^2(\mathcal{C})},$$

where the multindex $\alpha_i = (a_1^i, ..., a_N^i)$, $a_j^i = 0$ or 1, $\vec{\xi}^{\alpha_i} = \xi_1^{a_1^i} ... \xi_N^{a_N^i}$, $\sum_{i=1}^{k} \alpha_i = (1, ..., 1)$ and the sum is taken over all possible $\alpha = (\alpha_1, ..., \alpha_k)$.

For $\xi \in \mathcal{C}, X \in \mathcal{C}$

$$|\langle X\xi, F\rangle_{L^2(\mathcal{C})}| \leqslant \|\xi\| \|X\| \|F\|_{L^2(\mathcal{C})}.$$

The number of all such multindices α is k^N therefore

$$\left|(D^N p)(X)(\xi_1 \otimes ... \otimes \xi_N)\right| \leqslant k^N \|\xi_1\| ... \|\xi_N\| \|X\|_{L^2(\mathcal{C})}^k \prod_{i=1}^{k} \|F_i\|_{L^2(\mathcal{C})}.$$

Denote

$$\|c\|_1 = \int_0^1 \|c(t)\| dt.$$

Thus

$$\int_{\Delta_N(s)} \left|(D^N p)(g(s_1))(c(s_1) \otimes ... \otimes c(s_N))\right| d\vec{s}$$

$$\leqslant \prod_{i=1}^{k} \|F_i\|_{L^2(\mathcal{C})} \sup_{0 \leqslant t \leqslant s} \|(g(t)\|_{L^2(\mathcal{C})}^k k^N \int_{\Delta_N(s)} \|c(s_1)\| ... \|c(s_N)\| d\vec{s}$$

$$= \prod_{i=1}^{k} \|F_i\|_{L^2(\mathcal{C})} \|c\|_1^N \sup_{0\leqslant t\leqslant s} \|(g(t)\|_{L^2(\mathcal{C})}^k \frac{k^N}{N!} \xrightarrow[N\to\infty]{} 0.$$

□

In addition to the space of holomorphic polynomials we consider another space of holomorphic functions.

Definition 7.4. $\mathcal{H}^t(G_\infty)$ will denote the space of continuous functions on G_∞ whose restrictions to G_n are holomorphic for every n and $\|f\|_{t,\infty} = \sup_n\{\|f\|_{t,n}\} = \lim_{n\to\infty} \|f\|_{t,n} < \infty$, where $\|f\|_{t,n}$ is the L^2-norm with respect to the heat kernel measure on G_n.

Functions in the space $\mathcal{H}^t(G_\infty)$ and their derivatives satisfy pointwise estimates similar to the ones proved in [5] for Hilbert-Schmidt complex groups. We omit the estimates in the current paper since we do not need them to prove our main results.

Let us now describe the Taylor map. Suppose f is a function from G_{CM} to $\mathbb{C}$ and f has a derivative which is complex linear. Then this derivative is unique. Let $(Df)(X)$ denote the unique element of $\mathcal{C}_Q^*$ (as a complex space) such that

$$(Df)(X)(\xi) = (\tilde{\xi}f)(X) = \frac{d}{dt}\Big|_{t=0} f(Xe^{t\xi}), \qquad \xi \in \mathcal{C}_Q, X \in G_{G_{CM}}.$$

Similarly $(D^k f)(X)$ denotes the unique element of $({\mathcal{C}_Q}^{\otimes k})^*$ such that

$$(D^k f)(X)(\beta) = (\tilde{\beta}f)(X), \qquad \beta \in \mathcal{C}_Q^{\otimes k}, X \in G_{CM}.$$

We will use the following notation

$$(1-D)_X^{-1} f = \sum_{k=0}^{\infty} (D^k f)(X).$$

We call the series of all derivatives $(1-D)_X^{-1} f = \sum_{k=0}^{\infty} (D^k f)(X)$ the image of f under the Taylor map. Let us also describe J_t^0, a noncommutative analog of the bosonic Fock space. This space will play a role of space of Taylor coefficients in Theorem 4.

