Stochastic and Geometric Analysis of Two Infinite-dimensional Groups

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Abstract

The two groups I studied in this dissertation are $\mathrm{Diff}(S^1)$, the group of orientation-preserving C^∞-diffeomorphisms of the circle, and $\mathrm{Sp}(\infty)$, an infinite-dimensional symplectic group arising from certain symplectic representation of the group $\mathrm{Diff}(S^1)$. In Chapter 1, I constructed Brownian motion on $\mathrm{Diff}(S^1)$ associated with a very strong metric of the Lie algebra $\mathrm{diff}(S^1)$. In Chapter 2, I first studied the relationship between $\mathrm{Diff}(S^1)$ and $\mathrm{Sp}(\infty)$ and found that they are not isomorphic with each other, then I constructed a Brownian motion on the group $\mathrm{Sp}(\infty)$. In Chapter 3, I computed the Ricci curvature of the group $\mathrm{Sp}(\infty)$ associated with a certain inner product on the Lie algebra $\mathfrak{sp}(\infty)$.

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APPROVAL PAGE

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Stochastic and Geometric Analysis of Two Infinite-dimensional Groups

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Chapter 1

Construction of Brownian Motion on $\mathbf{Diff}(S^1)$

1.1 Introduction

Definition 1.1.1. Let $\mathrm{Diff}(S^1)$ be the group of orientation preserving C^∞-diffeomorphisms of S^1. Let $\mathrm{diff}(S^1)$ be the space of C^∞-vector fields on S^1.

The central extension of $\mathrm{Diff}(S^1)$ is the famous Virasoro group. Both the Virasoro group, the group $\mathrm{Diff}(S^1)$, the quotient groups $\mathrm{Diff}(S^1)/S^1$ and $\mathrm{Diff}(S^1)/SU(1,1)$ arise naturally in many places in mathematical physics and have been extensively studied for a long time [25, 6, 19, 21, 2, 3, 8, 13].

The space $\mathrm{diff}(S^1)$ can be identified with the space of C^∞-functions on S^1. Therefore, $\mathrm{diff}(S^1)$ carries a natural Fréchet space structure. In addition, $\mathrm{diff}(S^1)$ is an infinite-dimensional Lie algebra: for any $f, g \in \mathrm{diff}(S^1)$, the Lie bracket is given by $[f,g] = f'g - fg'$. Thus, the group $\mathrm{Diff}(S^1)$ associated with the Lie algebra $\mathrm{diff}(S^1)$ becomes an infinite-dimensional Fréchet Lie group [22].

One of the research goals of the stochastic analysis on the group $\mathrm{Diff}(S^1)$ is to construct a Brownian motion on it. Because Brownian motions on the group $\mathrm{Diff}(S^1)$ will induce measures on it, and once we establish quasi-invariance properties of the measures, we can study unitary representations of the group $\mathrm{Diff}(S^1)$.

In general, to construct a Brownian motion on a Lie group, one might solve a Stratonovich stochastic differential equation (SDE) on such a group [20, 16]. The method is best illustrated for a finite dimensional compact Lie group.

Let G be a finite dimensional compact Lie group. Denote by $\mathfrak{g}$ the Lie algebra of G identified with the tangent space T_eG to the group G at the identity element $e \in G$. Let $L_g : G \to G$ be the left translation of G by an element $g \in G$, and let $(L_g)_* : \mathfrak{g} \to T_gG$ be the differential of L_g. If we choose a metric on $\mathfrak{g}$ and let W_t be the standard Brownian motion on $\mathfrak{g}$ corresponding to this metric, we can develop the Brownian motion W_t onto G by solving a Stratonovich stochastic differential equation

$$\delta \widetilde{X}_t = (L_{\widetilde{X}_t})_* \delta W_t \tag{1.1.1}$$

where δ stands for the Stratonovich differential. The solution $\widetilde{X}_t$ is a Markov process on G whose generator is the Laplace operator on G. We call $\widetilde{X}_t$ the Brownian motion on the group G.

In the case when G is an infinite-dimensional Hilbert Lie group, that is, the tagent space has a Hilbert space structure, one can solve Equation (1.1.1) by using the theory of stochastic differential equations in Hilbert spaces as developed by G. DaPrato and J. Zabczyk in [7]. Using this method, M. Gordina [10, 11, 12] has constructed Brownian motions on several Hilbert-Schmidt groups. In Chapter 2, using the same method, I will also construct a Brownian motion on the infinite-dimensional symplectic group $\mathrm{Sp}(\infty)$. These constructions rely on the fact that these Hilbert-Schmidt groups are Hilbert Lie groups.

In the present case, we would like to replace G by $\mathrm{Diff}(S^1)$ and $\mathfrak{g}$ by $\mathrm{diff}(S^1)$ and solve Equation (1.1.1) correspondingly. But because the group $\mathrm{Diff}(S^1)$ is a Fréchet Lie group, which is not a Hilbert Lie group, Equation (1.1.1) does not even make sense as it stands. First, we need to interpret the Brownian motion W_t in the Fréchet space $\mathrm{diff}(S^1)$ appropriately. Second, we are lacking a well developed stochastic differential equation theory in Fréchet spaces to make sense of Equation (1.1.1).

In 1999, P. Malliavin [21] first constructed a canonical Brownian motion on $\mathrm{Homeo}(S^1)$, the group of Hölderian homeomorphisms of S^1. In 2002, S. Fang [8] gave a detailed construction of this canonical Brownian motion on the group $\mathrm{Homeo}(S^1)$. Their constructions were essentially carried out by interpreting and solving the same Equation (1.1.1) on the group $\mathrm{Diff}(S^1)$.

To define the Brownian motion W_t in Equation (1.1.1), Malliavin and Fang chose the $H^{3/2}$ metric of the Lie algebra $\mathrm{diff}(S^1)$. Basically, this metric uses the set

$$\{n^{-3/2}\cos(n\theta), m^{-3/2}\sin(m\theta) | m,n = 1,2,3,\cdots\},$$

which is a subset of the Lie algebra $\mathrm{diff}(S^1)$, as an orthonormal basis to form a Hilbert space $H^{3/2}$. Then they defined W_t to be the cylindrical Brownian motion in $H^{3/2}$ whose covariance operator is the identity operator on $H^{3/2}$. But since the coefficients $n^{-3/2}$ and $m^{-3/2}$ do not decrease rapidly enough, the Hilbert space $H^{3/2}$ is not contained in the Lie algebra $\mathrm{diff}(S^1)$. Therefore, the Brownian motion W_t they defined on $H^{3/2}$ does not live in $\mathrm{diff}(S^1)$ either. This is the essential reason why the canonical Brownian motion they constructed lives in a larger group $\mathrm{Homeo}(S^1)$, but not in the group $\mathrm{Diff}(S^1)$.

To interpret and solve Equation (1.1.1), Fang [8] treated it as a family of stochastic differential equations on S^1: for each $\theta \in S^1$, Fang considered the equation

$$\delta \widetilde{X}_{\theta,t} = (L_{\widetilde{X}_{\theta,t}})_* \delta W_{\theta,t}, \tag{1.1.2}$$

which is a stochastic differential equation on S^1. By solving the above equation for each $\theta \in S^1$, Fang obtained a family of solutions $\widetilde{X}_{\theta,t}$ parameterized by θ. Then he used a Kolmogorov type argument to show that the family $\widetilde{X}_{\theta,t}$ is Hölderian continuous in the variable θ. Using this method, he proved that for each $t \geq 0$, $\widetilde{X}_{\theta,t}$ is a Hölderian homeomorphism of S^1. Thus, he constructed the canonical Brownian motion on the group $\mathrm{Homeo}(S^1)$. But this Kolmogorov type argument cannot be pushed further to show that $\widetilde{X}_{\theta,t}$ is differentiable in θ. Therefore, Fang's method does not seem to be suitable to construct a Brownian motion that lives in the group $\mathrm{Diff}(S^1)$, rather than in

Homeo (S^1).

In this chapter, my goal is to construct a Brownian motion that lives exactly in the group $\mathrm{Diff}(S^1)$. To achieve this, I need another way to interpret and solve Equation (1.1.1). The idea is described as follows.

First, instead of the $H^{3/2}$ metric that Malliavin and Fang used, I will choose a very "strong" metric on the Lie algebra $\mathrm{diff}(S^1)$ (In some sense, we can call it H^∞ metric): let $\{\lambda(n)\}_{n=1}^\infty$ be a sequence of rapidly decreasing positive numbers. I use the set

$$\{\lambda(n)\cos(n\theta), \lambda(m)\sin(m\theta)|m,n = 1,2,3,\cdots\},$$

which is a subset of the Lie algebra $\mathrm{diff}(S^1)$, as an orthonormal basis to form a Hilbert space H_λ. Then I will define the Brownian motion W_t to be the cylindrical Brownian motion in H_λ whose covariance operator is the identity operator on H_λ. Because the coefficients $\lambda(n)$ are rapidly decreasing, the Hilbert space H_λ is a *subspace* of the Lie algebra $\mathrm{diff}(S^1)$. Therefore, the Brownian motion W_t lives in the Lie algebra $\mathrm{diff}(S^1)$, and the solution to Equation (1.1.1) will have a better chance to live in the group $\mathrm{Diff}(S^1)$.

Second, in contrast to Fang's method of interpreting Equation (1.1.1) "pointwise" as a family of stochastic differential equations on S^1, I will interpret it as a sequence of stochastic differential equations on a sequence of "Hilbert" spaces. To do this, I will embed the group $\mathrm{Diff}(S^1)$ into an affine space $\widetilde{\mathrm{diff}}(S^1)$ that is isomorphic to the Lie algebra $\mathrm{diff}(S^1)$. Let H^k be the kth Sobolev space over S^1. It is a separable Hilbert space. Let $\widetilde{H}^k$ be the corresponding affine space that is isomorphic to H^k. For the precise definition of the space $\widetilde{\mathrm{diff}}(S^1)$ and $\widetilde{H}^k$, see Section 1.2. It is well known that the space $\mathrm{diff}(S^1)$ is the intersection of the Sobolev spaces H^k. Similarly, $\widetilde{\mathrm{diff}}(S^1)$ is the intersection of the affine spaces $\widetilde{H}^k$. Now we have the embedding

$$\mathrm{Diff}(S^1) \subseteq \widetilde{\mathrm{diff}}(S^1) \subseteq \widetilde{H}^k, \quad k = 1,2,3,\cdots \tag{1.1.3}$$

Thus, I can interpret Equation (1.1.1) as a sequence of stochastic differential equations on the sequence of affine spaces $\{\widetilde{H}^k\}_{k=1}^\infty$ each of which is isomorphic to the Hilbert space H^k. These stochastic differential equations can be solved by DaPrato and Zabczyk's method [7].

In accordance with the notations used by DaPrato and Zabczyk in [7], in the rest of this chapter, I will denote the operator $(L_{\widetilde{X}_t})_*$ in Equation (1.1.1) by $\widetilde{\Phi}(\widetilde{X}_t)$. The operator $\widetilde{\Phi}$ will be discussed in detail in the next section. After adding the initial condition, I can now re-write Equation (1.1.1) as

$$\delta\widetilde{X}_t = \widetilde{\Phi}(\widetilde{X}_t)\delta W_t, \quad \widetilde{X}_0 = id \tag{1.1.4}$$

where id is the identity element in $\mathrm{Diff}(S^1)$.

Equation (1.1.4) is interpreted as a stochastic differential equation in the "Hilbert" space $\widetilde{H}^k$. To use DaPrato and Zabczyk's method to solve this equation, I will also need to establish the Lipschitz condition of the operator $\widetilde{\Phi}$. This will be done in Section 1.2. It turns out that the operator $\widetilde{\Phi}$ is *locally* Lipschitz. So the explosion time of the solution, which is a key part of the problem, needs to be discussed. This will be done in Section 1.3.

After solving Equation (1.1.4 in $\widetilde{H})^k$ for each k, it is relatively easy to prove that the solution lives in the affine space $\widetilde{\mathrm{diff}}(S^1)$ (Proposition 1.3.17). By the embedding (1.1.3), the group $\mathrm{Diff}(S^1)$ is a subset of the affine space $\widetilde{\mathrm{diff}}(S^1)$. I need to push one step further to prove that the solution actually lives in the group $\mathrm{Diff}(S^1)$.

In general, to prove a process lives in a group rather than in an ambient space, one needs to construct an inverse process. To construct the inverse process, usually one needs to solve another stochastic differential equation – the SDE for the inverse process [10, 14]. In my case, I have derived the SDE for the inverse process:

$$\delta \widetilde{Y}_t = \widetilde{\Psi}(\widetilde{Y}_t)\delta W_t \tag{1.1.5}$$

where $\widetilde{\Psi}$ is an operator such that for $\tilde{g} \in \mathrm{Diff}(S^1)$ and $f \in \mathrm{diff}(S^1)$, $\widetilde{\Psi}(\tilde{g})f = D\tilde{g} \cdot f$, where $D = d/d\theta$ and "$\cdot$" is the pointwise multiplication of two functions. Because the operator D causes loss of one degree of smoothness, the method I use to interpret and solve Equation (1.1.4) does not apply to Equation (1.1.5). This causes some problems, and I was forced to give up this method of solving the inverse SDE. But I mananged to get around the problem by using a different method.

I first observe that an element $\tilde{f} \in \widetilde{\mathrm{diff}}(S^1)$ belongs to $\mathrm{Diff}(S^1)$ if and only if $\tilde{f}'(\theta) > 0$ for all $\theta \in S^1$. Based on this observation, I can show that the solution is contained in the group $\mathrm{Diff}(S^1)$ up to a stopping time. Then I can "concatenate" this small piece of solution with another small piece of solution to make a new solution up to a longer stopping time. The key idea is Proposition (1.3.14) and the remark following it (Remark 1.3.15). Finally, I am able to prove the main theorem (Theorem 1.3.19) of this chapter. Basically, it says that Equation (1.1.4) has a unique solution that lives exactly in the group $\mathrm{Diff}(S^1)$, and furthermore, the solution is non-explosive.

The work in this chapter is written in [26], and has been accepted by Potential Analysis for publication.

1.2 An interpretation of Equation (1.1.4)

1.2.1 The group Diff(S^1) and the Lie algebra diff(S^1)

Let $\mathrm{Diff}(S^1)$ be the group of orientation preserving C^∞ diffeomorphisms of S^1, and $\mathrm{diff}(S^1)$ be the space of C^∞ vector fields on S^1. We have the following identifications for the space $\mathrm{diff}(S^1)$:

$$\begin{aligned}\mathrm{diff}(S^1) &\cong \{f : S^1 \to \mathbb{R} : f \in C^\infty\} \\ &\cong \{f : \mathbb{R} \to \mathbb{R} : f \in C^\infty, f(x) = f(x+2\pi), \text{ for all } x \in \mathbb{R}\}\end{aligned} \tag{1.2.1}$$

Using this identification, we see that the space $\mathrm{diff}(S^1)$ has a Fréchet space structure. In addition, this space has a Lie algebra structure, namely, for $f, g \in \mathrm{diff}(S^1)$ the Lie bracket is given by

$$[f, g] = f'g - fg',$$

where f' and g' are derivatives with respect to the variable $\theta \in S^1$. Therefore, the group $\mathrm{Diff}(S^1)$ is a Fréchet Lie group as defined in [22].

Using the above identification 1.2.1, we also have an identification for $\mathrm{Diff}(S^1)$

$$\mathrm{Diff}(S^1) \cong \{\tilde{f} : \mathbb{R} \to \mathbb{R} : \tilde{f} = id + f, f \in \mathrm{diff}(S^1), \tilde{f}' > 0\}, \tag{1.2.2}$$

where id is the identity function from $\mathbb{R}$ to $\mathbb{R}$. Note that the set on the right hand side of the above identification is a group with the group multiplication being composition of functions. We define that for $\tilde{f}, \tilde{g} \in \mathrm{Diff}(S^1)$, $\tilde{f}\tilde{g} = \tilde{g} \circ \tilde{f}$. Under this identification, the left translation of $\mathrm{Diff}(S^1)$ is given by $L_{\tilde{g}}\tilde{f} = \tilde{g}\tilde{f} = \tilde{f} \circ \tilde{g}$.

Definition 1.2.1. Define

$$\widetilde{\mathrm{diff}}(S^1) = \{\tilde{f} : \mathbb{R} \to \mathbb{R} | \tilde{f} = id + f, f \in \mathrm{diff}(S^1)\} \tag{1.2.3}$$

The space $\widetilde{\mathrm{diff}}(S^1)$ is an affine space which is isomorphic to the vector space $\mathrm{diff}(S^1)$. We denote the isomorphism by $\sim$, that is, $\sim: \mathrm{diff}(S^1) \to \widetilde{\mathrm{diff}}(S^1)$, $f \mapsto \tilde{f} = id + f$. Comparing (1.2.2 and (1.2.1)), we have the embedding

$$\mathrm{Diff}(S^1) \subseteq \widetilde{\mathrm{diff}}(S^1). \tag{1.2.4}$$

With this embedding, the differential of a left translation $L_{\tilde{g}}$ becomes $(L_{\tilde{g}})_* : \mathrm{diff}(S^1) \to \mathrm{diff}(S^1)$, and is given by $(L_{\tilde{g}})_* f = f \circ \tilde{g}$ for $f \in \mathrm{diff}(S^1)$. This can be easily seen by the following calculation:

$$\left.\frac{d}{dt}\right|_{t=0} (\tilde{g}(\theta) + t f(\tilde{g}(\theta))) = f(\tilde{g}(\theta))$$

The following proposition is an immediate observation from the identification (1.2.2) and definition of $\widetilde{\mathrm{diff}}(S^1)$ given by (1.2.1). Yet, it plays a key role in proving the main theorem (Theorem 1.3.19) of this chapter.

Proposition 1.2.2. *An element $\tilde{f} \in \widetilde{\mathit{diff}}(S^1)$ belongs to $\mathit{Diff}(S^1)$ if and only if $\tilde{f}' > 0$, or equivalently $f' > -1$.*

1.2.2 The Hilbert space H_λ and the Brownian motion W_t

To define the Brownian motion W_t in Equation (1.1.4), We need to choose a metric on the Lie algebra $\mathrm{diff}(S^1)$. Comparing with the $H^{3/2}$ metric that P. Malliavin and S. Fang chose, the metric we choose in the following definition is a very "strong" metric.

Definition 1.2.3. Let $\mathscr{S}$ be the set of *even* functions $\lambda : \mathbb{Z} \to (0, \infty)$ such that $\lim_{n\to\infty} |n|^k \lambda(n) = 0$ for all $k \in \mathbb{N}$. For $\lambda \in \mathscr{S}$, $n \in \mathbb{Z}$, let $\hat{e}_n = \hat{e}_n^{(\lambda)} \in \mathrm{diff}(S^1)$ be defined by

$$\hat{e}_n^{(\lambda)}(\theta) = \begin{cases} \lambda(n)\cos(n\theta), & n \geq 0 \\ \lambda(n)\sin(|n|\theta), & n < 0 \end{cases}$$

Let H_λ be the Hilbert space with the set $\{\hat{e}_n^{(\lambda)}\}_{n\in\mathbb{Z}}$ as an orthonormal basis.

Note that the function λ is rapidly decreasing, therefore the Hilbert space H_λ defined above is a *proper subspace* of $\mathrm{diff}(S^1)$. Also note that $\mathrm{diff}(S^1) = \bigcup_{\lambda \in \mathscr{S}} H_\lambda$.

Let $\alpha, \lambda \in \mathscr{S}$ be defined by $\lambda(n) = |n|\alpha(n)$, and let H_α and H_λ be the corresponding Hilbert subspaces of $\mathrm{diff}(S^1)$. Then we have $H_\alpha \subset H_\lambda$, and the inclusion map $\iota : H_\alpha \hookrightarrow H_\lambda$ that sends $\hat{e}_n^{(\alpha)}$ to $\hat{e}_n^{(\alpha)} = \frac{1}{|n|}\hat{e}_n^{(\lambda)}$ is a Hilbert-Schmidt operator. The adjoint operator $\iota^* : H_\lambda \to H_\alpha$ that sends $\hat{e}_n^{(\lambda)}$ to $\frac{1}{|n|}\hat{e}_n^{(\alpha)}$ is also a Hilbert-Schmidt operator. The operator $Q_\lambda = \iota\iota^* : H_\lambda \to H_\lambda$ is a trace class operator on H_λ, and $H_\alpha = Q_\lambda^{1/2} H_\lambda$.

Definition 1.2.4. Let W_t be a Brownian motion defined by

$$W_t = \sum_{n\in\mathbb{Z}} B_t^{(n)} \hat{e}_n^{(\alpha)} = \sum_{n\in\mathbb{Z}} \frac{1}{|n|} B_t^{(n)} \hat{e}_n^{(\lambda)}$$

where $\{B_t^{(n)}\}_{n\in\mathbb{Z}}$ are mutually independent standard $\mathbb{R}$-valued Brownian motions.

Remark 1.2.5. We see that W_t is a cylindrical Brownian motion on H_α whose covariance operator is the identity operator on H_α. Also, W_t is a Brownian motion on H_λ whose covariance operator is the operator Q_λ.

1.2.3 The Sobolev space H^k and the affine space $\widetilde{H}^k$

Now we turn to the Sobolev spaces over S^1. Let us first recall some basic properties of the Sobolev spaces over S^1 found for example in [1].

Let k be a non-negative integer.

Definition 1.2.6. Let C^k be the space of k-times continuously differentiable real-valued functions on S^1, and H^k be the kth Sobolev space on S^1.

Recall that H^k consists of functions $f : S^1 \to \mathbb{R}$ such that $f^{(k)} \in L^2$, where $f^{(k)}$ is the kth derivative of f in distributional sense. The Sobolev space H^k has a norm given by

$$\|f\|_{H^k}^2 = \|f\|_{L^2}^2 + \|f^{(k)}\|_{L^2}^2$$

The Sobolev space H^k is a separable Hilbert space, and C^k is a dense subspace of H^k. We will make use of the following standard properties of the spaces H^k.

Theorem 1.2.7 ([1]). *Let m, k be two non-negative integers.*

1. *If $m \le k$ and $f \in H^k$, then $\|f\|_{H^m} \le \|f\|_{H^k}$.*
2. *If $m < k$ and $f \in H^k$, then there exists a constant c_k such that $\|f^{(m)}\|_{L^\infty} \le c_k \|f\|_{H^k}$.*
3. *$H^{k+1} \subseteq H^k$ for all $k = 0, 1, 2, \cdots$, and $\mathrm{diff}(S^1) = \bigcap_{k=0}^\infty H^k$.*

□

An element $f \in H^k$ can be identified with a 2π-periodic function from $\mathbb{R}$ to $\mathbb{R}$. Let id be the identity function from $\mathbb{R}$ to $\mathbb{R}$. It makes sense to talk about the function $\tilde{f} = id + f$. Similar to the definition of $\widetilde{\text{diff}}(S^1)$, we can define $\widetilde{H}^k$ as follows.

Definition 1.2.8. Define

$$\widetilde{H}^k = \{\tilde{f} : \mathbb{R} \to \mathbb{R} : \tilde{f} = id + f, f \in H^k\}$$

The space $\widetilde{H}^k$ is an affine space that is isomorphic to the Sobolev space H^k. We denote the isomorphism by $\sim$, that is, $\sim: H^k \to \widetilde{H}^k$, $f \mapsto \tilde{f} = id + f$. The image of C^k under the isomorphism, denoted by $\widetilde{C}^k$, is a dense subspace of the affine space $\widetilde{H}^k$. An element $\tilde{f} \in \widetilde{H}^k$ can be identified as a function from S^1 to S^1. By item (3) in Theorem 1.2.7, we have $\widetilde{H}^{k+1} \subseteq \widetilde{H}^k$ and $\widetilde{\text{diff}}(S^1) = \bigcap_k \widetilde{H}^k$.

Now we have the following embeddings:

$$\text{Diff}(S^1) \subseteq \widetilde{\text{diff}}(S^1) \subseteq \cdots \subseteq \widetilde{H}^3 \subseteq \widetilde{H}^2 \subseteq \widetilde{H}^1, \tag{1.2.5}$$

and we can interpret Equation (1.1.4) as a sequence of stochastic differential equations on the sequence of affine spaces $\{\widetilde{H}^k\}_{k=1}^{\infty}$.

1.2.4 The operator $\widetilde{\Phi}$ and Φ

For $\tilde{g} \in \text{Diff}(S^1)$, let $(L_{\tilde{g}})_*$ be the differential of the left translation. In accordance with the notation used by DaPrato and Zabczyk in [7], we denote $(L_{\tilde{g}})_*$ by $\widetilde{\Phi}(\tilde{g})$.

Initially, $\widetilde{\Phi} : \text{Diff}(S^1) \to (\text{diff}(S^1) \to \text{diff}(S^1))$, which means $\widetilde{\Phi}$ takes an element $\tilde{g} \in \text{Diff}(S^1)$ and becomes a linear transformation $\widetilde{\Phi}(\tilde{g})$ from $\text{diff}(S^1)$ to $\text{diff}(S^1)$. Because we want to interpret Equation (1.1.4) as an SDE on $\widetilde{H}^k$ and use DaPrato and Zabczyk's theory [7], we need the operator $\widetilde{\Phi}$ to be extended as $\widetilde{\Phi} : \widetilde{H}^k \to (H_\lambda \to H^k)$, which means $\widetilde{\Phi}$ takes an element $\tilde{g} \in \widetilde{H}^k$ and becomes a linear transformation $\widetilde{\Phi}(\tilde{g})$ from H_λ to H^k [7].

Let $L(H_\lambda, H^k)$ be the space of linear transformations from H_λ to H^k. Define a mapping

$$\widetilde{\Phi} : \widetilde{C}^k \to L(H_\lambda, H^k) \tag{1.2.6}$$

such that if $\tilde{f} \in \widetilde{C}^k$, $g \in H_\lambda$, then $\widetilde{\Phi}(\tilde{f})(g) = g \circ \tilde{f}$. The mapping $\widetilde{\Phi}$ is easily seen to be well defined. Sometimes, it is easier to work with the vector space C^k. So we similarly define a mapping

$$\Phi : C^k \to L(H_\lambda, H^k) \tag{1.2.7}$$

such that if $f \in C^k$, $g \in H_\lambda$, then $\Phi(f)(g) = g \circ \tilde{f}$, where $\tilde{f} = id + f$ is the image of f under the isomorphism $\sim$.

Let $L^2(H_\lambda, H^k)$ denote the space of Hilbert-Schmidt operators from H_λ to H^k. The space $L^2(H_\lambda, H^k)$ is a separable Hilbert space. For $T \in L^2(H_\lambda, H^k)$, the norm of T is given by

$$\|T\|^2_{L^2(H_\lambda, H^k)} = \sum_{n \in \mathbb{Z}} \|T \hat{e}_n^{(\lambda)}\|^2_{H^k}$$

where $\hat{e}_n^{(\lambda)}$ is defined in Definition (1.2.3).

To use DaPrato and Zabczyk's theory [7], we need $\widetilde{\Phi}$ to be $\widetilde{\Phi}: \widetilde{H}^k \to L^2(H_\lambda, H^k)$ or equivalently, we need Φ to be $\Phi: H^k \to L^2(H_\lambda, H^k)$. We will also need some Lipschitz condition of $\widetilde{\Phi}$ and Φ. These are proved in proposition (1.2.10) and (1.2.12). Both propositions need the Faà di Bruno's formula for higher derivatives of a composition function.

Theorem 1.2.9 (Faà di Bruno's formula [17])**.**

$$f(g(x))^{(n)} = \sum_{k=0}^{n} f^{(k)}(g(x)) B_{n,k}(g'(x), g''(x), \cdots, g^{(n-k+1)}(x)), \tag{1.2.8}$$

where $B_{n,k}$ is the Bell polynomial

$$B_{n,k}(x_1, \cdots, x_{n-k+1}) = \sum \frac{n!}{j_1! \cdots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}},$$

and the summation is taken over all sequences of $\{j_1, \cdots, j_{n-k+1}\}$ of nonnegative integers such that $j_1 + \cdots + j_{n-k+1} = k$ and $j_1 + 2j_2 + \cdots + (n-k+1)j_{n-k+1} = n$.

We remark that after expanding expression (1.2.8), $f(g(x))^{(n)}$ can be viewed as a summation of several terms, each of which has the form

$$f^{(j)}(g(x)) m(g', g'', \cdots, g^{(n)})$$

where $j \le n$ and $m(g', g'', \cdots, g^{(n)})$ is a *monomial* in $g', g'', \cdots, g^{(n)}$. Also observe that, the only term that involves the highest derivative of g is $f'(g(x))g^{(n)}(x)$.

Proposition 1.2.10. *For any $f \in C^k$, $k = 0, 1, 2, \cdots$, $\Phi(f) \in L^2(H_\lambda, H^k)$.*

Proof.

$$\begin{aligned}\|\Phi(f)\|^2_{L^2(H_\lambda, H^k)} &= \sum_{n\in\mathbb{Z}} \|\Phi(f)(\hat{e}_n)\|^2_{H^k} \\ &= \sum_{n\in\mathbb{Z}} \|\hat{e}_n(id+f)\|^2_{L^2} + \|\hat{e}_n(id+f)^{(k)}\|^2_{L^2},\end{aligned}$$

where $\hat{e}_n$ is defined in Definition (1.2.3) and we have suppressed the index λ here. $\hat{e}_n(id+f)$ denotes the function $\hat{e}_n$ composed with $id+f$, and $\hat{e}_n(id+f)^{(k)}$ is the kth derivative of $\hat{e}_n(id+f)$.

First, we have

$$\|\hat{e}_n(id+f)\|^2_{L^2} \le \lambda(n)^2.$$

We apply Faà di Bruno's formula (1.2.8) to $\hat{e}_n(id+f)^{(k)}$, and then expand it to a summation of several terms. We are going to deal with the terms with and without $f^{(k)}$, the highest derivative of f, separately. So we write the summaion as

$$\hat{e}_n(id+f)^{(k)} = \text{... terms without } f^{(k)} \text{...} + \hat{e}'_n(id+f) f^{(k)}, \tag{1.2.9}$$

where each term without $f^{(k)}$ has the form

$$\hat{e}_n^{(j)}(id+f)m(f',f'',\cdots,f^{(k-1)})$$

with $j \le k$ and $m(f',f'',\cdots,f^{(k-1)})$ a *monomial* in $f',f'',\cdots,f^{(k-1)}$. Let d be the degree of the monomial $m(f',f'',\cdots,f^{(k-1)})$. Then from Faà di Bruno's formula we see that $d \le k$ for all monomials.

By Definition (1.2.3) of $\hat{e}_n$ and using item (2) in Theorem (1.2.7), we have

$$\begin{aligned} &\|\hat{e}_n^{(j)}(id+f)m(f',f'',\cdots,f^{(k-1)})\|_{L^2} \\ &\le \|\hat{e}_n^{(j)}(id+f)\|_{L^\infty}\|m(f',f'',\cdots,f^{(k-1)})\|_{L^\infty} \\ &\le \lambda(n)|n|^k c_k^k \|f\|_{H^k}^k. \end{aligned} \tag{1.2.10}$$

For the last term in expression (1.2.9), we have

$$\begin{aligned} \|\hat{e}_n'(id+f)f^{(k)}\|_{L^2} &\le \|\hat{e}_n'(id+f)\|_{L^\infty}\|f^{(k)}\|_{L^2} \\ &\le \lambda(n)|n|\|f\|_{H^k} \le \lambda(n)|n|^k c_k^k \|f\|_{H^k}^k. \end{aligned} \tag{1.2.11}$$

By (1.2.10) and (1.2.11), we have

$$\|\hat{e}_n(id+f)^{(k)}\|_{L^2}^2 \le K\lambda(n)^2|n|^{2k}c_k^{2k}\|f\|_{H^k}^{2k},$$

where K is the number of terms in expression (1.2.9), which depends on k but does not depend on n. Therefore,

$$\|\Phi(f)\|_{L^2(H_\lambda,H^k)}^2 \le \sum_{n\in\mathbb{Z}}\left(\lambda(n)^2 + K\lambda(n)^2|n|^{2k}c_k^{2k}\|f\|_{H^k}^{2k}\right)$$

Because $\lambda(n)$ is rapidly decreasing (Definition 1.2.3), $\sum_{n\in\mathbb{Z}}\lambda(n)^2|n|^{2k} < \infty$. Therefore, we have

$$\|\Phi(f)\|_{L^2(H_\lambda,H^k)}^2 < \infty$$

□

Now Φ can be viewed as a mapping $\Phi : C^k \to L^2(H_\lambda, H^k)$. Similarly, $\widetilde{\Phi}$ can be viewed as a mapping $\widetilde{\Phi} : \widetilde{C}^k \to L^2(H_\lambda, H^k)$. To use DaPrato and Zabczyk's theory [7], we will need the Lipschitz condition of Φ and $\widetilde{\Phi}$. It turns out that they are *locally* Lipschitz. Let us recall the concept of local Lipschitzness.

Definition 1.2.11. Let A and B be two normed linear spaces with norm $\|\cdot\|_A$ and $\|\cdot\|_B$ respectively. A mapping $f : A \to B$ is said to be *locally Lipschitz* if for $R > 0$, and $x, y \in A$ such that $\|x\|, \|y\| \le R$, we have

$$\|f(x) - f(y)\|_B \le C_R\|x-y\|_A,$$

where C_N is a constant which in general depends on N.

Proposition 1.2.12. *For any $k = 0,1,2,\cdots$, $\Phi : C^k \to L^2(H_\lambda, H^k)$ is locally Lipschitz.*

Proof. Let $R > 0$, and $f, g \in C^k$ be such that $\|f\|_{H^k}, \|g\|_{H^k} \leq R$. We have

$$\begin{aligned}
&\|\Phi(f)-\Phi(g)\|^2_{L^2(H_\lambda, H^k)} \\
&= \sum_{n\in\mathbb{Z}} \|[\Phi(f)-\Phi(g)]\hat{e}_n\|^2_{H^k} = \sum_{n\in\mathbb{Z}} \|\hat{e}_n(id+f)-\hat{e}_n(id+g)\|^2_{H^k} \\
&= \sum_{n\in\mathbb{Z}} \|\hat{e}_n(id+f)-\hat{e}_n(id+g)\|^2_{L^2} + \|\hat{e}_n(id+f)^{(k)}-\hat{e}_n(id+g)^{(k)}\|^2_{L^2},
\end{aligned}$$

where $\hat{e}_n$ is defined in Definition (1.2.3) and we have suppressed the index λ here. $\hat{e}_n(id+f)$ and $\hat{e}_n(id+g)$ denote the function $\hat{e}_n$ composed with $id+f$ and $id+g$ respectively. $\hat{e}_n(id+f)^{(k)}$ and $\hat{e}_n(id+g)^{(k)}$ are the kth derivatives of $\hat{e}_n(id+f)$ and $\hat{e}_n(id+g)$ respectively.

First, by the mean value theorem we have

$$\begin{aligned}
&\|\hat{e}_n(id+f)-\hat{e}_n(id+g)\|_{L^2} = \|\hat{e}'_n(id+\xi)(f-g)\|_{L^2} \\
&\leq \|\hat{e}'_n(id+\xi)\|_{L^\infty}\|f-g\|_{L^2} \leq \lambda(n)|n|\|f-g\|_{H^k}
\end{aligned}$$

We apply Faà di Bruno's formula (1.2.8) to $\hat{e}_n(id+f)^{(k)}$, and then expand it to a summation of several terms. We are going to deal with the terms with and without $f^{(k)}$, the highest derivative of f, separately. So we write the summaion as

$$\hat{e}_n(id+f)^{(k)} = \text{... terms without } f^{(k)}\text{...} + \hat{e}'_n(id+f)f^{(k)}, \tag{1.2.12}$$

where each term without $f^{(k)}$ has the form

$$\hat{e}_n^{(j)}(id+f)m(f', f'', \cdots, f^{(k-1)})$$

with $j \leq k$ and $m(f', f'', \cdots, f^{(k-1)})$ a *monomial* in $f', f'', \cdots, f^{(k-1)}$. Let d be the degree of the monomial $m(f', f'', \cdots, f^{(k-1)})$. Then from Faà di Bruno's formula we see that $d \leq k$ for all monomials. By replacing f with g in (1.2.12), we obtain

$$\hat{e}_n(id+g)^{(k)} = \text{... terms without } g^{(k)}\text{...} + \hat{e}'_n(id+g)g^{(k)} \tag{1.2.13}$$

Next, we need a simple observation: suppose $A_1A_2A_3...$ and $B_1B_2B_3...$ are two monomials with the same number of factors. By telescoping, we can put $A_1A_2A_3... - B_1B_2B_3...$ into the form

$$(A_1-B_1)A_2A_3... + B_1(A_2-B_2)A_3... + B_1B_2(A_3-B_3)... + \cdots$$

Using this observation, we can put $\hat{e}_n(id+f)^{(k)} - \hat{e}_n(id+g)^{(k)}$ into the form

$$\begin{aligned}
\hat{e}_n(id+f)^{(k)} - \hat{e}_n(id+g)^{(k)} &= \text{...terms without } f^{(k)} \text{ and } g^{(k)}\text{...} \\
&+ \left(\hat{e}'_n(id+f)-\hat{e}'_n(id+g)\right)f^{(k)} + \hat{e}'_n(id+g)\left(f^{(k)}-g^{(k)}\right)
\end{aligned} \tag{1.2.14}$$

In expression (1.2.14), there are two types of terms without $f^{(k)}$ and $g^{(k)}$. One type has the form

$$\left(\hat{e}_n^{(j)}(id+f)-\hat{e}_n^{(j)}(id+g)\right)m_A(f',\cdots,f^{(k-1)},g',\cdots,g^{(k-1)}), \tag{1.2.15}$$

where $j \le k$ and m_A is a monomial in $f',\cdots,f^{(k-1)},g',\cdots,g^{(k-1)}$. We denote such a term by A. Another type has the form

$$\hat{e}_n^{(i)}(id+g)\left(f^{(j)}-g^{(j)}\right)m_B(f',\cdots,f^{(k-1)},g',\cdots,g^{(k-1)}) \tag{1.2.16}$$

where $i,j \le k$ and m_B is a monomial in $f',\cdots,f^{(k-1)},g',\cdots,g^{(k-1)}$. We denote such a term by B.

Now we want to find an L^2 bound of each term in (1.2.14). For the term A, by the mean value theorem we have

$$[\hat{e}_n^{(j)}(id+f)-\hat{e}_n^{(j)}(id+g)]=\hat{e}_n^{(j+1)}(id+\xi)(f-g).$$

By Definition (1.2.3) of $\hat{e}_n$, and using Item (1) and (2) in Theorem (1.2.7), we have

$$\begin{aligned}\|A\|_{L^2} &\le \|\hat{e}_n^{(j+1)}(id+\xi)\|_{L^\infty}\|m_A\|_{L^\infty}\|f-g\|_{L^2}\\ &\le \lambda(n)|n|^{k+1}c_k^kN^k\|f-g\|_{H^k}.\end{aligned} \tag{1.2.17}$$

For the term B, we have

$$\begin{aligned}\|B\|_{L^2} &\le \|\hat{e}_n^{(i)}(id+g)\|_{L^\infty}\|m_B\|_{L^\infty}\|f^{(j)}-g^{(j)}\|_{L^2}\\ &\le \lambda(n)|n|^{k}c_k^kN^k\|f-g\|_{H^k}.\end{aligned} \tag{1.2.18}$$

For the last two terms in expression (1.2.14), using Item (1) and (2) in Theorem (1.2.7) again, we have

$$\begin{aligned}&\|[\hat{e}_n'(id+f)-\hat{e}_n'(id+g)]f^{(k)}\|_{L^2}\\ &=\|\hat{e}_n''(id+\xi)(f-g)f^{(k)}\|_{L^2}\le\|\hat{e}_n''(id+\xi)\|_{L^\infty}\|f-g\|_{L^\infty}\|f^{(k)}\|_{L^2}\\ &\le\|\hat{e}_n''(id+\xi)\|_{L^\infty}c_k\|f-g\|_{H^k}\|f\|_{H^k}\le\lambda(n)|n|^2c_kN\|f-g\|_{H^k}\end{aligned} \tag{1.2.19}$$

and

$$\|\hat{e}_n'(id+g)[f^{(k)}-g^{(k)}]\|_{L^2}\le\lambda(n)|n|\|f-g\|_{H^k}. \tag{1.2.20}$$

By (1.2.17–1.2.20), we see that $\lambda(n)|n|^{k+1}c_k^kN^k\|f-g\|_{H^k}$ is a common L^2 bound for all terms in (1.2.14). So,

$$\|\hat{e}_n(id+f)^{(k)}-\hat{e}_n(id+g)^{(k)}\|_{L^2}\le K\lambda(n)|n|^{k+1}c_k^kN^k\|f-g\|_{H^k} \tag{1.2.21}$$

where K is the number of terms in expression (1.2.14), which depends on k but does not depend on n.