Let $\mathfrak{h}$ be a complex Lie algebra with a Hermitian inner product on it. Then $T(\mathfrak{h})$ will denote the algebraic tensor algebra over $\mathfrak{h}$ as a complex vector space and $T'(\mathfrak{h})$ will denote the algebraic dual of $T(\mathfrak{h})$. Define a norm on $T(\mathfrak{h})$ by

$$|\beta|_t^2 = \sum_{k=0}^{n} \frac{k!}{t^k} |\beta_k|^2, \beta = \sum_{k=0}^{n} \beta_k, \beta_k \in \mathfrak{h}^{\otimes k}, k = 0, 1, 2, ..., t > 0. \tag{7.1}$$

Here $|\beta_k|$ is the cross norm on $\mathfrak{h}^{\otimes k}$ arising from the inner product on $\mathfrak{h}^{\otimes k}$ determined by the norm $|\cdot|$ on $\mathfrak{h}$. $T_t(\mathfrak{h})$ will denote the completion of $T(\mathfrak{h})$ in this norm. The topological dual of $T_t(\mathfrak{h})$ may be identified with the subspace $T_t^*(\mathfrak{h})$ of $T'(\mathfrak{h})$ consisting of such $\alpha \in T'(\mathfrak{h})$ that the norm

$$|\alpha|_t^2 = \sum_{k=0}^{\infty} \frac{t^k}{k!} |\alpha_k|^2, \alpha = \sum_{k=0}^{\infty} \alpha_k, \alpha_k \in (\mathfrak{h}^{\otimes k})^*, k = 0, 1, 2, ..., t > 0 \tag{7.2}$$

is finite. Here $|\alpha_k|$ is the norm on $(\mathfrak{h}^{\otimes k})^*$ dual to the cross norm on $\mathfrak{h}^{\otimes k}$.

There is a natural pairing for any $\alpha \in T'(\mathfrak{h})$ and $\beta \in T(\mathfrak{h})$ denoted by

$$\langle \alpha, \beta \rangle = \sum_{k=0}^{\infty} \langle \alpha_k, \beta_k \rangle, \alpha = \sum_{k=0}^{\infty} \alpha_k, \beta = \sum_{k=0}^{n} \beta_k, \alpha_k \in (\mathfrak{h}^{\otimes k})^*, \beta_k \in \mathfrak{h}^{\otimes k}, k = 0, 1, 2, ...$$

Denote by $J(\mathfrak{h})$ the two-sided ideal in $T(\mathfrak{h})$ generated by $\{\xi \otimes \eta - \eta \otimes \xi - [\xi, \eta], \xi, \eta \in \mathfrak{h}\}$. Let $J^0(\mathfrak{h}) = \{\alpha \in T'(\mathfrak{h}) : \alpha(J) = 0\}$. Finally, let $J_t^0(\mathfrak{h}) = T_t^*(\mathfrak{h}) \cap J^0(\mathfrak{h})$. We will denote $J_t^0(\mathcal{C})$ by J_t^0 .

The coefficients $\frac{k!}{t^k}$ in the norm on the tensor algebra $T(\mathfrak{h})$ are related to the fact that the measure we consider is the heat kernel measure.

The first part of the following theorem generalizes the Driver-Gross isometry [3] to our infinite-dimensional case. The isometry from the second part uses finite-dimensional approximations to the process X_t to show that for holomorphic polynomials the L^2-norm (with respect to the heat kernel measure) is equal to the limit norm on $\mathcal{H}^t(G_{CM})$. We omit the proof which is essentially the same as for Theorems 4.4 and 8.7 in [5].

Theorem 4.
(1) *$\mathcal{H}^t(G_\infty)$ is a Hilbert space with respect to $\|\cdot\|_{t,\infty}$ and the Taylor map at the identity $(1-D)_I^{-1}$ is an isometry from $\mathcal{H}^t(G_\infty)$ into J_t^0.*
(2) *The embedding of $\mathcal{HP}$ into $\mathcal{H}^t(G_\infty)$ can be extended to an isometry from $\mathcal{H}L^2(\mu_t)$ into $\mathcal{H}^t(G_\infty)$*

One of the remarkable features of the Cameron-Martin group is that elements of the space $\mathcal{H}L^2(\mu_t)$ have so called holomorphic skeletons defined on G_{CM}. We provide only a sketch of proof since it is the same as for Theorem 8.5 in [5].