Finally,

$$
\begin{aligned}
&\|\Phi(f)-\Phi(g)\|^2_{L^2(H_\lambda,H^k)}\\
&\leq \sum_{n\in\mathbb{Z}} \lambda(n)^2|n|^2\|f-g\|^2_{H^k}+K^2\lambda(n)^2|n|^{2k+2}c_k^{2k}R^{2k}\|f-g\|^2_{H^k}\\
&\leq Kc_k^kR^k\|f-g\|_{H^k}\left(\sum_{n\in\mathbb{Z}}\lambda(n)^2|n|^{2k+2}\right)^{1/2}
\end{aligned}
$$

Let

$$
C_R=\left(\sum_{n\in\mathbb{Z}}\lambda(n)^2|n|^2+K^2\lambda(n)^2|n|^{2k+2}c_k^{2k}R^{2k}\right)^{1/2},
$$

Because $\lambda(n)$ is rapidly decreasing (Definition 1.2.3), $\sum_{n\in\mathbb{Z}}\lambda(n)^2|n|^{2k}<\infty$. So C_R is a finite number that depends on R and k. Therefore,

$$
\|\Phi(f)-\Phi(g)\|_{L^2(H_\lambda,H^k)}\leq C_R\|f-g\|_{H^k} \tag{1.2.22}
$$

□

By the above proposition, $\Phi : C^k \to L^2(H_\lambda, H^k)$ is locally Lipschitz. So Φ is uniformly continuous on C^k. But C^k is a dense subspace of H^k (see subsection 2.3). Therefore, we can extend the domain of Φ from C^k to H^k, and obtain a mapping $\Phi : H^k \to L^2(H_\lambda, H^k)$. Similarly, we can also extend the domain of $\widetilde{\Phi}$ from $\widetilde{C}^k$ to $\widetilde{H}^k$, and obtain a mapping $\widetilde{\Phi} : \widetilde{H}^k \to L^2(H_\lambda, H^k)$. After extension, Φ and $\widetilde{\Phi}$ are still locally Lipschitz.

Definition 1.2.13. Define $\widetilde{\Phi} : \widetilde{H}^k \to L^2(H_\lambda, H^k)$ to be the extension of $\widetilde{\Phi} : \widetilde{C}^k \to L^2(H_\lambda, H^k)$ from $\widetilde{C}^k$ to $\widetilde{H}^k$, and $\Phi : H^k \to L^2(H_\lambda, H^k)$ to be the extension of $\Phi : C^k \to L^2(H_\lambda, H^k)$ from C^k to H^k. By the remark in the previous paragraph, Φ and $\widetilde{\Phi}$ are still locally Lipschitz.

1.3 A Brownian motion on Diff(S^1)

In this section, we fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ equipped with a filtration $\mathscr{F}_* = \{\mathscr{F}_t, t \geq 0\}$ that is right continuous and such that each $\mathscr{F}_t$ is complete with respect to $\mathbb{P}$.

Equation (1.1.4) is now interpreted as a Stratonovich stochastic differential equation on $\widetilde{H}^k$ for each $k = 0, 1, 2, \cdots$. Let us fix such a k.

1.3.1 Changing Equation (1.1.4) into the Itô form

To solve Equation (1.1.4), we first need to change it into the Itô form. Here we follow the treatment of S. Fang in [8]. In Definition 1.2.4, $W_t = \sum_{n\in\mathbb{Z}} B_t^{(n)} \hat{e}_n^{(\alpha)}$, where α is a rapidly decreasing *even* function as described in Definition 1.2.3. Using the definition of $\widetilde{\Phi}$, W_t, and $\hat{e}_n^{(\alpha)}$, we can write

Equation (1.1.4) as

$$\delta\widetilde{X}_t = \alpha(0) + \sum_{n=1}^{\infty} \alpha(n)\cos(n\widetilde{X}_t)\delta B_t^{(n)} + \sum_{m=-1}^{-\infty} \alpha(m)\sin(-m\widetilde{X}_t)\delta B_t^{(m)}. \tag{1.3.1}$$

Using the stochastic contraction of $dB_t^{(n)} \cdot dB_t^{(m)} = \delta_{mn}dt$ for $m,n \in \mathbb{Z}$, we have

$$d\cos(n\widetilde{X}_t) \cdot dB_t^{(n)} = -\alpha(n) \cdot n \cdot \sin(n\widetilde{X}_t)\cos(n\widetilde{X}_t)dt, \quad n = 1,2,\cdots$$
$$d\sin(-m\widetilde{X}_t) \cdot dB_t^{(m)} = \alpha(m)(-m)\sin(-m\widetilde{X}_t)\cos(-m\widetilde{X}_t)dt, \quad m = -1,-2,\cdots$$

So the stochastic contraction of the right hand side of (1.3.1) is zero because α is an even function. Therefore Equation (1.3.1) can be written in the following Itô form:

$$d\widetilde{X}_t = \alpha(0) + \sum_{n=1}^{\infty} \alpha(n)\cos(n\widetilde{X}_t)dB_t^{(n)} + \sum_{m=-1}^{-\infty} \alpha(m)\sin(-m\widetilde{X}_t)dB_t^{(m)} \tag{1.3.2}$$

Using the definition of W_t and $\widetilde{\Phi}$ again, Equation (1.3.2) becomes

$$d\widetilde{X}_t = \widetilde{\Phi}(\widetilde{X}_t)dW_t$$

Therefore, Equation (1.1.4) is equivalent to the following Itô stochastic differential equation

$$d\widetilde{X}_t = \widetilde{\Phi}(\widetilde{X}_t)dW_t, \quad \widetilde{X}_0 = id \tag{1.3.3}$$

This equation is considered in the affine space $\widetilde{H}^k$.

If we write $\widetilde{X}_t = id + X_t$ with X_t a process with values in the Sobolev space H^k and use the definition of Φ (see subsection 2.4), Equation (1.3.3) is equivalent to the following equation

$$dX_t = \Phi(X_t)dW_t, \quad X_0 = 0 \tag{1.3.4}$$

This equation is considered in the Sobolev space H^k.

1.3.2 Truncated stochastic differential equation

By Proposition (1.2.12) the operator Φ is locally Lipschitz. To use G. DaPrato and J. Zabczyk's theory [7], we need to "truncate" the operator Φ: Let $R > 0$. Let $\Phi_R : H^k \to L^2(H_\alpha, H^k)$ be defined by

$$\Phi_R(x) = \begin{cases} \Phi(x), & \|x\|_{H^k} \le R \\ \Phi(Rx/\|x\|_{H^k}), & \|x\|_{H^k} > R \end{cases} \tag{1.3.5}$$

Then Φ_R is globally Lipschitz. Let us consider the following "truncated" stochastic differential equation

$$dX_t = \Phi_R(X_t)dW_t, \quad X_0 = 0 \tag{1.3.6}$$

in the Sobolev space H^k. The following defintion is in accordance with G. DaPrato and J. Zabczyk's treatments (p.182 in [7]).

Definition 1.3.1. Let $T > 0$. An $\mathscr{F}_*$-adapted H^k-valued process X_t with continuous sample paths is said to be a mild solution to Equation (1.3.6) up to time T if

$$\int_0^T \|X_s\|_{H^k}^2 ds < \infty, \qquad \mathbb{P}\text{-a.s.}$$

and for all $t \in [0,T]$, we have

$$X_t = X_0 + \int_0^t \Phi_R(X_s) dW_s, \qquad \mathbb{P}\text{-a.s.}$$

For Equation (1.3.6), a strong solution is the same as a mild solution. The solution X_t is said to be unique up to time T if for any other solution Y_t, the two processes X_t and Y_t are equivalent up to time T, that is, the stopped processes $X_{t\wedge T}$ and $Y_{t\wedge T}$ are equivalent.

Remark 1.3.2. In the above definition, we require a solution to have continuous sample paths.

Proposition 1.3.3. *For each $T > 0$, there is a unique solution $X^{(T)}$ to Equation (1.3.6) up to time T.*

Proof. The proof is a simple application of Theorem 7.4, p.186 from [7]. We need to check the conditions to use Theorem 7.4 from [7]. By definition of Φ_R, we see that Φ_R satisfies the following growth condition:

$$\|\Phi_R(x)\|_{L^2(H_\alpha,H^k)}^2 \le C(1+\|x\|_{H^k}^2), \qquad x \in H^k$$

for some constant C. All other conditions to use Theorem 7.4 from [7] are easily verified. Therefore, we have the conclusion. □

Let us choose a sequence $\{T_n\}_{n=1}^\infty$ such that $T_n \uparrow \infty$, and let each $X^{(T_n)}$ be the unique solution to Equation (1.3.6) up to time T_n. By the uniqueness of the solution, and by the continuity of sample paths, for $1 \le i < j$, the sample paths of $X^{(T_j)}$ coincide with the sample paths of $X^{(T_i)}$ up to time T_i almost surely. To be precise, we have, for almost all $\omega \in \Omega$,

$$X^{(T_j)}(t,\omega) = X^{(T_i)}(t,\omega), \qquad \text{for all } t \in [0,T_i]$$

Therefore, we can extend the sample paths to obtain a process X^R: For almost all $\omega \in \Omega$, let

$$X^R(t,\omega) = \lim_{n\to\infty} X^{(T_n)}(t,\omega) \qquad \text{for all } t \in [0,\infty)$$

Then the process X^R is a unique solution with continuous sample paths to Equation (1.3.6) up to time T for all $T > 0$.

Remark 1.3.4. The above construction of the process X^R is independent of the choice of the sequence $\{T_n\}_{n=1}^\infty$: Let $\{S_n\}_{n=1}^\infty$ be another sequence such that $S_n \uparrow \infty$. Let Y^R be the process contructed as above but using the sequence $\{S_n\}_{n=1}^\infty$. Then X^R and Y^R are equivalent up to T for all $T > 0$. Therefore, they are equivalent.

Definition 1.3.5. For every $R > 0$, we define X^R to be the H^k-valued process with continuous sample paths as constructed above. Define

$$\tau_R = \inf\{t : \|X^R(t)\|_{H^k} \geq R\} \tag{1.3.7}$$

1.3.3 Solutions up to stopping times

Let us consider Equation (1.3.4) in the Sobolev space H^k. The following definition is in accordance with E. Hsu's treatments in [16].

Definition 1.3.6. Let τ be an $\mathscr{F}_*$-stopping time. An $\mathscr{F}_*$-adapted process X_t with continuous sample paths is said to be a solution to Equation (1.3.4) up to time τ if for all $t \geq 0$

$$X_{t\wedge\tau} = X_0 + \int_0^{t\wedge\tau} \Phi(X_s)dW_s$$

The solution X_t is said to be unique up to τ if for any other solution Y_t, the two processes X_t and Y_t are equivalent up to τ, that is, the stopped processes $X_{t\wedge\tau}$ and $Y_{t\wedge\tau}$ are equivalent.

Remark 1.3.7. We can similarly define an $\widetilde{H}^k$-valued process being the unique solution to Equation (1.3.3) up to a stopping time τ. Clearly, we have the following: If X_t is the solution to Equation (1.3.4) up to a stopping time τ, then the $\widetilde{H}^k$-valued process $\widetilde{X}_t = id + X_t$ is the solution to Equation (1.3.3) up to time τ and vice versa.

Remark 1.3.8. If X_t is a solution to Equation (1.3.4) up to τ, then it is also a solution up to σ for any $\mathscr{F}_*$-stopping time σ such that $\sigma \leq \tau$ a.s.

Proposition 1.3.9. *Let $R > 0$. Let X^R and τ_R be defined as in Definition (1.3.5). Then X^R is the unique solution to Equation (1.3.4) up to τ_R.*

Proof. Because X^R is the unique solution to Equation (1.3.6) up to T for all $T > 0$, we have

$$X_t^R = \int_0^t \Phi_R(X_s^R)dW_s$$

for all $t \geq 0$. By the definition of Φ_R, we have $\Phi_R(X_s^R) = \Phi(X_s^R)$ for $s \leq \tau_R$. So,

$$X_{t\wedge\tau_R}^R = \int_0^{t\wedge\tau_R} \Phi_R(X_s^R)dW_s = \int_0^{t\wedge\tau_R} \Phi(X_s^R)dW_s$$

Therefore, X^R is a solution to Equation (1.3.4) up to τ_R.

Suppose Y_t is another solution to Equation (1.3.4) up to τ_R. Then Y_t is also a solution to Equation (1.3.6) up to τ_R. But X_t^R is the unique solution to Equation (1.3.6) up to T for all $T > 0$. Therefore, Y_t and X_t^R are equivalent up to τ_R. $\square$

Let us choose a sequence $\{R_n\}_{n=1}^\infty$ such that $R_n \uparrow \infty$, and let X^{R_n} and τ_{R_n} be defined as in Definition (1.3.5). For $1 \leq i < j$, we have $\Phi_{R_i}(x) = \Phi_{R_j}(x)$ for $\|x\|_{H^k} \leq R_i$. Thus, X^{R_j} is also a solution to

Equation (1.3.4 up to τ_{R_i}). Therefore, by the uniqueness of solution and by the continuity of sample paths of solution, the sample paths of X^{R_j} coincide with the sample paths of X^{R_i} almost surely. To be precise, we have, for almost all $\omega \in \Omega$,

$$X^{R_j}(t,\omega) = X^{R_i}(t,\omega), \quad \text{for all } t \in [0, \tau_{R_i}(\omega)]$$

Consequently, $\{\tau_{R_n}\}_{n=1}^{\infty}$ is an increasing sequence of stopping times. Let

$$\tau_\infty = \lim_{n\to\infty} \tau_{R_n} \tag{1.3.8}$$

Now we can extend the sample paths of X^{R_n} to obtain a process X^∞: For almost all $\omega \in \Omega$, let

$$X^\infty(t,\omega) = \lim_{n\to\infty} X^{R_n}(t,\omega) \quad \text{for all } 0 \le t < \tau_\infty(\omega)$$

Then the process X^∞ is a unique solution with continuous sample paths to Equation (1.3.4) up to time τ_R for all $R > 0$. Also, the stopping time τ_R defined in Definition (1.3.5) is realized by the process X^∞:

$$\tau_R = \inf\{t : \|X^\infty(t)\|_{H^k} \ge R\}$$

Remark 1.3.10. The above constructions of the process X^∞ and the stopping time τ_∞ are independent of the choice of the sequence $\{R_n\}_{n=1}^{\infty}$: Let $\{S_n\}_{n=1}^{\infty}$ be another sequence such that $S_n \uparrow \infty$. Let σ_∞ be the stopping time and Y^∞ be the process contructed as above but using the sequence $\{S_n\}_{n=1}^{\infty}$. First, we can combine the two sequences $\{R_n\}_{n=1}^{\infty}$ and $\{S_n\}_{n=1}^{\infty}$ to form a new sequence $\{K_n\}_{n=1}^{\infty}$ such that $K_n \uparrow \infty$. Let γ_∞ be the stopping time constructed as above but using the sequence $\{K_n\}_{n=1}^{\infty}$. Then $\tau_\infty = \sigma_\infty = \gamma_\infty$. Also, X^∞ and Y^∞ are equivalent up to τ_{R_n} and τ_{S_n} for all $n = 1, 2, \cdots$. Therefore, they are equivalent up to τ_∞.

Definition 1.3.11. We define X^∞ to be the H^k-valued process and τ_∞ to be the stopping time as constructed above. We call τ_∞ the explosion time of the process X^∞. We also define the $\widetilde{H}^k$-valued process $\widetilde{X}^\infty$ to be $\widetilde{X}^\infty = id + X^\infty$.

We can slightly extend Definition (1.3.6) and make the following definition:

Definition 1.3.12. Let τ be an $\mathscr{F}_*$-stopping time. An $\mathscr{F}_*$-adapted process X_t with continuous sample paths is said to be a solution to Equation (1.3.4) up to time τ if there is an increasing sequence of $\mathscr{F}_*$-stopping time $\{\tau_n\}_{n=1}^{\infty}$ such that $\tau_n \uparrow \tau$ and X_t is a solution to Equation (1.3.4) up to time τ_n in the sense of Definition (1.3.6) for all $n = 1, 2, \cdots$. The solution X_t is said to be unique up to τ if it is unique up to τ_n for all $n = 1, 2, \cdots$.

We have proved the following proposition:

Proposition 1.3.13. *Let k be a non-negative integer. The process X^∞ as defined in Definition (1.3.11) is the unique solution with continuous sample paths to Equation (1.3.4) up to the explosion time τ_∞.*

1.3.4 The main result

In this subsection, we will prove that the explosion time τ_∞ defined in Definition (1.3.11) is infinity almost surely. We will also prove that the process $\widetilde{X}^\infty$ defined in Definition (1.3.11) lives in the group Diff(S^1). The key idea to both proofs is the following proposition:

Proposition 1.3.14. *Let $\widetilde{X}_t$ be an $\mathscr{F}_*$-adapted $\widetilde{H}^k$-valued process with continuous sample paths and τ an $\mathscr{F}_*$-stopping time. If $\widetilde{X}_t$ is a solution to*

$$d\widetilde{X}_t = \widetilde{\Phi}(\widetilde{X}_t)dW_t, \quad \widetilde{X}_0 = id$$

up to τ, then $\widetilde{X}_t \circ \tilde{\xi}$ is a solution to

$$d\widetilde{X}_t = \widetilde{\Phi}(\widetilde{X}_t)dW_t, \quad \widetilde{X}_0 = \tilde{\xi}$$

up to τ, where $\tilde{\xi}$ is a bounded $\widetilde{H}^k$-valued random variable and "$\circ$" is the composition of two functions.

Proof. By assumption

$$\widetilde{X}_{t\wedge\tau} = id + \int_0^{t\wedge\tau} \widetilde{\Phi}(\widetilde{X}_s)dW_s$$

By definition of the operator $\widetilde{\Phi}$ (see subsection 2.4), this can be written as

$$\widetilde{X}_{t\wedge\tau} = id + \int_0^{t\wedge\tau} dW_s \circ \widetilde{X}_s$$

So

$$\widetilde{X}_{t\wedge\tau} \circ \tilde{\xi} = \tilde{\xi} + \int_0^{t\wedge\tau} dW_s \circ \widetilde{X}_s \circ \tilde{\xi}$$

that is

$$\widetilde{X}_{t\wedge\tau} \circ \tilde{\xi} = \tilde{\xi} + \int_0^{t\wedge\tau} \widetilde{\Phi}(\widetilde{X}_s \circ \tilde{\xi})dW_s$$

Therefore, $\widetilde{X}_t \circ \tilde{\xi}$ is a solution to

$$d\widetilde{X}_t = \widetilde{\Phi}(\widetilde{X}_t)dW_t, \quad \widetilde{X}_0 = \tilde{\xi}$$

up to τ. □

Remark 1.3.15. (**Concatenating procedure**.) Let $R > 0$. Let $\tilde{\xi} = \widetilde{X}^\infty(\tau_R)$. Then $\tilde{\xi}$ is an $\widetilde{H}^k$-valued bounded random variable. Let $W_t' = W_{t+\tau_R} - W_{\tau_R}$. Similar to the construction of X^∞ and $\widetilde{X}^\infty$, we can construct Y^∞ and $\widetilde{Y}^\infty$, where $\widetilde{Y}^\infty$ is a solution to the following equation

$$d\widetilde{X}_t = \widetilde{\Phi}(\widetilde{X}_t)dW_t', \quad \widetilde{X}_0 = id$$

up to the stopping time

$$\tau_R' = \inf\{t : \|Y^\infty(t)\|_{H^k} \geq R\}$$

By the strong Markov property of the Brownian motion W_t, we have $W_t' = W_t$ in distribution, and they are independent of each other for all $t \geq 0$. Therefore, $\tau_R = \tau_R'$ in distribution, and they are independent of each other. By Proposition (1.3.14), $\widetilde{Y}^\infty \circ \tilde{\xi}$ is the solution up to time τ_R' to the following equation

$$d\widetilde{X}_t = \widetilde{\Phi}(\widetilde{X}_t)dW_t', \quad \widetilde{X}_0 = \tilde{\xi}$$

Because $\tilde{\xi} = \widetilde{X}^\infty(\tau_R)$, we can *concatenate* the two processes $\widetilde{X}^\infty$ and $\widetilde{Y}^\infty$ to form a new process $\widetilde{Z}^\infty$ as follows:

$$\widetilde{Z}_t^\infty = \begin{cases} \widetilde{X}_t^\infty, & \text{for } t \leq \tau_R \\ \widetilde{Y}_{t-\tau_R}^\infty \circ \tilde{\xi}, & \text{for } t > \tau_R \end{cases} \tag{1.3.9}$$

By the choice of W_t', we see that the process $\widetilde{Z}^\infty$ is a solution to Equation (1.3.3) up to time $\tau_R + \tau_R'$. By the uniqueness of solution, $\widetilde{Z}^\infty$ is equivalent to $\widetilde{X}^\infty$ up to time $\tau_R + \tau_R'$.

We can carry out this "concatenating" procedure over and over again. Thus, for any $n \in \mathbb{N}$, we can construct a process $\widetilde{Z}^\infty$ which is a solutionn to Equation (1.3.3) and is equivalent to $\widetilde{X}^\infty$ up to time $\tau_R + \tau_R' + \cdots + \tau_R^{(n)}$ with $\tau_R, \tau_R', \cdots$ being identical in distribution and mutually independent with each other.

Proposition 1.3.16. *Let τ_∞ be the explosion time of the process X^∞ defined as in Definition (1.3.11). Then $\tau_\infty = \infty$ almost surely.*

Proof. We can carry out the above "concatenating" procedure as many times as we want. Thus, for any $n \in \mathbb{N}$, we can construct a process $\widetilde{Z}^\infty$ which is a solutionn to Equation (1.3.3) and is equivalent to $\widetilde{X}^\infty$ up to time $\tau_R + \tau_R' + \cdots + \tau_R^{(n)}$.

By the triangle inequality in H^k, we have

$$\tau_R + \tau_R' + \cdots + \tau_R^{(n)} \leq \tau_{nR} \leq \tau_\infty,$$

On the other hand, because $\tau_R, \tau_R', \cdots$ have the same distributions and are mutually independent with each other,

$$\lim_{n\to\infty} \tau_R + \tau_R' + \cdots + \tau_R^{(n)} = \infty \quad \text{a.s.}$$

Therefore, the explosion time $\tau_\infty = \infty$ almost surely. □

Proposition 1.3.17. *Let X^∞ be the H^k-valued process defined in Defintion (1.3.11). Then X^∞ actually lives in the space diff(S^1).*

Proof. The construction of X^∞ in subsection 3.3 is for a fixed k. But the method is valid for all $k = 0, 1, 2, \cdots$. Let us denote by $X^{k,\infty}$ the H^k-valued process as constructed in subsection 3.3. Because Equation (1.3.4) takes the same form in each space H^k, $k = 0, 1, 2, \cdots$, also, $H^{k+1} \subseteq H^k$, we see that the H^{k+1}-valued process $X^{k+1,\infty}$ is also a solution to Equation (1.3.4) in the space H^k. By uniqueness of the solution, $X^{k+1,\infty}$ is equivalent to $X^{k,\infty}$. Therefore, we can also say the solution $X^{k,\infty}$ to Equation (1.3.4) in the space H^k is also the solution to Equation (1.3.4 in the space H^{k+1}). By induction, the solution $X^{k,\infty}$ actually lives in H^{k+i} for all $i = 0, 1, 2, \cdots$. Therefore it lives in $\bigcap_{i=0}^\infty H^{k+i} = \text{diff}(S^1)$. □

By the above proposition, the $\widetilde{H}^k$-valued process $\widetilde{X}^\infty$ lives in the affine space $\widetilde{\text{diff}}(S^1)$. In the next proposition we will prove that $\widetilde{X}^\infty$ actually lives in the group $\text{Diff}(S^1)$. The key to the proof is Proposition (1.2.2) together with the "concatenating" procedure (remark 1.3.15).

Proposition 1.3.18. *The process $\widetilde{X}^\infty$ defined in Definition (1.3.11) lives in the group Diff(S^1).*

Proof. Let us fix a $k \geq 2$. Suppose $\tilde{f} \in \widetilde{H}^k$. By item (2) in Theorem 2.5, $\|f'\|_{L^\infty} \leq c_k \|f\|_{H^k}$. Thus, by controling the H^k-norm of f we can control the L^∞-norm of f'. When $\|f'\|_{L^\infty} < 1$, we have $f' > -1$, or equivalently, $\tilde{f}' > 0$. If we also know that $\tilde{f}$ is C^∞, then by Proposition (1.2.2), we can conclude that $\tilde{f}$ is actually a diffeomorphism of S^1. The process X^∞ has values in the R-ball

$$B(0,R) = \{x \in H^k : \|x\|_{H^k} \leq R\}$$

up to time τ_R. Let us choose R so that $f \in B(0,R)$ implies $\|f'\|_{L^\infty} < 1$. Then up to τ_R, the first derivative $\|X^\infty(t,\omega)^{(1)}\|_{L^\infty} < 1$ almost surely. So up to τ_R, $X^\infty(t,\omega)^{(1)} > -1$, or equivalently $\widetilde{X}^\infty(t,\omega)^{(1)} > 0$ almost surely. Also by Proposition (1.3.17), $\widetilde{X}^\infty$ lives in the affine space $\widetilde{\text{diff}}(S^1)$, which means: every element $\widetilde{X}^\infty(t,\omega)$ is C^∞. Therefore, by Proposition (1.2.2), $\widetilde{X}^\infty$ lives in the group $\text{Diff}(S^1)$ up to time τ_R.

In the "concatenating" procedure (see remark 3.13), the process $\widetilde{Y}^\infty$ lives in the group $\text{Diff}(S^1)$ up to time τ'_R for the same reason. Because $\xi = \widetilde{X}^\infty(\tau_R)$, it is now a $\text{Diff}(S^1)$-valued random variable. So the composition $\widetilde{Y}^\infty \circ \tilde{\xi}$ lives in $\text{Diff}(S^1)$ up to time τ'_R. By concatenation, the process $\widetilde{Z}^\infty$ lives in $\text{Diff}(S^1)$ up to time $\tau_R + \tau'_R$. Because $\widetilde{X}^\infty$ is equivalent to $\widetilde{Z}^\infty$ up to time $\tau_R + \tau'_R$, we have the process $\widetilde{X}^\infty$ lives in $\text{Diff}(S^1)$ up to time $\tau_R + \tau'_R$. We can carry out this "concatenating" procedure over and over again. Therefore, the process $\widetilde{X}^\infty$ lives in $\text{Diff}(S^1)$ up to the explosion time τ_∞ which is infinity by Proposition (1.3.16). □

Putting together Propositions (1.3.13), (1.3.16) and (1.3.18), we have proved the main result of this chapter:

Theorem 1.3.19. *There is a unique $\widetilde{H}^k$-valued solution with continuous sample paths to Equation (1.3.3) for all $k = 0,1,2,\cdots$. Furthermore, the solution is non-explosive and lives in the group Diff(S^1).*

Remark 1.3.20. The solution in the above theorem is the Brownian motion on the group $\text{Diff}(S^1)$ that we are seeking for.

Chapter 2

Stochastic Analysis of Infinite-dimensional Symplectic Group $\mathrm{Sp}(\infty)$

2.1 Introduction

The group $\mathrm{Sp}(\infty)$ arises from the study of the group $\mathrm{Diff}(S^1)$. It was first defined by G. Segal [24], and was further studied by H. Airault and P. Malliavin in [3]. Roughly speaking, $\mathrm{Sp}(\infty)$ is the symplectic representation group of $\mathrm{Diff}(S^1)$ on a certain infinite-dimensional complex vector space equipped with a symplectic and inner product structure. There are some extra requirements in the definition of $\mathrm{Sp}(\infty)$. The intention is to make the group $\mathrm{Sp}(\infty)$ as small as possible. Ideally, if the group $\mathrm{Sp}(\infty)$ is isomorphic to the group $\mathrm{Diff}(S^1)$, then the study of $\mathrm{Diff}(S^1)$ will be exactly the same as the study of $\mathrm{Sp}(\infty)$. Unfortunately, we discover that they are not isomorphic with each other (Theorem 2.4.6).

In this chapter, we describe in detail the symplectic representation of $\mathrm{Diff}(S^1)$ which gives an embedding of $\mathrm{Diff}(S^1)$ into $\mathrm{Sp}(\infty)$. One of the main results is Theorem (2.4.6), where we describe the embedding of $\mathrm{Diff}(S^1)$ into $\mathrm{Sp}(\infty)$ and prove that the map is not surjective.

In this chapter, we also construct a Brownian motion on $\mathrm{Sp}(\infty)$ (Theorem 2.6.17). The group $\mathrm{Sp}(\infty)$ can be represented as an infinite-dimensional matrix group. For such matrix groups, the method of[10, 12] can be used to construct a Brownian motion living in the group. The construction relies on the fact that these groups can be embedded into a larger Hilbert space of Hilbert-Schmidt operators. One of the advantages of Hilbert-Schmidt groups is that one can associate an infinite-dimensional Lie algebra to such a group, and this Lie algebra is a Hilbert space. This is not the case with $\mathrm{Diff}(S^1)$, as an infinite-dimensional Lie algebra associated with $\mathrm{Diff}(S^1)$ is not a Hilbert space with respect to the inner product compatible with the symplectic structure on $\mathrm{Diff}(S^1)$.

In the construction of the Brownian motion on $\mathrm{Sp}(\infty)$, in order for the Brownian motion to live in the group $\mathrm{Sp}(\infty)$, we are forced to choose a non-Ad-invariant inner product on the Lie algebra of $\mathrm{Sp}(\infty)$. This fact has a potential implication for this Brownian motion not to be quasi-invariant for the appropriate choice of the Cameron-Martin subgroup of $\mathrm{Sp}(\infty)$. This is in contrast to results in

[2].

The work in this chapter is written in [14] and is published in Communications of Stochastic Analysis.

2.2 The spaces H and $\mathbb{H}_\omega$

Definition 2.2.1. Let H be the space of complex-valued C^∞ functions on the unit circle S^1 with the mean value 0. Define a bilinear form ω on H by

$$\omega(u,v) = \frac{1}{2\pi}\int_0^{2\pi} uv' d\theta, \qquad \text{for any } u,v \in H.$$

Remark 2.2.2. By using integration by parts, we see that the form ω is anti-symmetric, that is, $\omega(u,v) = -\omega(v,u)$ for any $u,v \in H$.

Next we define an inner product $(\cdot,\cdot)_\omega$ on H which is compatible with the form ω. First, we introduce a complex structure on H, that is, a linear map J on H such that $J^2 = -id$. Then the inner product is defined by $(u,v)_\omega = \pm\omega(u,J\bar{v})$, where the sign depends on the choice of J. The complex structure J in this context is called the Hilbert transform.

Definition 2.2.3. Let $\mathbb{H}_0$ be the Hilbert space of complex-valued L^2 functions on S^1 with the mean value 0 equipped with the inner product

$$(u,v) = \frac{1}{2\pi}\int_0^{2\pi} u\bar{v} d\theta, \qquad \text{for any } u,v \in \mathbb{H}_0.$$

Notation 2.2.4. Denote $\hat{e}_n = e^{in\theta}, n \in \mathbb{Z}\backslash\{0\}$, and $\mathscr{B}_H = \{\hat{e}_n,\ n \in \mathbb{Z}\backslash\{0\}\}$. Let $\mathbb{H}^+$ and $\mathbb{H}^-$ be the closed subspaces of $\mathbb{H}_0$ spanned by $\{\hat{e}_n : n > 0\}$ and $\{\hat{e}_n : n < 0\}$, respectively. By π^+ and π^- we denote the projections of $\mathbb{H}_0$ onto subspaces $\mathbb{H}^+$ and $\mathbb{H}^-$, respectively. For $u \in \mathbb{H}_0$, we can write $u = u_+ + u_-$, where $u_+ = \pi^+(u)$ and $u_- = \pi^-(u)$.

Definition 2.2.5. Define the **Hilbert transformation** J on $\mathscr{B}_H$ by

$$J : \hat{e}_n \mapsto i\operatorname{sgn}(n)\hat{e}_n$$

where $\operatorname{sgn}(n)$ is the sign of n, and then extended by linearity to $\mathbb{H}_0$.

Remark 2.2.6. In the above definition, J is defined on the space $\mathbb{H}_0$. We need to address the issue whether it is well–defined on the *subspace* H. That is, if $J(H) \subseteq H$. We will see that if we modify the space H a little bit, for example, if we let $C_0^1(S^1)$ be the space of complex-valued C^1 functions on the circle with mean value zero, then J is *not* well–defined on $C_0^1(S^1)$. This problem really lies in the heart of Fourier analysis. To see this, we need to characterize J by using the Fourier transform.

Notation 2.2.7. For $u \in \mathbb{H}_0$, let $\mathscr{F} : u \mapsto \hat{u}$ be the **Fourier transformation** with $\hat{u}(n) = (u,\hat{e}_n)$. Let $\hat{J}$ be a transformation on $l^2(\mathbb{Z}\backslash\{0\})$ defined by $\left(\hat{J}\hat{u}\right)(n) = i\operatorname{sgn}(n)\hat{u}(n)$ for any $\hat{u} \in l^2(\mathbb{Z}\backslash\{0\})$.

The Fourier transformation $\mathscr{F} : \mathbb{H}_0 \to l^2(\mathbb{Z}\backslash\{0\})$ is an isomorphism of Hilbert spaces, and $J = \mathscr{F}^{-1} \circ \hat{J} \circ \mathscr{F}$.

Proposition 2.2.8. *The Hilbert transformation J is well–defined on H, that is $J(H) \subseteq H$.*

Proof. The key of the proof is the fact that functions in H can be completely characterized by their Fourier coefficients. To be precise, let $u \in \mathbb{H}_0$ be continuous. Then u is C^∞ if and only if $\lim_{n\to\infty} n^k \hat{u}(n) = 0$ for any $k \in \mathbb{N}$. From this fact, it follows immediately that J is well–defined on H, because J only changes the signs of the Fourier coefficients of a function $u \in H$.

For completeness of exposition, we give a proof of this fact. Though the statement is probably a standard fact in the Fourier analysis, we found it proven only in one direction in [18].

We first assume that u is C^∞. Then $u(\theta) = u(0) + \int_0^\theta u'(t)dt$. So

$$\begin{aligned}\hat{u}(n) &= \frac{1}{2\pi}\Big(\int_0^{2\pi}\int_0^{2\pi} u'(t)\chi_{[0,\theta]}dt\Big)e^{-in\theta}d\theta = \frac{1}{2\pi}\int_0^{2\pi}\Big(\int_t^{2\pi} e^{-in\theta}d\theta\Big)u'(t)dt \\ &= -\frac{1}{2\pi in}\int_0^{2\pi} u'(t) - u'(t)e^{-int}dt = \frac{\widehat{u'}(n)}{in},\end{aligned}$$

where we have used Fubini's theorem and the continuity of u'. Now, u' is itself C^∞, so we can apply the procedure again. By induction, we get $\hat{u}(n) = \frac{\widehat{u^{(k)}}(n)}{(in)^k}$. But from the general theory of Fourier analysis, $\widehat{u^{(k)}}(n) \to 0$ as $n \to \infty$. Therefore $n^k \hat{u}(n) \to 0$ as $n \to \infty$.

Conversely, assume u is such that for any k, $n^k \hat{u}(n) \to 0$ as $n \to \infty$. Then the Fourier series of u converges uniformly. Also by assumption that u is continuous, the Fourier series converges to u for all $\theta \in S^1$ (see Corollary I.3.1 in [18]). So we can write $u(\theta) = \sum_{n\neq 0} \hat{u}(n)e^{in\theta}$.

Fix a point $\theta \in S^1$,

$$u'(\theta) = \frac{d}{dt}\Big|_{t=\theta} \sum_{n\neq 0} \hat{u}(n)e^{int} = \lim_{t\to\theta}\lim_{N\to\infty}\sum_{n=-N}^{N} \hat{u}(n)\frac{e^{int} - e^{in\theta}}{t-\theta}.$$

Note that the derivatives of $\cos nt$ and $\sin nt$ are all bounded by $|n|$. So by the mean value theorem, $|\cos nt - \cos n\theta| \leq |n||t-\theta|$, and $|\sin nt - \sin n\theta| \leq |n||t-\theta|$. So

$$\Big|\frac{e^{int} - e^{in\theta}}{t-\theta}\Big| \leq 2|n|, \qquad \text{for any } t, \theta \in S^1.$$

Therefore, by the growth condition on the Fourier coefficients $\hat{u}$, we have

$$\lim_{N\to\infty}\sum_{n=-N}^{N} \hat{u}(n)\frac{e^{int} - e^{in\theta}}{t-\theta}$$

converges at the fixed $\theta \in S^1$ and the convergence is uniform in $t \in S^1$. Therefore we can interchange the two limits, and obtain

$$\Big(\sum_{n\neq 0} \hat{u}(n)e^{in\theta}\Big)' = \sum_{n\neq 0} \hat{u}(n)ine^{in\theta},$$

which means we can differentiate term by term. So the Fourier coefficients of u' are given by $\hat{u'}(n) = in\hat{u}(n)$. Clearly, $\hat{u'}$ satisfies the same condition as $\hat{u}$: $n^k\hat{u'}(n) \to 0$ as $n \to \infty$. By induction, u is j-times differentiable for any j. Therefore, u is C^∞. $\square$

Proposition 2.2.9. *Let $C_0^1(S^1)$ be the space of complex-valued C^1 functions on the circle with the mean value zero. Then the Hilbert transformation J is* not *well defined on $C_0^1(S^1)$, i.e., $J(C_0^1(S^1)) \nsubseteq C_0^1(S^1)$.*

Proof. Let $C(S^1)$ be the space of continuous functions on the circle. In [18], it is shown that there exists a function in $C(S^1)$ such that the corresponding Fourier series does not converges *uniformly* [18, Theorem II.1.3], and therefore there exists an $f \in C(S^1)$ such that $Jf \notin C(S^1)$ [18, Theorem II.1.4]. Now take $u = f - f_0$ where f_0 is the mean value of f. Then u is a continuous function on the circle with the mean value zero, and Ju is *not* continuous.

Using Notation 2.2.4 let us write $u = u_+ + u_-$. Then we can use the relation

$$iu + Ju = 2iu_+ \quad \text{and} \quad iu - Ju = 2iu_-.$$

to see that u_+ and u_- are *not* continuous. Integrating $u = u_+ + u_-$, we have

$$\int_0^t u(\theta)d\theta = \int_0^t u_+(\theta)d\theta + \int_0^t u_-(\theta)d\theta.$$

Denote the three functions in the above equation by v, v_1, v_2. By theorem I.1.6 in [18],

$$\hat{v}(n) = \frac{\hat{u}(n)}{in}, \quad \text{and} \quad \hat{v_1}(n) = \frac{\hat{u}_+(n)}{in}, \hat{v_2}(n) = \frac{1}{in}\hat{u}_-(n) \text{ for } n \neq 0.$$

Let $g = v - v_0$ where v_0 is the mean value of v. Then $g \in C_0^1(S^1)$. Write $g = g_+ + g_-$ 2.2.4. Then $g_+ = v_1 - (v_1)_0$ and $g_- = v_2 - (v_2)_0$ where $(v_1)_0$ and $(v_2)_0$ are the mean values of v_1 and v_2 respectively. Then $g_+, g_- \notin C_0^1(S^1)$ since $v_1' = u_+, v_2' = u_-$ are *not* continuous.