Theorem 5. *Suppose*

$$p_n \xrightarrow[n\to\infty]{L^2(L^2(\mathcal{C}),\mu_t)} f, \qquad p_n \in \mathcal{HP}.$$

Then there is a holomorphic function $\tilde{f}$, a skeleton of f, on G_{CM} such that

$$p_n(x) \to \tilde{f}(x) \text{ for any } x \in G_{CM}.$$

Sketch of proof. By Theorem 4 the Taylor map is an isometry from $\mathcal{H}L^2(\mu_t)$ to J_t^0. Thus we can find the limit of $(1-D)_I^{-1} p_n$ in J_t^0. Denote the limit by $\alpha = \sum \alpha_m$. Let us define

$$\tilde{f}(x) = \sum_{m=0}^{\infty} \int_{\Delta_m(1)} \alpha_m(c(s_1) \otimes ... \otimes c(s_m)) d\vec{s}.$$

Then one can use Driver's formula (Lemma 7.3) for p_n to prove that $p_n(x) \to \tilde{f}(x)$ for any $x \in G_{CM}$. What is left to check is that the skeleton $\tilde{f}(x)$ is a holomorphic function. □

H. Sugita in [13] constructed holomorphic skeletons for elements in L^2 on an abstract Wiener space. Those skeletons also are defined only on a "small" set, namely, the Cameron-Martin subspace, though the construction is different. The isometry in the second part of Theorem 4 is actually the restriction of the image of the skeleton map $f \mapsto \tilde{f}$ to G_∞.

In conclusion let us explain why we change the inner product $\langle\cdot,\cdot\rangle_{L^2(\mathcal{C})}$ to the one determined by a trace-class operator Q. First of all, Q is the covariance operator for the Wiener process W_t. The fact that Q is trace class implies that the process W_t lives in $L^2(\mathcal{C})$. There is one more reason for using an inner product different from $\langle \cdot, \cdot \rangle_{L^2(\mathcal{C})}$. In what follows we show that if Q is the identity operator then the space J_t^0 is "almost" trivial. That is, if we use the invariant inner product $\langle \cdot, \cdot \rangle_{L^2(\mathcal{C})}$, the only holomorphic functions square integrable with respect to the heat kernel measure are functions of the form $f(X) = g(\langle X, I \rangle_{L^2(\mathcal{C})})$, where g is a holomorphic function on $\mathbb{C}$. Thus essentially the only holomorphic functions square integrable with respect to the heat kernel measure are holomorphic functions of one variable. In [4] we proved the following theorem.

Theorem 6. *Suppose $\mathfrak{h}$ is a Lie algebra with an inner product $\langle \cdot, \cdot \rangle$. Assume that there is an orthonormal basis $\{\xi_k\}_{k=1}^{\infty}$ of $\mathfrak{g}$ such that for any k there are nonzero $\alpha_k \in \mathbb{C}$ and an infinite set of distinct pairs (i_m, j_m) satisfying $\xi_k = \alpha_k[\xi_{i_m}, \xi_{j_m}]$. Then $J_t^0(\mathfrak{h})$ is isomorphic to $\mathbb{C}$.*

Let $\mathfrak{h}$ be the weak closure of $Span\{B_{x_{i_1}} B_{x_{i_2}} ... B_{x_{i_n}},\ i_1 < i_2 < ... < i_n, n > 0\}$. Note that $\mathfrak{g}$ is a Lie algebra, since $[B_{x_{i_1}} B_{x_{i_2}} ... B_{x_{i_n}}, B_{x_{j_1}} B_{x_{j_2}} ... B_{x_{j_m}}] \in \mathfrak{g}$. Also $\mathcal{C} = \{\mathbb{C}I\} \oplus \mathfrak{h}$ as Lie algebras. At the same time, $\mathfrak{h}$ is not an algebra.

Corollary 7.5. *$J_t^0(\mathfrak{h})$ is isomorphic to $\mathbb{C}$ and $J_t^0(\mathcal{C})$ is isomorphic to $\sum_{n=0}^{\infty} \mathbb{C}^{\otimes n}$.*

Proof. Note that

$$[B_{x_{i_1}} B_{x_{i_2}} ... B_{x_{i_{2n-1}}} B_{x_j}, B_{x_j}] = 2 B_{x_{i_1}} B_{x_{i_2}} ... B_{x_{i_{2n-1}}},$$

$$[B_{x_{i_1}} B_{x_{i_2}} ... B_{x_{i_{2n-1}}} B_{x_j}, B_{x_j} B_{x_{i_{2n}}}] = 2 B_{x_{i_1}} B_{x_{i_2}} ... B_{x_{i_{2n}}}$$

for any j. Therefore J_t^0 is trivial by Theorem 6. □

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