By the relation

$$ig + Jg = 2ig_+ \quad \text{and} \quad ig - Jg = 2ig_-,$$

we see that $Jg \notin C_0^1(S^1)$. $\square$

Notation 2.2.10. Define an $\mathbb{R}$-bilinear form $(\cdot,\cdot)_\omega$ on H by

$$(u,v)_\omega = -\omega(u, J\bar{v}) \qquad \text{for any } u, v \in H.$$

Proposition 2.2.11. *$(\cdot,\cdot)_\omega$ is an inner product on H.*

Proof. We need to check that $(\cdot,\cdot)_\omega$ satisfies the following properties (1) $(\lambda u, v)_\omega = \lambda(u,v)_\omega$ for $\lambda \in \mathbb{C}$; (2) $(v,u)_\omega = \overline{(u,v)_\omega}$; (3) $(u,u)_\omega > 0$ unless $u = 0$.

(1) for $\lambda \in \mathbb{C}$,

$$(\lambda u, v)_\omega = -\omega(\lambda u, J\bar{v}) = -\lambda \cdot \omega(u, J\bar{v}) = \lambda \cdot (u,v)_\omega.$$

To prove (2) and (3), we need some simple facts: $H^+ = \pi^+(H) \subseteq H$ and $H^- = \pi^-(H) \subseteq H$, and $H = H^+ \oplus H^-$. If $u \in H^+, v \in H^-$, then $(u,v) = 0$. If $u \in H^+$, then $\bar{u} \in H^-, Ju = iu, Ju \in H^+$. If $u \in H^-$, then $\bar{u} \in H^+, Ju = -iu, Ju \in H^-$. $J\bar{u} = \overline{Ju}$. $\widehat{u'}(n) = in\hat{u}(n)$. In particular, if $u \in H^+$, then $u' \in H^+$; if $u \in H^-$, then $u' \in H^-$.

(2) By definition,

$$\begin{aligned}(v,u)_\omega &= -\omega(v,J\bar{u}) = \omega(J\bar{u},v) = \frac{1}{2\pi}\int (J\bar{u})v'd\theta \\ \overline{(u,v)_\omega} &= -\overline{\omega(u,J\bar{v})} = \overline{\omega(J\bar{v},u)} = \frac{1}{2\pi}\int \overline{J\bar{v}}\bar{u}'d\theta = \frac{1}{2\pi}\int (Jv)\bar{u}'d\theta.\end{aligned}$$

Write $u = u_+ + u_-$ and $v = v_+ + v_-$ as in Notation 2.2.4. Using the above fact, we can show that the above two quantities are equal to each other.

(3) Write $u = u_+ + u_-$, then

$$(u,u)_\omega = \frac{1}{2\pi}\int (-i\overline{u_+}u'_+ + i\overline{u_-}u'_-)d\theta = \sum_{n\neq 0} |n||\hat{u}(n)|^2.$$

Therefore, $(u,u)_\omega > 0$ unless $u = 0$. $\square$

Definition 2.2.12. Let $\mathbb{H}_\omega$ be the completion of H under the norm $\|\cdot\|_\omega$ induced by the inner product $(\cdot,\cdot)_\omega$. Define

$$\mathscr{B}_\omega = \left\{\tilde{e}_n = \frac{1}{\sqrt{n}}e^{in\theta}, n > 0\right\} \cup \left\{\tilde{e}_n = \frac{1}{i\sqrt{|n|}}e^{in\theta}, n < 0\right\}.$$

Remark 2.2.13. $\mathbb{H}_\omega$ is a Hilbert space. Also the norm $\|\cdot\|_\omega$ induced by the inner product $(\cdot,\cdot)_\omega$ is *strictly* stronger than the norm $\|\cdot\|$ induced by the inner product $(\cdot,\cdot)$. So $\mathbb{H}_\omega$ can be identified as a *proper* subspace of $\mathbb{H}_0$. The inner product $(\cdot,\cdot)_\omega$ or the norm induced by it is sometimes called the $H^{1/2}$ metric or the $H^{1/2}$ norm on the space H.

One can verify that $\mathscr{B}_\omega$ is an orthonormal basis of $\mathbb{H}_\omega$. From the definition of the inner product $(\cdot,\cdot)_\omega$, we have the relation $\omega(u,v) = (u,\overline{Jv})_\omega$ for any $u,v \in H$. This can be used to *extend* the form ω to $\mathbb{H}_\omega$.

Finally, from the non–degeneracy of the inner product $(\cdot,\cdot)_\omega$, we see that the form $\omega(\cdot,\cdot)$ on $\mathbb{H}_\omega$ is also non–degenerate.

2.3 The infinite-dimensional symplectic group Sp(∞)

Definition 2.3.1. Let $B(\mathbb{H}_\omega)$ be the space of **bounded operators** on $\mathbb{H}_\omega$ equipped with the operator norm. For an operator $A \in B(\mathbb{H}_\omega)$,

1. $\bar{A}$ is the *conjugate* of A if $\bar{A}u = \overline{A\bar{u}}$ for any $u \in \mathbb{H}_\omega$.

2. $A^\dagger$ is the *adjoint* of A if $(Au,v)_\omega = (u,A^\dagger v)_\omega$ for any $u,v \in \mathbb{H}_\omega$.

3. $A^T = \bar{A}^\dagger$ is *transpose* of A.

4. $A^\#$ is the *symplectic adjoint* of A if $\omega(Au, v) = \omega(u, A^\# v)$ for any $u, v \in \mathbb{H}_\omega$.

5. A is said to *preserve the form* ω if $\omega(Au, Av) = \omega(u, v)$ for any $u, v \in \mathbb{H}_\omega$.

In the orthonormal basis $\mathscr{B}_\omega$, an operator $A \in B(\mathbb{H}_\omega)$ can be represented by an infinite dimensional matrix, still denoted by A, with (m,n)th entry equal to $A_{m,n} = (A\tilde{e}_n, \tilde{e}_m)_\omega$.

Remark 2.3.2. If we represent an operator $A \in B(\mathbb{H}_\omega)$ by a matrix $\{A_{m,n}\}_{m,n\in\mathbb{Z}\setminus\{0\}}$, the indices m and n are allowed to be both positive and negative following Definition 2.2.12 of $\mathscr{B}_\omega$.

The next proposition collects some simple facts about operations on $B(\mathbb{H}_\omega)$ introduced in Definition 2.3.1.

Proposition 2.3.3. *Let $A, B \in B(\mathbb{H}_\omega)$. Then*

1. $\overline{\tilde{e}_n} = i\tilde{e}_{-n}$, $J\tilde{e}_n = isgn(n)\tilde{e}_n$, $(\tilde{e}_n)' = in\tilde{e}_n$;

2. $(\bar{A})_{m,n} = \overline{A_{-m,-n}}$;

3. $(A^\dagger)_{m,n} = \overline{A_{n,m}}$;

4. $\bar{A}^\dagger = \overline{A^\dagger}$, *and* $(A^T)_{m,n} = A_{-n,-m}$;

5. *if* $A = \bar{A}$, *then* $(A^\#)_{m,n} = sgn(mn)\overline{A_{n,m}}$;

6. $\overline{AB} = \bar{A}\bar{B}$, $(AB)^\dagger = B^\dagger A^\dagger$, $(AB)^T = B^T A^T$, $(AB)^\# = B^\# A^\#$;

7. *If A is invertible, then* $\bar{A}, A^T, A^\dagger, A^\#$ *are all invertible, and* $(\bar{A})^{-1} = \overline{A^{-1}}$, $(A^T)^{-1} = (A^{-1})^T$, $(A^\dagger)^{-1} = (A^{-1})^\dagger$, $(A^\#)^{-1} = (A^{-1})^\#$;

8. $(\pi^+)_{m,n} = \frac{1}{2}(\delta_{mn} + sgn(m)\delta_{mn})$, $(\pi^-)_{m,n} = \frac{1}{2}(\delta_{mn} - sgn(m)\delta_{mn})$, $\overline{\pi^+} = \pi^-$, $\overline{\pi^-} = \pi^+$, $(\pi^+)^T = \pi^-$, $(\pi^-)^T = \pi^+$, $(\pi^+)^\dagger = \pi^+$, $(\pi^-)^\dagger = \pi^-$;

9. $J_{m,n} = isgn(m)\delta_{mn}$, $\bar{J} = J$, $J = i(\pi^+ - \pi^-)$, $J^T = -J$, $J^\dagger = -J$, $J^2 = -id$;

10. $(A^\#)_{m,n} = sgn(mn)A_{-n,-m}$.

Proof. All of these properties can be checked by straight forward calculations. We only prove (10).

$$
\begin{aligned}
(A^\#)_{m,n} &= (A^\#\tilde{e}_n, \tilde{e}_m)_\omega = -\omega(A^\#\tilde{e}_n, J\overline{\tilde{e}_m}) = \omega(J\overline{\tilde{e}_m}, A^\#\tilde{e}_n) \\
&= \omega(AJ\overline{\tilde{e}_m}, \tilde{e}_n) = -\omega(\tilde{e}_n, AJ\overline{\tilde{e}_m}) = -\omega(\tilde{e}_n, J(-J)AJ\overline{\tilde{e}_m}) \\
&= -\omega(\tilde{e}_n, J\overline{(-J\bar{A}J\tilde{e}_m)}),
\end{aligned}
$$

where in the last equality we used property (6), $\overline{AB}=\bar{A}\bar{B}$, and property (9), $\bar{J}=J$, so that $\overline{-J\bar{A}J\tilde{e}_m} = -\bar{J}\bar{\bar{A}}\bar{J}\overline{\tilde{e}_m} = -JAJ\overline{\tilde{e}_m}$. Therefore,

$$
\begin{aligned}
(A^{\#})_{m,n} &= -\omega(\tilde{e}_n, J\overline{(-J\bar{A}J\tilde{e}_m)}) = (\tilde{e}_n, -J\bar{A}J\tilde{e}_m)_\omega = -(\tilde{e}_n, J\bar{A}J\tilde{e}_m)_\omega \\
&= -(J^\dagger\tilde{e}_n, \bar{A}J\tilde{e}_m)_\omega = -(-J\tilde{e}_n, \bar{A}J\tilde{e}_m)_\omega = (i\operatorname{sgn}(n)\tilde{e}_n, \bar{A}i\operatorname{sgn}(m)\tilde{e}_m)_\omega \\
&= \operatorname{sgn}(mn)(\tilde{e}_n, \bar{A}\tilde{e}_m)_\omega = \operatorname{sgn}(mn)\overline{(\bar{A}\tilde{e}_m, \tilde{e}_n)_\omega} = \operatorname{sgn}(mn)\overline{(\bar{A})_{n,m}} \\
&= \operatorname{sgn}(mn)A_{-n,-m}.
\end{aligned}
$$

□

Notation 2.3.4. For $A \in B(\mathbb{H}_\omega)$, let $a = \pi^+A\pi^+$, $b = \pi^+A\pi^-$, $c = \pi^-A\pi^+$, and $d = \pi^-A\pi^-$, where $a : \mathbb{H}_\omega^+ \to \mathbb{H}_\omega^+$, $b : \mathbb{H}_\omega^- \to \mathbb{H}_\omega^+$, $c : \mathbb{H}_\omega^+ \to \mathbb{H}_\omega^-$, $d : \mathbb{H}_\omega^- \to \mathbb{H}_\omega^-$. Then $A = a+b+c+d$ can be represented as the following block matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If $A, B \in B(\mathbb{H}_\omega)$, then the block matrix representation for AB is exactly the multiplication of block matrices for A and B.

Proposition 2.3.5. *Suppose $A \in B(\mathbb{H}_\omega)$ with the matrix $\{A_{m,n}\}_{m,n\in\mathbb{Z}\setminus\{0\}}$. Then the following are equivalent:*

1. *$A = \bar{A}$;*
2. *if $u = \bar{u}$, then $Au = \overline{Au}$;*
3. *$A_{m,n} = \overline{A_{-m,-n}}$ (2.3.2);*
4. *as a block matrix, A has the form $\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$.*

Proof. Equivalence of (1), (3) and (4) follows from Proposition2.3.3 and Notation2.3.4. First we show that (1) is equivalent to (2).

[(1)$\Longrightarrow$(2)]. If $u = \bar{u}$, then $Au = \bar{A}u = \overline{A\bar{u}} = \overline{Au}$.

[(2)$\Longrightarrow$(1)]. Let $u = \tilde{e}_n + \overline{\tilde{e}_n}$, and $v = \tilde{e}_{-n} + \overline{\tilde{e}_{-n}}$. Then u, v are real-valued functions on the circle. Using Proposition 2.3.3 we have $\overline{\tilde{e}_n} = i\tilde{e}_{-n}$, and therefore $Au = \overline{Au}$ and $Av = \overline{Av}$ imply

$$
\begin{aligned}
A\tilde{e}_n + iA\tilde{e}_{-n} &= \overline{A\tilde{e}_n} - i\overline{A\tilde{e}_{-n}} \\
A\tilde{e}_n - iA\tilde{e}_{-n} &= -\overline{A\tilde{e}_n} - i\overline{A\tilde{e}_{-n}}.
\end{aligned}
$$

Solving the above two equations for $A\tilde{e}_n$, we have

$$A\tilde{e}_n = -i\overline{A\tilde{e}_{-n}} = \overline{A\overline{\tilde{e}_n}} = \bar{A}\tilde{e}_n$$

with this being true for any $n \neq 0$, and so $A = \bar{A}$. □

Proposition 2.3.6. *Let $A \in B(\mathbb{H}_\omega)$. The following are equivalent:*

1. *A preserves the form ω;*
2. *$\omega(Au, Av) = \omega(u, v)$ for any $u, v \in \mathbb{H}_\omega$;*
3. *$\omega(A\tilde{e}_m, A\tilde{e}_n) = \omega(\tilde{e}_m, \tilde{e}_n)$ for any $m, n \neq 0$;*
4. *$A^T JA = J$;*
5. *$\sum_{k \neq 0} sgn(mk) A_{k,m} A_{-k,-n} = \delta_{m,n}$ for any $m, n \neq 0$.*

If we further assume that $A = \bar{A}$, then the following two are equivalent to the above:

(I) *$a^T \bar{a} - b^\dagger b = \pi^-$ and $a^T \bar{b} - b^\dagger a = 0$;*

(II) *$\sum_{k \neq 0} sgn(mk) A_{k,m} \overline{A_{k,n}} = \delta_{m,n}$ for any $m, n \neq 0$.*

Proof. Equivalence of (1),(2) and (3) follows directly from Definition 2.3.1. Let us check the equivalency of (2) and (4). First assume that (2) holds. By Remark 2.2.13 we have $\omega(u, v) = (u, J\bar{v})_\omega$, and therefore

$$\omega(Au, Av) = (Au, J\overline{Av})_\omega = (u, A^\dagger J\overline{Av})_\omega.$$

By assumption, $\omega(Au, Av) = \omega(u, v)$ for any $u, v \in \mathbb{H}_\omega$. So by the non-degeneracy of the inner product $(\cdot, \cdot)_\omega$, we have $A^\dagger J\overline{Av} = J\bar{v}$ for any $v \in \mathbb{H}_\omega$. By definition of $\bar{A}$, we have $\overline{Av} = \bar{A}\bar{v}$. So $A^\dagger J\bar{A}\bar{v} = J\bar{v}$ for any $v \in \mathbb{H}_\omega$, or $A^\dagger J\bar{A} = J$. Taking conjugation of both sides and using $\bar{J} = J$, we see that $A^T JA = J$.

Every step above is reversible, therefore we have implication in the other direction as well.

Now we check the equivalency of (3) and (5). First, by Remark 2.2.13 $\omega(u, v) = (u, J\bar{v})_\omega$ and Proposition 2.3.3

$$\omega(\tilde{e}_m, \tilde{e}_n) = (\tilde{e}_m, J\overline{\tilde{e}_n})_\omega = -\mathrm{sgn}(m)\delta_{m,-n}.$$

On the other hand, by the continuity of the form $\omega(\cdot, \cdot)$ in both variables, we have

$$\begin{aligned}\omega(A\tilde{e}_m, A\tilde{e}_n) &= \omega\Big(\sum_k A_{k,m}\tilde{e}_k, \sum_k A_{l,n}\tilde{e}_l\Big)\\ &= \sum_{k,l} A_{k,m}A_{l,n}(-\mathrm{sgn}(k))\delta_{k,-l} = -\sum_k \mathrm{sgn}(k)A_{k,m}A_{-k,n}.\end{aligned}$$

Now assuming $\omega(A\tilde{e}_m, A\tilde{e}_n) = \omega(\tilde{e}_m, \tilde{e}_n)$, we have

$$-\sum_k \mathrm{sgn}(k)A_{k,m}A_{-k,n} = -\mathrm{sgn}(m)\delta_{m,-n}, \text{ for any } m, n \neq 0.$$

By multiplying by $\mathrm{sgn}(m)$ both sides, and replacing $-n$ with n, we get (5). Conversely, note that every step above is reversible, therefore we have implication in the other direction.

We have proved equivalence of (1)-(5). Now assume $A = \bar{A}$. To prove equivalence of (4) and (I), just notice that as block matrices, A, A^T and J have the form

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \quad \begin{pmatrix} a^\dagger & b^T \\ b^\dagger & a^T \end{pmatrix}, \quad \text{and} \quad i\begin{pmatrix} \pi^+ & 0 \\ 0 & -\pi^- \end{pmatrix}.$$

Equivalence of (5) and (II) follows from the relation $A_{-k,-n} = \overline{A_{k,n}}$. □

Proposition 2.3.7. *Let $A \in B(\mathbb{H}_\omega)$. If A preserves the form ω, then the following are equivalent:*

1. *A is invertible.*

2. *$AJA^T = J$.*

3. *A^T preserves the form ω.*

4. *$\sum_k sgn(mk) A_{m,k} A_{-n,-k} = \delta_{m,n}$ for any $m, n \neq 0$.*

If we further assume that $A = \bar{A}$, then the following are equivalent to the above:

(I) *$\bar{a}a^T - \bar{b}b^T = \pi^-$ and $\bar{b}a^\dagger - \bar{a}b^\dagger = 0$.*

(II) *$\sum_k sgn(mk) A_{m,k} \overline{A_{n,k}} = \delta_{m,n}$ for any $m, n \neq 0$.*

Proof. We will use several times the fact that if A preserves ω, then $A^T JA = J$.

[(1)⇒(2)] Multiplying on the left by $(A^T)^{-1}$ and multiplying on the right by A^{-1} both sides, we get $J = (A^T)^{-1} JA^{-1}$, and so $(A^{-1})^T JA^{-1} = J$. Taking inverse of both sides, and using $J^{-1} = -J$, we have $A^T JA = J$.

[(2)⇒(1)] As J is injective, so is $A^T JA$, and therefore A is injective. On the other hand, by assumption $AJA^T = J$. As J is surjective, so AJA^T is surjective too. This implies that A is surjective, and therefore A is invertible.

Equivalence of (2) and (3) follows from $(A^T)^T = A$ and Proposition 2.3.6. Equivalence of (3) and (4) follows directly from Proposition 2.3.6 and the fact that $(A^T)_{m,n} = A_{-n,-m}$.

Now assume that $A = \bar{A}$. Then equivalence of (3) and (I)can be checked by using multiplication of block matrices as in the proof of Proposition 2.3.6. Finally (4) is equivalent to (II) as if $A = \bar{A}$, then $A_{-m,-n} = \overline{A_{m,n}}$. □

Corollary 2.3.8. *Let $A \in B(\mathbb{H}_\omega)$ and $A = \bar{A}$. Then the following are equivalent:*

1. *A preserves the form ω and is invertible;*

2. *$A^\# A = A^\# A = id$;*

Proof. By Proposition 2.3.3

$$(A^\# A)_{m,n} = \sum_{k \neq 0} (A^\#)_{m,k} A_{k,n} = \sum_{k \neq 0} \text{sgn}(mk) A_{k,n} \overline{A_{k,m}},$$
$$(AA^\#)_{m,n} = \sum_{k \neq 0} A_{m,k} (A^\#)_{k,n} = \sum_{k \neq 0} \text{sgn}(nk) A_{m,k} \overline{A_{n,k}}.$$

Therefore, by (II) in Proposition 2.3.6 and (II) in Proposition 2.3.7 we have equivalence. $\square$

Definition 2.3.9. Define a (semi)norm $\|\cdot\|_2$ on $B(\mathbb{H}_\omega)$ such that for $A \in B(\mathbb{H}_\omega)$, $\|A\|_2^2 = \mathrm{Tr}(b^\dagger b) = \|b\|_{\mathrm{HS}}$, where $b = \pi^+ A \pi^-$. That is, the norm $\|A\|_2$ is just the Hilbert-Schmidt norm of the block b.

Definition 2.3.10. An **infinite-dimensional symplectic group** $\mathrm{Sp}(\infty)$ is the set of bounded operators A on H such that

1. A is invertible;
2. $A = \bar{A}$;
3. A preserves the form ω;
4. $\|A\|_2 < \infty$.

Remark 2.3.11. Condition (2) in Definition 2.3.10 says that an element in $\mathrm{Sp}(\infty)$ has the following form:

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$$

Condition (4) in Definition 2.3.10 says that the block b is a Hilbert-Schmidt matrix.

Remark 2.3.12. If A is a bounded operator on H, then A can be extended to a bounded operator on $\mathbb{H}_\omega$. Therefore, we can equivalently define $\mathrm{Sp}(\infty)$ to be the set of operators $A \in B(\mathbb{H}_\omega)$ such that

1. A is invertible;
2. $A = \bar{A}$;
3. A preserves the form ω;
4. $\|A\|_2 < \infty$.
5. A is invariant on H, i.e., $A(H) \subseteq H$.

Remark 2.3.13. By Corollary 2.3.8, the definition of $\mathrm{Sp}(\infty)$ is also equivalent to

1. $A = \bar{A}$;
2. $A^\# A = AA^\# = id$;
3. $\|A\|_2 < \infty$.

Proposition 2.3.14. *$Sp(\infty)$ is a group.*

Proof. First we show that if $A \in \mathrm{Sp}(\infty)$, then $A^{-1} \in \mathrm{Sp}(\infty)$. By the assumption on A, it is easy to verify that A^{-1} satisfies (1), (2), (3) and (5) in Remark 2.3.12. We need to show that A^{-1} satisfies the condition (4), i.e. $\|A^{-1}\|_2 < \infty$. Suppose

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \quad \text{and} \quad A^{-1} = \begin{pmatrix} a' & b' \\ \overline{b'} & \overline{a'} \end{pmatrix},$$

where by our assumptions all blocks are bounded operators, and in addition b is a Hilbert-Schmidt operator. We want to prove b' is also a Hilbert-Schmidt operator. $AA^{-1} = I$ and $A^{-1}A = I$ imply that

$$ab' = -b\overline{a'}, \quad a'a + b'\bar{b} = I.$$

The last equation gives $a'ab' + b'\bar{b}b' = b'$, and so

$$b' = a'ab' + b'\bar{b}b' = -a'b\overline{a'} + b'\bar{b}b'$$

which is a Hilbert-Schmidt operator as b and $\bar{b}$ are Hilbert-Schmidt. Therefore $\|A^{-1}\|_2 < \infty$ and $A^{-1} \in \mathrm{Sp}(\infty)$.

Next we show that if $A, B \in \mathrm{Sp}(\infty)$, then $AB \in \mathrm{Sp}(\infty)$. By the assumption on A and B, it is easy to verify that AB satisfies (1), (2), (3) and (5) in Remark 2.3.12. We need to show that AB satisfies the condition (4), i.e. $\|AB\|_2 < \infty$. Suppose

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \text{ and } B = \begin{pmatrix} c & d \\ \bar{d} & \bar{c} \end{pmatrix},$$

where all blocks are bounded, and $\|b\|_{\mathrm{HS}}, \|d\|_{\mathrm{HS}} < \infty$. Then

$$AB = \begin{pmatrix} ac + b\bar{d} & ad + b\bar{c} \\ \bar{b}c + \bar{a}\bar{d} & \bar{b}d + \bar{a}\bar{c} \end{pmatrix}.$$

Then

$$\|AB\|_2^2 = \|ad + b\bar{c}\|_{\mathrm{HS}} \leqslant \|ad\|_2 + \|b\bar{c}\|_{\mathrm{HS}} < \infty,$$

since both ad and $b\bar{c}$ are Hilbert-Schmidt operators. Therefore $\|AB\|_2 < \infty$ and $AB \in \mathrm{Sp}(\infty)$.

□

2.4 Symplectic Representation of Diff(S^1)

Definition 2.4.1. Let $\mathrm{Diff}(S^1)$ be the group of orientation preserving C^∞ diffeomorphisms of S^1. $\mathrm{Diff}(S^1)$ acts on H as follows

$$(\phi.u)(\theta) = u(\phi^{-1}(\theta)) - \frac{1}{2\pi}\int_0^{2\pi} u(\phi^{-1}(\theta))d\theta.$$

Note that if $u \in H$ is real-valued, then $\phi.u$ is real-valued as well.

Proposition 2.4.2. *The action of Diff(S^1) on H gives a group homomorphism*

$$\Phi : \mathit{Diff}(S^1) \to \mathit{Aut}H$$

defined by $\Phi(\phi)(u) = \phi.u$, for $\phi \in \mathit{Diff}(S^1)$ and $u \in H$, where $\mathit{Aut}H$ is the group of automorphisms on H.

Proof. Let $u \in H$, then $\phi.u$ is a C^∞ function with the mean value 0, and so $\phi.u \in H$. It is also clear that $\phi.(u+v) = \phi.u + \phi.v$ and $\phi.(\lambda u) = \lambda\phi.u$. So Φ is well–defined as a map from $\mathrm{Diff}(S^1)$ to $\mathrm{End}H$, the space of endomorphisms on H. Now let us check that Φ is a group homomorphism. Suppose $\phi, \psi \in \mathrm{Diff}(S^1)$ and $u \in H$, then

$$\begin{aligned}\Phi(\phi\psi)(u)(\theta) &= u\big((\phi\psi)^{-1}(\theta)\big) - \frac{1}{2\pi}\int_0^{2\pi} u\big((\phi\psi)^{-1}(\theta)\big)d\theta \\ &= u\big((\psi^{-1}\phi^{-1})(\theta)\big) - \frac{1}{2\pi}\int_0^{2\pi} u\big((\psi^{-1}\phi^{-1})(\theta)\big)d\theta.\end{aligned}$$

On the other hand,

$$\begin{aligned}\Phi(\phi)\Phi(\psi)(u)(\theta) &= \Phi(\phi)\left[u(\psi^{-1}(\theta)) - \frac{1}{2\pi}\int_0^{2\pi} u(\psi^{-1}(\theta))d\theta\right] \\ &= \Phi(\phi)\left[u(\psi^{-1}(\theta))\right] = u\big((\psi^{-1}\phi^{-1})(\theta)\big) - \frac{1}{2\pi}\int_0^{2\pi} u\big((\psi^{-1}\phi^{-1})(\theta)\big)d\theta.\end{aligned}$$

So $\Phi(\phi\psi) = \Phi(\phi)\Phi(\psi)$. In particular, the image of Φ is in the $\mathrm{Aut}H$. □

Lemma 2.4.3. *Any $\phi \in$ Diff(S^1) preserves the form ω, that is, $\omega(\phi.u, \phi.v) = \omega(u,v)$ for any $u, v \in H$.*

Proof. By Definition 2.4.1 $\phi.u = u(\psi) - u_0, \phi.v = v(\psi) - v_0$, where $\psi = \phi^{-1}$ and u_0, v_0 are the constants. Then

$$\begin{aligned}\omega(\phi.u, \phi.v) &= \omega(u(\psi) - u_0, v(\psi) - v_0) \\ &= \frac{1}{2\pi}\int_0^{2\pi}\big(u(\psi(\theta)) - u_0\big)\big(v(\psi(\theta)) - v_0\big)'d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} u(\psi)v'(\psi)\psi'(\theta)d\theta - \frac{1}{2\pi}\int_0^{2\pi} u_0 v(\psi(\theta))d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} u(\psi)v'(\psi)d\psi \\ &= \omega(u,v).\end{aligned}$$

□

We are going to prove that a diffeomorphism $\phi \in \mathrm{Diff}(S^1)$ acts on H as a bounded linear map, and that $\Phi(\phi)$ is in $\mathrm{Sp}(\infty)$. The next lemma is a generalization of a proposition in a paper of G. Segal [24].

Lemma 2.4.4. *Let $\psi \neq id \in$ Diff(S^1) and $\phi = \psi^{-1}$. Let*

$$I_{n,m} = (\psi.e^{im\theta}, e^{in\theta}) = \frac{1}{2\pi}\int_0^{2\pi} e^{im\phi - in\theta}d\theta.$$

Then

1. $\sum_{n>0,m<0} |n||I_{n,m}|^2 < \infty$, *and* $\sum_{m>0,n<0} |n||I_{n,m}|^2 < \infty$.

2. For sufficiently large $|m|$ there is a constant C independent of m such that

$$\sum_{n\neq 0} |n||I_{n,m}|^2 < C|m|. \tag{2.4.1}$$

Proof. Let

$$m_{\phi'} = \min\{\phi'(\theta)|\theta \in S^1\}; \quad \text{and} \quad M_{\phi'} = \max\{\phi'(\theta)|\theta \in S^1\}.$$

Since ϕ is a diffeomorphism, we have $0 < m_{\phi'} < M_{\phi'} < \infty$.

Take four points a, b, c, d on the unit circle such that a corresponds to $m_{\phi'}$ in the sense $\tan(a) = m_{\phi'}$, b corresponds to $M_{\phi'}$ in the sense $\tan(b) = M_{\phi'}$, c is opposite to a, i.e., $c = a + \pi$, d is opposite to b, i.e., $d = b + \pi$. The four points on the circle are arranged in the counter-clockwise order, and $0 < a < b < \frac{\pi}{2}$, $\pi < c < d < \frac{3}{2}\pi$.

Let $\tau \in S^1$ such that $\tau \neq \frac{\pi}{4}, \frac{5}{4}\pi$. Define a function ϕ_τ on S^1 by

$$\phi_\tau(\theta) = \frac{\cos\tau \cdot \phi(\theta) - \sin\tau \cdot \theta}{\cos\tau - \sin\tau}.$$

We will show that if $\tau \in (b, c)$ or $\tau \in (d, a)$, then ϕ_τ is an orientation preserving diffeomorphism of S^1, where (b, c) is the open arc from the point b to the point c, and (d, a) is the open arc from the point d to the point a.

Clearly ϕ_τ is a C^∞ function on S^1. Also, $\phi_\tau(0) = 0$ and $\phi_\tau(2\pi) = 2\pi$. Taking derivative with respect to θ, we have

$$\phi'_\tau(\theta) = \frac{\cos\tau \cdot \phi'(\theta) - \sin\tau}{\cos\tau - \sin\tau}.$$

By the choice of τ, we can prove that $\phi'_\tau(\theta) > 0$. Therefore, ϕ_τ is an orientation preserving diffeomorphism as claimed.

Let $m, n \in \mathbb{Z}\backslash\{0\}$. Let $\tau_{mn} = \mathrm{Arg}(m + in)$, i.e., the argument of the complex number $m + in$, considered to be in $[0, 2\pi]$. Then we have $m\phi - n\theta = (m - n)\phi_{\tau_{mn}}$.

If $\tau_{mn} \in (b, c)$, then $\phi_{\tau_{mn}}$ is a diffeomorphism. Let $\psi_{\tau_{mn}} = \phi_{\tau_{mn}}^{-1}$. Then

$$I_{n,m} = \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)\phi_{\tau_{mn}}} d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)\theta} \psi'_{\tau_{mn}}(\theta) d\theta,$$

where the last equality is by change of variable. On integration by parts k times, we have

$$I_{n,m} = \left(\frac{1}{i(m-n)}\right)^k \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)\theta} \psi_{\tau_{mn}}^{(k+1)}(\theta) d\theta.$$

Let $\alpha = [\alpha_0, \alpha_1]$ be a closed arc contained in the arc (b, c). Let S_α be the set of all pairs of nonzero integers (m, n) such that $\alpha_0 < \tau_{mn} < \alpha_1$, where $\tau_{mn} = \mathrm{Arg}(m + in)$. We are going to consider an upper bound of the sum $\sum_{(m,n)\in S_\alpha} |n||I_{n,m}|^2$.

For the pair (m,n), if $|m-n|=p$, the condition $\alpha_0<\tau_{mn}<\alpha_1$ gives us both an upper bound and a lower bound for n:

$$\frac{m_{\phi'}}{m_{\phi'}-1}p \le n \le \frac{M_{\phi'}}{M_{\phi'}-1}p.$$

So $|n|\le C_1 p$ where C_1 is a constant which does not depend on the pair (m,n). Also, the number of pairs $(m,n)\in S_\alpha$ such that $|m-n|=p$ is bounded by $C_2 p$ for some constant C_2. Let $C_3 = \max\left\{|\psi_\tau^{(k+1)}(\theta)| : \theta\in S^1, \tau\in[\alpha_0,\alpha_1]\right\}$. Then

$$|I_{n,m}| \le C_3 \Big|\frac{1}{i(m-n)}\Big|^k \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)\theta} d\theta = C_3 p^{-k}.$$

Therefore,

$$\begin{aligned}\sum_{(m,n)\in S} |n||I_{n,m}|^2 &= \sum_p \sum_{(m,n)\in S_\alpha; |m-n|=p} |n||I_{n,m}|^2 \\ &\le \sum_p C_1 p\cdot C_3^2 p^{-2k}\cdot C_2 p = C_\alpha \sum_p p^{-(2k-2)},\end{aligned}$$

where the constant C_α depends on the arc α.

Similarly, for a closed arc $\beta=[\beta_0,\beta_1]$ contained in the arc (d,a), we have

$$\sum_{(m,n)\in S_\beta} |n||I_{n,m}|^2 \le C_\beta \sum_p p^{-(2k-2)},$$

where the constant C_β depends on the arc β.

Now let $\alpha=[\frac{\pi}{2},\pi]$, and $\beta=[\frac{3}{2}\pi, 2\pi]$. Then α is contained in (b,c) and β is contained in (d,a). We have

$$\sum_{n>0,m<0} |n||I_{n,m}|^2 = C_\alpha\cdot\sum_p p^{-(2k-2)} < \infty$$

and

$$\sum_{n<0,m>0} |n||I_{n,m}|^2 = C_\beta\cdot\sum_p p^{-(2k-2)} < \infty,$$

which proves (1) of the lemma.

To prove (2), we let $\alpha=[\alpha_0,\alpha_1]$ be a closed arc contained in the arc (b,c) such that $b<\alpha_0<\frac{\pi}{2}$ and $\pi<\alpha_1<c$, and $\beta=[\beta_0,\beta_1]$ be a closed arc contained in the arc (d,a) such that $d<\beta_0<\frac{3}{2}\pi$ and $0<\beta_1<a$. Then we have

$$\sum_{(m,n)\in S_\alpha} |n||I_{n,m}|^2 + \sum_{(m,n)\in S_\beta} |n||I_{n,m}|^2 \leqslant C_{\alpha\beta}$$

for some constant $C_{\alpha\beta}$.

Let $m > 0$ be sufficiently large, and N_m be the largest integer less than or equal to $m\tan(\alpha_0)$,

$$\sum_{0<n\leqslant N_m} |I_{n,m}|^2 \leqslant \sum_{n\neq 0} |I_{n,m}|^2.$$

Note that $I_{n,m}$ is the nth Fourier coefficient of $\psi.e^{im\theta}$. Therefore,

$$\sum_{n\neq 0} |I_{n,m}|^2 = \|\psi.e^{im\theta}\|_{L^2}$$

which is bounded by a constant K. Therefore,

$$\sum_{0<n\leqslant N_m} |n||I_{n,m}|^2 \leqslant Km\tan(\alpha_0).$$

On the other hand,

$$\sum_{n<0} |n||I_{n,m}|^2 + \sum_{n>N_m} |n||I_{n,m}|^2 \leqslant \sum_{(m,n)\in S_\alpha} |n||I_{n,m}|^2 + \sum_{(m,n)\in S_\beta} |n||I_{n,m}|^2 = C_{\alpha\beta}.$$

Therefore,

$$\sum_{n\neq 0} |n||I_{n,m}|^2 \leqslant C_{\alpha\beta} + Km\tan(\alpha_0) \leqslant mC_+,$$

where C_+ can be chosen to be, for example, $K\tan(\alpha_0) + C_{\alpha\beta}$, which is independent of m.

Similarly, for $m < 0$ with sufficiently large $|m|$

$$\sum_{n\neq 0} |n||I_{n,m}|^2 \leqslant mC_-.$$

Let $C = \max\{C_+, C_-\}$. Then we have, for sufficiently large $|m|$,

$$\sum_{n\neq 0} |n||I_{n,m}|^2 \leqslant |m|C,$$

which proves (2) of the lemma.

□

Lemma 2.4.5. *For any $\psi \in Diff(S^1)$, $\Phi(\psi) \in B(H)$, the space of bounded linear maps on H. Moreover,*

$$\|\Phi(\psi)\| \leqslant C,\ \|\Phi(\psi)\|_2 \leqslant C,$$

where C is the constant in Equation 2.4.1.

Proof. First observe that the operator norm of $\Phi(\psi)$ is

$$\|\Phi(\psi)\| = \sup\{\|\psi.u\|_\omega \ \mid\ u \in H, \|u\|_\omega = 1\}.$$

For any $u \in H$, let $\hat{u}$ be its Fourier coefficients, that is $\hat{u}(n) = (u, \hat{e}_n)$, and let $\tilde{u}$ be defined by

$\tilde{u} = (u, \tilde{e}_n)_\omega$ (2.2.10,2.2.12). It can be verified that the relation between $\hat{u}$ and $\tilde{u}$ is: if $n > 0$, then $\tilde{u}(n) = \sqrt{n}\hat{u}(n)$; if $n < 0$, then $\tilde{u}(n) = i\sqrt{|n|}\hat{u}(n)$. We have

$$\|u\|_\omega^2 = (u,u)_\omega = (\tilde{u},\tilde{u})_{l^2} = \sum_{n\neq 0} |\tilde{u}(n)|^2 = \sum_{n\neq 0} |n||\hat{u}(n)|^2.$$

Let $\phi = \psi^{-1}$. We have $u(\phi) = \sum_{m\neq 0} \hat{u}(m)e^{im\phi}$. Using the notation $I_{n,m}$ (2.4.4), we have

$$\begin{aligned}
\|\psi.u\|_\omega^2 &= \sum_{n\neq 0} |n||\widehat{\psi.u}(n)|^2 = \sum_{n\neq 0} |n|\Big|\frac{1}{2\pi}\int_0^{2\pi} u(\phi(\theta))e^{-in\theta}d\theta\Big|^2 \\
&= \sum_{n\neq 0} |n|\Big|\frac{1}{2\pi}\int_0^{2\pi} \sum_{m\neq 0} \hat{u}(m)e^{im\phi}e^{-in\theta}d\theta\Big|^2 \\
&= \sum_{n\neq 0} |n|\Big|\sum_{m\neq 0} \hat{u}(m)\frac{1}{2\pi}\int_0^{2\pi} e^{im\phi-in\theta}d\theta\Big|^2 \\
&= \sum_{n\neq 0} |n|\Big|\sum_{m\neq 0} \hat{u}(m)I_{n,m}\Big|^2 \\
&\leqslant \sum_{m,n\neq 0} |n||\hat{u}(m)|^2|I_{n,m}|^2 = \sum_{m\neq 0} |\hat{u}(m)|^2 \sum_{n\neq 0} |n||I_{n,m}|^2 \\
&= \sum_{|m|\leqslant M_0} |\hat{u}(m)|^2 \sum_{n\neq 0} |n||I_{n,m}|^2 + \sum_{|m|>M_0} |\hat{u}(m)|^2 \sum_{n\neq 0} |n||I_{n,m}|^2,
\end{aligned}$$

where the constant M_0 in the last equality is a positive integer large enough so that we can apply part (2) of Lemma 2.4.4. It is easy to see that the first term in the last equality is finite. For the second term we use Lemma 2.4.4

$$\sum_{|m|>M_0} |\hat{u}(m)|^2 \sum_{n\neq 0} |n||I_{n,m}|^2 \leqslant C \sum_{|m|>M_0} |\hat{u}(m)|^2|m| \leqslant C.$$

Thus for any $u \in H$ with $\|u\|_\omega = 1$, $\|\psi.u\|_\omega$ is uniformly bounded. Therefore, $\Phi(\psi)$ is a bounded operator on H.

Now we can use Lemma 2.4.4 again to estimate the norm $\|\Phi(\psi)\|_2$

$$\begin{aligned}
\|\Phi(\psi)\|_2 &= \sum_{n>0,m<0} |(\psi.\tilde{e}_m, \tilde{e}_n)_\omega|^2 = \sum_{n>0,m<0} |n||(\psi.\hat{e}_m, \hat{e}_n)|^2 \\
&= \sum_{n>0,m<0} |n||I_{n,m}|^2 < \infty.
\end{aligned}$$

□

Theorem 2.4.6. $\Phi : \mathit{Diff}(S^1) \to Sp(\infty)$ *is a group homomorphism. Moreover,* Φ *is injective, but not surjective.*

Proof. Combining Lemma 2.4.3 and Lemma 2.4.5 we see that for any diffeomorphism $\psi \in \mathrm{Diff}(S^1)$

the map $\Phi(\psi)$ is an invertible bounded operator on H, it preserves the form ω, and $\|\Phi(\psi)\|_2 < \infty$. In addition, by our remark after Definition 2.4.1 $\psi.u$ is real-valued, if u is real-valued. Therefore, Φ maps $\mathrm{Diff}(S^1)$ into $\mathrm{Sp}(\infty)$.

Next, we first prove that Φ is injective. Let $\psi_1, \psi_2 \in \mathrm{Diff}(S^1)$, and denote $\phi_1 = \psi_1^{-1}, \phi_2 = \psi_2^{-1}$. Suppose $\Phi(\psi_1) = \Phi(\psi_2)$, i.e. $\psi_1.u = \psi_2.u$, for any $u \in H$. In particular, $\psi_1.e^{i\theta} = \psi_2.e^{i\theta}$. Therefore

$$e^{i\phi_1} - C_1 = e^{i\phi_2} - C_2,$$

where $C_1 = \frac{1}{2\pi}\int_0^{2\pi} e^{i\phi_1} d\theta$, and $C_2 = \frac{1}{2\pi}\int_0^{2\pi} e^{i\phi_2} d\theta$. Note that $e^{i\phi_1}$ and $e^{i\phi_2}$ have the same image as maps from S^1 to $\mathbb{C}$. This implies $C_1 = C_2$, since otherwise $e^{i\phi_1} = e^{i\phi_2} + (C_1 - C_2)$ and $e^{i\phi_1}$ and $e^{i\phi_2}$ would have had different images. Therefore, we have $e^{i\phi_1} = e^{i\phi_2}$. But the function $e^{i\tau} : S^1 \to S^1$ is an injective function, so $\phi_1 = \phi_2$. Therefore $\psi_1 = \psi_2$, and so Φ is injective.

To prove that Φ is not surjective, we will construct an operator $A \in \mathrm{Sp}(\infty)$ which can not be written as $\Phi(\psi)$ for any $\psi \in \mathrm{Diff}(S^1)$. Let the linear map A be defined by the corresponding matrix $\{A_{m,n}\}_{m,n\in\mathbb{Z}}$ with the entries

$$\begin{aligned}
&A_{1,1} = A_{-1,-1} = \sqrt{2} \\
&A_{1,-1} = i, A_{-1,1} = -i \\
&A_{m,m} = 1, \quad \text{for } m \neq \pm 1
\end{aligned}$$

with all other entries being 0.

First we show that $A \in \mathrm{Sp}(\infty)$. For any $u \in H$, we can write $u = \sum_{n\neq 0} \tilde{u}(n)\tilde{e}_n$. Then A acting on u changes only $\tilde{e}_1$ and $\tilde{e}_{-1}$. Therefore, $Au \in H$, and clearly A is a well–defined bounded linear map on H to H. Moreover, $\|A\|_2 < \infty$. It is clear that $A_{m,n} = \overline{A_{-m,-n}}$, and therefore $A = \bar{A}$ by Proposition 2.3.3. Moreover, A preserves the form ω by part(II) of Proposition 2.3.6, as

$$\sum_{k\neq 0} \mathrm{sgn}(mk) A_{k,m}\overline{A_{k,n}} = \delta_{m,n}.$$

Finally, A is invertible, since $\{A_{k,m}\}_{m,n\in\mathbb{Z}}$ is, with the inverse $\{B_{k,m}\}_{m,n\in\mathbb{Z}}$ given by

$$\begin{aligned}
&B_{1,1} = B_{-1,-1} = \sqrt{2} \\
&B_{1,-1} = -i, B_{-1,1} = i \\
&B_{m,m} = 1, \quad \text{for } m \neq \pm 1
\end{aligned}$$

with all other entries being 0. Next we show that $A \neq \Phi(\psi)$ for any $\psi \in \mathrm{Diff}(S^1)$. First observe that if we look at any basis element $\tilde{e}_1 = e^{i\theta}$ as a function from S^1 to $\mathbb{C}$, then the image of this function lies on the unit circle. Clearly, when acted by a diffeomorphism $\phi \in \mathrm{Diff}(S^1)$, the image of the function $\phi.e^{i\theta}$ is still a circle with radius 1. But if we consider $A\tilde{e}_1$ as a function from S^1 to $\mathbb{C}$, we will show that the image of the function $A\tilde{e}_1 : S^1 \to \mathbb{C}$ is not a circle. Therefore, $A \neq \Phi(\psi)$ for any $\psi \in \mathrm{Diff}(S^1)$. Indeed, by definition of A we have

$$A\tilde{e}_1 = \sqrt{2}\tilde{e}_1 - i\tilde{e}_{-1}.$$

Let us write it as a function on S^1

$$A\tilde{e}_1(\theta) = \sqrt{2}e^{i\theta} - e^{-i\theta} = (\sqrt{2}-1)\cos\theta + i(\sqrt{2}+1)\sin\theta,$$

and then we see that the image lies on an ellipse, which is not the unit circle

$$\frac{x^2}{(\sqrt{2}-1)^2} + \frac{y^2}{(\sqrt{2}+1)^2} = 1.$$

□

2.5 The Lie algebra associated with Diff(S^1)

Let diff(S^1) be the space of smooth vector fields on S^1. Elements in diff(S^1) can be identified with smooth functions on S^1. The space diff(S^1) is a Lie algebra with the following Lie bracket

$$[X,Y] = XY' - X'Y, \quad X, Y \in \mathrm{diff}(S^1),$$

where X' and Y' are derivatives with respect to θ.

Let $X \in \mathrm{diff}(S^1)$, and ρ_t be the corresponding flow of diffeomorphisms. We define an action of diff(S^1) on H as follows: for $X \in \mathrm{diff}(S^1)$ and $u \in H$, $X.u$ is a function on S^1 defined by

$$(X.u)(\theta) = \left.\frac{d}{dt}\right|_{t=0} [(\rho_t.u)(\theta)],$$

where ρ_t acts on u via the representation $\Phi : \mathrm{Diff}(S^1) \to \mathrm{Sp}(\infty)$.

The next proposition shows that the action is well–defined, and also gives an explicit formula of $X.u$.

Proposition 2.5.1. *Let $X \in$ diff(S^1). Then*

$$(X.u)(\theta) = u'(\theta)(-X(\theta)) - \frac{1}{2\pi}\int_0^{2\pi} u'(\theta)(-X(\theta))d\theta,$$

that is, $X.u$ is the function $-u'X$ with the 0th Fourier coefficient replaced by 0.

Proof. Let ρ_t be the flow that corresponds to X, and λ_t be the flow that corresponds to $-X$. Then λ_t is the inverse of ρ_t for all t.

$$(X.u)(\theta) = \left.\frac{d}{dt}\right|_{t=0} [(\rho_t.u)(\theta)] = \left.\frac{d}{dt}\right|_{t=0} \left[u(\lambda_t(\theta)) - \frac{1}{2\pi}\int_0^{2\pi} u(\lambda_t(\theta))d\theta\right].$$

Using the chain rule, we have

$$\left.\frac{d}{dt}\right|_{t=0} u(\lambda_t(\theta)) = u'(\theta)(-\widetilde{X}(\theta)),$$

and
$$\left.\frac{d}{dt}\right|_{t=0} \frac{1}{2\pi}\int_0^{2\pi} u(\lambda_t(\theta))d\theta = \frac{1}{2\pi}\int_0^{2\pi} u'(\theta)(-X(\theta))d\theta.$$

□

Notation 2.5.2. We consider diff(S^1) as a subspace of the space of real-valued L^2 functions on S^1. The space of real-valued L^2 functions on S^1 has an orthonormal basis

$$\mathscr{B} = \{X_l = \cos(m\theta), Y_k = \sin(k\theta), l = 0,1,...,k = 1,2,...\}$$

which is contained in diff(S^1).

Let us consider how these basis elements act on H.

Proposition 2.5.3. *For any $l = 0,1,...,k = 1,2,...$ the basis elements X_l, Y_k act on H as linear maps. In the basis $\mathscr{B}_\omega$ of H, they are represented by infinite-dimensional matrices with (m,n)th entries equal to*

$$(X_l)_{m,n} = (X_l.\tilde{e}_n, \tilde{e}_m)_\omega = s(m,n)\frac{1}{2}\sqrt{|mn|}(\delta_{m-n,l} + \delta_{n-m,l})$$
$$(Y_k)_{m,n} = (Y_k.\tilde{e}_n, \tilde{e}_m)_\omega = s(m,n)(-i)\frac{1}{2}\sqrt{|mn|}(\delta_{m-n,k} - \delta_{n-m,k})$$

where $m,n \neq 0$,

$$s(m,n) = \begin{cases} -i & m,n > 0 \\ 1 & m > 0, n < 0 \\ 1 & m < 0, n > 0 \\ i & m,n < 0. \end{cases}$$

Proof. By Proposition 2.5.1 and a simple verification depending on the signs of m,n we see that

$$X_l.e^{in\theta} = -ine^{in\theta}\cos(l\theta) = -\frac{1}{2}in\left[e^{i(n+l)\theta} + e^{i(n-l)\theta}\right]$$
$$Y_k.e^{in\theta} = -ine^{in\theta}\sin(k\theta) = -\frac{1}{2}n\left[e^{i(n+k)\theta} - e^{i(n-k)\theta}\right].$$

Indeed, recall that a basis element $\tilde{e}_n \in \mathscr{B}_\omega$ has the form

$$\tilde{e}_n = \begin{cases} \frac{1}{\sqrt{n}}e^{in\theta} & n > 0 \\ \frac{1}{i\sqrt{|n|}}e^{in\theta} & n < 0. \end{cases}$$

Suppose $m,n > 0$

$$X_l.\tilde{e}_n = \frac{1}{\sqrt{n}}X_l.e^{in\theta} = -\frac{1}{2}i\sqrt{n}\left[e^{i(n+l)\theta} + e^{i(n-l)\theta}\right],$$

and

$$(e^{i(n+l)\theta}, \tilde{e}_m)_\omega = \sqrt{m}\delta_{m-n,k}; \qquad (e^{i(n-l)\theta}, \tilde{e}_m)_\omega = \sqrt{m}\delta_{n-m,l}.$$

Therefore,

$$(X_l)_{m,n} = (X_l.\tilde{e}_n, \tilde{e}_m)_\omega = (-i)\frac{1}{2}\sqrt{|mn|}(\delta_{m-n,l} + \delta_{n-m,l}).$$

All other cases can be verified similarly. $\square$

Remark 2.5.4. Recall that $\mathbb{H}_\omega$ is the completion of H under the metric $(\cdot,\cdot)_\omega$. The above calculation shows that the trigonometric basis X_l, Y_k of $\mathrm{diff}(S^1)$ act on $\mathbb{H}_\omega$ as *unbounded* operators. They are densely defined on the subspace $H \subseteq \mathbb{H}_\omega$.

2.6 Brownian motion on Sp(∞)

Definition 2.6.1. As in [3], let $\mathfrak{sp}(\infty)$ be the space of infinite-dimensional matrices A which can be written as block matrices of the form

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$$

such that $a + a^\dagger = 0$, $b = b^T$, and b is a Hilbert-Schmidt operator.

Remark 2.6.2. The space $\mathfrak{sp}(\infty)$ has a structure of Lie algebra with the operator commutator as a Lie bracket. Following [3], we call $\mathfrak{sp}(\infty)$ the Lie algebra of the group $\mathrm{Sp}(\infty)$. An element of $\mathfrak{sp}(\infty)$ can be viewed as an operator on the space H or $\mathbb{H}_\omega$ defined in Section 2.2. Note that as the Lie algebra of the group $\mathrm{Sp}(\infty)$, $\mathfrak{sp}(\infty)$ may contain a lot of unbounded operators.

In the definition of Lie algebra $\mathfrak{sp}(\infty)$, the condition $a + a^\dagger = 0$ says that the block a is conjugate skew-symmetric. The condition $b = b^T$ says that the block b is symmetric. These are summarized in the following proposition.

Proposition 2.6.3. *Let $\{A_{m,n}\}_{m,n\in\mathbb{Z}\setminus\{0\}}$ be the matrix corresponding to an operator A. Then any $A \in \mathfrak{sp}(\infty)$ satisfies (1) $A_{m,n} = \overline{A_{-m,-n}}$; (2) $A_{m,n} + \overline{A_{n,m}} = 0$, for $m,n > 0$; (3) $A_{m,n} = A_{-n,-m}$, for $m > 0, n < 0$. Moreover, $A \in \mathfrak{sp}(\infty)$ if and only if (1) $A = \bar{A}$; (2) $\pi^+ A \pi^-$ is Hilbert-Schmidt; (3) $A + A^\# = 0$.*

Proof. The first part follows directly from definition of $\mathfrak{sp}(\infty)$. Then we can use this fact and the formula for the matrix entries of $A^\#$ in Proposition 2.3.3 to prove the second part. $\square$

Definition 2.6.4. Let HS be the space of Hilbert-Schmidt matrices with complex entries and indexed by $\mathbb{Z}\setminus\{0\} \times \mathbb{Z}\setminus\{0\}$. That is, the matrix $\{a_{mn}\} \in \mathrm{HS}$ if and only if $\sum_{m,n\in\mathbb{Z}\setminus\{0\}} |a_{mn}|^2 < \infty$. Let $\mathfrak{sp}_{\mathrm{HS}} = \mathfrak{sp}(\infty) \cap \mathrm{HS}$.

The space HS as a real Hilbert space has an orthonormal basis

$$\mathscr{B}_{\mathrm{HS}} = \{e_{mn}^{Re} : m,n \neq 0\} \cup \{e_{mn}^{Im} : m,n \neq 0\},$$

where e_{mn}^{Re} is a matrix with (m,n)-th entry 1 all other entries 0, and e_{mn}^{Im} is a matrix with (m,n)th entry i all other entries 0.

The space $\mathfrak{sp}_{\mathrm{HS}}$ is a closed subspace of HS, and therefore a real Hilbert space. According to the symmetry of the matrices in $\mathfrak{sp}_{\mathrm{HS}}$, we define a projection $\pi : \mathrm{HS} \to \mathfrak{sp}_{\mathrm{HS}}$, such that

$$\begin{aligned}
\pi(e_{mn}^{Re}) &= \frac{1}{2}\left(e_{mn}^{Re} - e_{nm}^{Re} + e_{-m,-n}^{Re} - e_{-n,-m}^{Re}\right), && \text{if } \mathrm{sgn}(mn) > 0 \\
\pi(e_{mn}^{Im}) &= \frac{1}{2}\left(e_{mn}^{Im} + e_{nm}^{Im} - e_{-m,-n}^{Im} - e_{-n,-m}^{Im}\right), && \text{if } \mathrm{sgn}(mn) > 0 \\
\pi(e_{mn}^{Re}) &= \frac{1}{2}\left(e_{mn}^{Re} + e_{-n,-m}^{Re} + e_{-m,-n}^{Re} + e_{n,m}^{Re}\right), && \text{if } \mathrm{sgn}(mn) < 0 \\
\pi(e_{mn}^{Im}) &= \frac{1}{2}\left(e_{mn}^{Im} + e_{-n,-m}^{Im} - e_{-m,-n}^{Im} - e_{nm}^{Im}\right), && \text{if } \mathrm{sgn}(mn) < 0
\end{aligned}$$

Notation 2.6.5. We choose $\mathscr{B}_{\mathfrak{sp}_{\mathrm{HS}}} = \pi(\mathscr{B}_{\mathrm{HS}})$ to be the orthonormal basis of $\mathfrak{sp}_{\mathrm{HS}}$.

Clearly, if $A \in \mathfrak{sp}_{\mathrm{HS}}$, then $|A|_{\mathfrak{sp}_{\mathrm{HS}}} = |A|_{\mathrm{HS}}$.

Definition 2.6.6. Let W_t be a Brownian motion on $\mathfrak{sp}_{\mathrm{HS}}$ which has the mean zero and covariance Q, where Q is assumed to be a positive symmetric trace class operator on H. We further assume that Q is diagonal in the basis $\mathscr{B}_{\mathfrak{sp}_{\mathrm{HS}}}$.

Remark 2.6.7. Q can also be viewed as a positive function on the set $\mathscr{B}_{\mathfrak{sp}_{\mathrm{HS}}}$, and the Brownian motion W_t can be written as

$$W_t = \sum_{\xi \in \mathscr{B}_{\mathfrak{sp}_{\mathrm{HS}}}} \sqrt{Q(\xi)} B_t^{\xi} \xi, \tag{2.6.1}$$

where $\{B_t^{\xi}\}_{\xi \in \mathscr{B}_{\mathfrak{sp}_{\mathrm{HS}}}}$ are standard real-valued mutually independent Brownian motions.

Our goal now is to construct a Brownian motion on the group $\mathrm{Sp}(\infty)$ using the Brownian motion W_t on $\mathfrak{sp}_{\mathrm{HS}}$. This is done by solving the Stratonovich stochastic differential equation

$$\delta X_t = X_t \delta W_t. \tag{2.6.2}$$

This equation can be written as the following Itô stochastic differential equation

$$dX_t = X_t dW_t + \frac{1}{2} X_t D dt, \tag{2.6.3}$$

where $D = \mathrm{Diag}(D_m)$ is a diagonal matrix with entries

$$D_m = -\frac{1}{4} \mathrm{sgn}(m) \sum_k \mathrm{sgn}(k) \left[Q_{mk}^{Re} + Q_{mk}^{Im}\right] \tag{2.6.4}$$

with $Q_{mk}^{Re} = Q(\pi(e_{mk}^{Re}))$ and $Q_{mk}^{Im} = Q(\pi(e_{mk}^{Im}))$.

Notation 2.6.8. Denote by $\mathfrak{sp}_{\mathrm{HS}}^{\mathrm{Q}} = Q^{1/2}(\mathfrak{sp}_{\mathrm{HS}})$ which is a subspace of $\mathfrak{sp}_{\mathrm{HS}}$. Define an inner product on $\mathfrak{sp}_{\mathrm{HS}}^{\mathrm{Q}}$ by $\langle u, v\rangle_{\mathfrak{sp}_{\mathrm{HS}}^{\mathrm{Q}}} = \langle Q^{-1/2}u, Q^{-1/2}v\rangle_{\mathfrak{sp}_{\mathrm{HS}}}$. Then $\mathscr{B}_{\mathfrak{sp}_{\mathrm{HS}}^{\mathrm{Q}}} = \{\hat{\xi} = Q^{1/2}\xi : \xi \in \mathscr{B}_{\mathfrak{sp}_{\mathrm{HS}}}\}$ is an orthonormal basis of the Hilbert space $\mathfrak{sp}_{\mathrm{HS}}^{\mathrm{Q}}$.

Notation 2.6.9. Let L_2^0 be the space of Hilbert-Schmidt operators from $\mathfrak{sp}_{\mathrm{HS}}^{\mathrm{Q}}$ to $\mathfrak{sp}_{\mathrm{HS}}$ with the norm

$$|\Phi|_{L_2^0}^2 = \sum_{\hat{\xi} \in \mathscr{B}_{\mathfrak{sp}_{\mathrm{HS}}^{\mathrm{Q}}}} |\Phi \hat{\xi}|_{\mathfrak{sp}_{\mathrm{HS}}}^2 = \sum_{\xi, \zeta \in \mathscr{B}_{\mathfrak{sp}_{\mathrm{HS}}}} Q(\xi) |\langle \Phi \xi, \zeta \rangle_{\mathfrak{sp}_{\mathrm{HS}}}|^2 = \mathrm{Tr}[\Phi Q \Phi^*],$$

where $Q(\xi)$ means Q evaluated at ξ as a positive function on $\mathscr{B}_{\mathfrak{sp}_{\mathrm{HS}}}$.

Lemma 2.6.10. *If $\Psi \in L(\mathfrak{sp}_{HS}, \mathfrak{sp}_{HS})$, a bounded linear operator from $\mathfrak{sp}_{HS}$ to $\mathfrak{sp}_{HS}$, then Ψ restricted on $\mathfrak{sp}_{HS}^Q$ is a Hilbert-Schmidt operator from $\mathfrak{sp}_{HS}^Q$ to $\mathfrak{sp}_{HS}$, and $|\Psi|_{L_2^0} \leqslant Tr(Q) \|\Psi\|^2$, where $\|\Psi\|$ is the operator norm of Ψ.*

Proof.

$$\begin{aligned} |\Psi|_{L_2^0}^2 &= \sum_{\hat{\xi} \in \mathscr{B}_{\mathfrak{sp}_{\mathrm{HS}}^{\mathrm{Q}}}} |\Psi \hat{\xi}|_{\mathfrak{sp}_{\mathrm{HS}}}^2 \leqslant \|\Psi\|^2 \sum_{\hat{\xi} \in \mathscr{B}_{\mathfrak{sp}_{\mathrm{HS}}^{\mathrm{Q}}}} |\hat{\xi}|_{\mathfrak{sp}_{\mathrm{HS}}}^2 \\ &= \|\Psi\|^2 \sum_{\xi \in \mathscr{B}_{\mathfrak{sp}_{\mathrm{HS}}}} \langle Q^{1/2} \xi, Q^{1/2} \xi \rangle_{\mathfrak{sp}_{\mathrm{HS}}} = \|\Psi\|^2 \sum_{\xi \in \mathscr{B}_{\mathfrak{sp}_{\mathrm{HS}}}} \langle Q \xi, \xi \rangle_{\mathfrak{sp}_{\mathrm{HS}}} = \|\Psi\|^2 \mathrm{Tr}(Q) \end{aligned}$$

□

Notation 2.6.11. Define $B : \mathfrak{sp}_{\mathrm{HS}} \to L_2^0$ by $B(Y)A = (I+Y)A$ for $A \in \mathfrak{sp}_{\mathrm{HS}}^{\mathrm{Q}}$, and $F : \mathfrak{sp}_{\mathrm{HS}} \to \mathfrak{sp}_{\mathrm{HS}}$ by $F(Y) = \frac{1}{2}(I+Y)D$.

Note that B is well–defined by Lemma 2.6.10. Also $D \in \mathfrak{sp}_{\mathrm{HS}}$, and so $F(Y) \in \mathfrak{sp}_{\mathrm{HS}}$ and F is well–defined as well.

Theorem 2.6.12. *The stochastic differential equation*

$$\begin{aligned} dY_t &= B(Y_t) dW_t + F(Y_t) dt \\ Y_0 &= 0 \end{aligned} \tag{2.6.5}$$

has a unique solution, up to equivalence, among the processes satisfying

$$P\left(\int_0^T |Y_s|_{\mathfrak{sp}_{HS}}^2 ds < \infty \right) = 1.$$

Proof. To prove this theorem we will use Theorem 7.4 from the book by G. DaPrato and J. Zabczyk [7] as it has been done in [10, 12]. It is enough to check

1. B is a measurable mapping.
2. $|B(Y_1) - B(Y_2)|_{L_2^0} \leqslant C_1 |Y_1 - Y_2|_{\mathfrak{sp}_{\mathrm{HS}}}$ for $Y_1, Y_2 \in \mathfrak{sp}_{\mathrm{HS}}$;
3. $|B(Y)|_{L_2^0}^2 \leqslant K_1 (1 + |Y|_{\mathfrak{sp}_{\mathrm{HS}}}^2)$ for any $Y \in \mathfrak{sp}_{\mathrm{HS}}$;
4. F is a measurable mapping.

5. $|F(Y_1) - F(Y_2)|_{\mathfrak{sp}_{\mathrm{HS}}} \leqslant C_2 |Y_1 - Y_2|_{\mathfrak{sp}_{\mathrm{HS}}}$ for $Y_1, Y_2 \in \mathfrak{sp}_{\mathrm{HS}}$;

6. $|F(Y)|^2_{\mathfrak{sp}_{\mathrm{HS}}} \leqslant K_2(1 + |Y|^2_{\mathfrak{sp}_{\mathrm{HS}}})$ for any $Y \in \mathfrak{sp}_{\mathrm{HS}}$.

Proof of 1. By the proof of 2, B is a continuous mapping, therefore it is measurable.
Proof of 2.

$$|B(Y_1) - B(Y_2)|^2_{L_2^0} = \sum_{\hat{\xi} \in \mathscr{B}_{\mathfrak{sp}^{\mathrm{Q}}_{\mathrm{HS}}}} |(Y_1 - Y_2)\hat{\xi}|^2_{\mathfrak{sp}_{\mathrm{HS}}} = \sum_{\xi \in \mathscr{B}_{\mathfrak{sp}_{\mathrm{HS}}}} Q(\xi)|(Y_1 - Y_2)\xi|^2_{\mathfrak{sp}_{\mathrm{HS}}}$$

$$\leqslant \sum_{\xi \in \mathscr{B}_{\mathfrak{sp}_{\mathrm{HS}}}} Q(\xi)\|\xi\|^2 |Y_1 - Y_2|^2_{\mathfrak{sp}_{\mathrm{HS}}} \leqslant \max_{\xi \in \mathscr{B}_{\mathfrak{sp}_{\mathrm{HS}}}} \|\xi\|^2 \left(\sum_{\xi \in \mathscr{B}_{\mathfrak{sp}_{\mathrm{HS}}}} Q(\xi) \right) |Y_1 - Y_2|^2_{\mathfrak{sp}_{\mathrm{HS}}}$$

$$= \mathrm{Tr} Q \left(\max_{\xi \in \mathscr{B}_{\mathfrak{sp}_{\mathrm{HS}}}} \|\xi\|^2 \right) |Y_1 - Y_2|^2_{\mathfrak{sp}_{\mathrm{HS}}} = C_1^2 |Y_1 - Y_2|^2_{\mathfrak{sp}_{\mathrm{HS}}},$$

where $\|\xi\|$ is the operator norm of ξ, which is uniformly bounded for all $\xi \in \mathscr{B}_{\mathfrak{sp}_{\mathrm{HS}}}$.
Proof of 3.

$$|B(Y_1)|^2_{L_2^0} = \sum_{\hat{\xi} \in \mathscr{B}_{\mathfrak{sp}^{\mathrm{Q}}_{\mathrm{HS}}}} |(I + Y)\hat{\xi}|^2_{\mathfrak{sp}_{\mathrm{HS}}} = \sum_{\xi \in \mathscr{B}_{\mathfrak{sp}_{\mathrm{HS}}}} Q(\xi)|(I + Y)\xi|^2_{\mathfrak{sp}_{\mathrm{HS}}}$$

$$\leqslant |(I + Y)\xi|^2_{\mathfrak{sp}_{\mathrm{HS}}} \sum_{\xi \in \mathscr{B}_{\mathfrak{sp}_{\mathrm{HS}}}} Q(\xi)\|\xi\|^2 \leq (1 + |Y|^2_{\mathfrak{sp}_{\mathrm{HS}}}) \cdot K_1.$$

Proof of 4. By the proof of 5, F is a continuous mapping, therefore it is measurable.
Proof of 5.

$$|F(Y_1) - F(Y_2)|_{\mathfrak{sp}_{\mathrm{HS}}} = |\frac{1}{2}(Y_1 - Y_2)D|_{\mathfrak{sp}_{\mathrm{HS}}} \leq \|\frac{1}{2}D\| |Y_1 - Y_2|_{\mathfrak{sp}_{\mathrm{HS}}}$$

Proof of 6.

$$|F(Y)|^2_{\mathfrak{sp}_{\mathrm{HS}}} = |\frac{1}{2}(I + Y)D|^2_{\mathfrak{sp}_{\mathrm{HS}}} \leq \|\frac{1}{2}D\|^2 |I + Y|^2_{\mathfrak{sp}_{\mathrm{HS}}} \leqslant K_2(1 + |Y|^2_{\mathfrak{sp}_{\mathrm{HS}}}).$$

□

Notation 2.6.13. Let $B^\# : \mathfrak{sp}_{\mathrm{HS}} \to L_2^0$ be the operator $B^\#(Y)A = A^\#(I + Y)$, and $F^\# : \mathfrak{sp}_{\mathrm{HS}} \to \mathfrak{sp}_{\mathrm{HS}}$ be the operator $F^\#(Y) = \frac{1}{2}D^\#(Y + I)$.

Proposition 2.6.14. *If Y_t is the solution to the stochastic differential equation*

$$dX_t = B(X_t)dW_t + F(X_t)dt$$
$$X_0 = 0,$$

where B and F are defined in Notation 2.6.11, then $Y_t^{\#}$ is the solution to the stochastic differential equation

$$dX_t = B^{\#}(X_t)dW_t + F^{\#}(X_t)dt \tag{2.6.6}$$
$$X_0 = 0,$$

where $B^{\#}$ and $F^{\#}$ are defined in Notation 2.6.13.

Proof. This follows directly from the property $(AB)^{\#} = B^{\#}A^{\#}$ for any A and B, which can be verified by using part (5) of Proposition 2.3.3. □

Lemma 2.6.15. *Let U and H be real Hilbert spaces. Let $\Phi : U \to H$ be a bounded linear map. Let $G : H \to H$ be a bounded linear map. Then*

$$Tr_H(G\Phi\Phi^*) = Tr_U(\Phi^* G\Phi)$$

Proof.

$$\mathrm{Tr}_H(G\Phi\Phi^*) = \sum_{i,j\in H;k\in U} G_{ij}\Phi_{jk}(\Phi^*)_{ki} = \sum_{i,j\in H;k\in U} G_{ij}\Phi_{jk}\Phi_{ik}$$
$$\mathrm{Tr}_U(\Phi^* G\Phi) = \sum_{i,j\in H;k\in U} (\Phi^*)_{ki}G_{ij}\Phi_{jk} = \sum_{i,j\in H;k\in U} G_{ij}\Phi_{jk}\Phi_{ik}.$$

Therefore $\mathrm{Tr}_H(G\Phi\Phi^*) = \mathrm{Tr}_U(\Phi^* G\Phi)$. □

Lemma 2.6.16.

$$\sum_{\xi\in\mathscr{B}_{\mathfrak{sp}_{HS}}} \left(Q^{1/2}\xi\right)\left(Q^{1/2}\xi\right)^{\#} = -D$$

Proof. If $\xi \in \mathscr{B}_{\mathfrak{sp}_{\mathrm{HS}}}$, then $\xi \in \mathfrak{sp}(\infty)$, so $\xi^{\#} = -\xi$. We will use the fact that

$$(e_{ij}^{Re} e_{kl}^{Re})_{pq} = \delta_{ip}\delta_{jk}\delta_{lq}$$

where e_{ij}^{Re} is the matrix with the (i,j)th entry being 1 and all other entries being zero. Using this fact, we see

1. for $\xi = \frac{1}{2}\left(e_{mn}^{Re} - e_{nm}^{Re} + e_{-m,-n}^{Re} - e_{-n,-m}^{Re}\right)$ with $\mathrm{sgn}(mn) > 0$,

$$\left(Q^{1/2}\xi\right)\left(Q^{1/2}\xi\right)^{\#} = -\frac{1}{4}Q_{mn}^{Re}\left[-e_{mm}^{Re} - e_{nn}^{Re} - e_{-m,-m}^{Re} - e_{-n,-n}^{Re}\right]$$

2. for $\xi = \frac{1}{2}\left(e_{mn}^{Im} + e_{nm}^{Im} - e_{-m,-n}^{Im} - e_{-n,-m}^{Im}\right)$ with $\mathrm{sgn}(mn) > 0$,

$$\left(Q^{1/2}\xi\right)\left(Q^{1/2}\xi\right)^{\#} = -\frac{1}{4}Q_{mn}^{Im}\left[-e_{mm}^{Re} - e_{nn}^{Re} - e_{-m,-m}^{Re} - e_{-n,-n}^{Re}\right]$$

3. for $\xi = \frac{1}{2}\left(e^{Re}_{mn} + e^{Re}_{-n,-m} + e^{Re}_{-m,-n} + e^{Re}_{n,m}\right)$ with $\operatorname{sgn}(mn) < 0$,

$$\left(Q^{1/2}\xi\right)\left(Q^{1/2}\xi\right)^{\#} = -\frac{1}{4}Q^{Re}_{mn}\left[e^{Re}_{mm} + e^{Re}_{nn} + e^{Re}_{-m,-m} + e^{Re}_{-n,-n}\right]$$

4. for $\xi = \frac{1}{2}\left(e^{Im}_{mn} + e^{Im}_{-n,-m} - e^{Im}_{-m,-n} - e^{Im}_{nm}\right)$ with $\operatorname{sgn}(mn) < 0$,

$$\left(Q^{1/2}\xi\right)\left(Q^{1/2}\xi\right)^{\#} = -\frac{1}{4}Q^{Im}_{mn}\left[e^{Re}_{mm} + e^{Re}_{nn} + e^{Re}_{-m,-m} + e^{Re}_{-n,-n}\right].$$

Each of the above is a diagonal matrix. The lemma can be proved by looking at the diagonal entries of the sum. □

Theorem 2.6.17. *Let Y_t be the solution to Equation 2.6.5. Then $Y_t + I \in Sp(\infty)$ for any $t > 0$ with probability* 1.

Proof. The proof is adapted from papers by M. Gordina [10, 12]. Let Y_t be the solution to Equation (2.6.5) and $Y_t^{\#}$ be the solution to Equation (2.6.6). Consider the process $\mathbf{Y}_t = (Y_t, Y_t^{\#})$ in the product space $\mathfrak{sp}_{\text{HS}} \times \mathfrak{sp}_{\text{HS}}$. It satisfies the following stochastic differential equation

$$d\mathbf{Y}_t = (B(Y_t), B^{\#}(Y_t^{\#}))dW + (F(Y_t), F^{\#}(Y_t^{\#}))dt.$$

Let G be a function on the Hilbert space $\mathfrak{sp}_{\text{HS}} \times \mathfrak{sp}_{\text{HS}}$ defined by $G(Y_1, Y_2) = \Lambda((Y_1 + I)(Y_2 + I))$, where Λ is a nonzero linear real bounded functional from $\mathfrak{sp}_{\text{HS}} \times \mathfrak{sp}_{\text{HS}}$ to $\mathbb{R}$. We will apply Itô's formula to $G(\mathbf{Y}_t) = G(Y_t, Y_t^{\#})$. Then $(Y_t + I)(Y_t^{\#} + I) = I$ if and only if $\Lambda((Y_t + I)(Y_t^{\#} + I) - I) = 0$ for any Λ.

In order to use Itô's formula we must verify that G and the derivatives G_t, $G_{\mathbf{Y}}$, $G_{\mathbf{YY}}$ are uniformly continuous on bounded subsets of $[0, T] \times \mathfrak{sp}_{\text{HS}} \times \mathfrak{sp}_{\text{HS}}$, where $G_{\mathbf{Y}}$ is defined as follows

$$G_{\mathbf{Y}}(\mathbf{Y})(\mathbf{S}) = \lim_{\varepsilon \to 0} \frac{G(\mathbf{Y} + \varepsilon\mathbf{S}) - G(\mathbf{Y})}{\varepsilon} \qquad \text{for any } \mathbf{Y}, \mathbf{S} \in \mathfrak{sp}_{\text{HS}} \times \mathfrak{sp}_{\text{HS}}$$

and $G_{\mathbf{YY}}$ is defined as follows

$$G_{\mathbf{YY}}(\mathbf{Y})(\mathbf{S} \otimes \mathbf{T}) = \lim_{\varepsilon \to 0} \frac{G_{\mathbf{Y}}(\mathbf{Y} + \varepsilon\mathbf{T})(\mathbf{S}) - G_{\mathbf{Y}}(\mathbf{Y})(\mathbf{S})}{\varepsilon}$$

for any $\mathbf{Y}, \mathbf{S}, \mathbf{T} \in \mathfrak{sp}_{\text{HS}} \times \mathfrak{sp}_{\text{HS}}$. Let us calculate G_t, $G_{\mathbf{Y}}$, $G_{\mathbf{YY}}$. Clearly, $G_t = 0$. It is easy to verify that for any $\mathbf{S} = (S_1, S_2) \in \mathfrak{sp}_{\text{HS}} \times \mathfrak{sp}_{\text{HS}}$

$$G_{\mathbf{Y}}(\mathbf{Y})(\mathbf{S}) = \Lambda(S_1(Y_2 + I) + (Y_1 + I)S_2)$$

and for any $\mathbf{S} = (S_1, S_2) \in \mathfrak{sp}_{\text{HS}} \times \mathfrak{sp}_{\text{HS}}$ and $\mathbf{T} = (T_1, T_2) \in \mathfrak{sp}_{\text{HS}} \times \mathfrak{sp}_{\text{HS}}$

$$G_{\mathbf{YY}}(\mathbf{Y})(\mathbf{S} \otimes \mathbf{T}) = \Lambda(S_1 T_2 + T_1 S_2).$$

So the condition is satisfied.

We will use the following notation

$$\begin{aligned}
G_{\mathbf{Y}}(\mathbf{Y})(\mathbf{S}) &= \langle \bar{G}_{\mathbf{Y}}(\mathbf{Y}), \mathbf{S} \rangle_{\mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}} \\
G_{\mathbf{YY}}(\mathbf{Y})(\mathbf{S} \otimes \mathbf{T}) &= \langle \bar{G}_{\mathbf{YY}}(\mathbf{Y})\mathbf{S}, \mathbf{T} \rangle_{\mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}},
\end{aligned}$$

where $\bar{G}_{\mathbf{Y}}(\mathbf{Y})$ is an element of $\mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}$ corresponding to the functional $G_{\mathbf{Y}}(\mathbf{Y})$ in $(\mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}})^*$ and $\bar{G}_{\mathbf{YY}}(\mathbf{Y})$ is an operator on $\mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}$ corresponding to the functional $G_{\mathbf{YY}}(\mathbf{Y}) \in ((\mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}) \otimes (\mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}))^*$.

Now we can apply Itô's formula to $G(\mathbf{Y}_t)$

$$\begin{aligned}
G(\mathbf{Y}_t) - G(\mathbf{Y}_0) = &\int_0^t \langle \bar{G}_{\mathbf{Y}}(\mathbf{Y}_s), \big(B(Y_s)dW_s, B^{\#}(Y_s^{\#})dW_s\big)\rangle_{\mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}} \\
&+ \int_0^t \langle \bar{G}_{\mathbf{Y}}(\mathbf{Y}_s), \big(F(Y_s), F^{\#}(Y_s^{\#})\big)\rangle_{\mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}} ds \\
&+ \int_0^t \frac{1}{2} \mathrm{Tr}_{\mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}} \Big[\bar{G}_{\mathbf{YY}}(\mathbf{Y}_s) \Big(B(Y_s)Q^{1/2}, B^{\#}(Y_s^{\#})Q^{1/2} \Big) \\
&\qquad \Big(B(Y_s)Q^{1/2}, B^{\#}(Y_s^{\#})Q^{1/2} \Big)^* \Big] ds.
\end{aligned}$$

Let us calculate the three integrands separately. The first integrand is

$$\begin{aligned}
&\langle \bar{G}_{\mathbf{Y}}(\mathbf{Y}_s), \big(B(Y_s)dW_s, B^{\#}(Y_s^{\#})dW_s\big)\rangle_{\mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}} \\
&\quad = \Big(B(Y_s)dW_s\Big)(Y_s^{\#} + I) + (Y_s + I)\Big(B^{\#}(Y_s^{\#})dW_s\Big) \\
&\quad = (Y_s + I)dW_s(Y_s^{\#} + I) + (Y_s + I)dW_s^{\#}(Y_s^{\#} + I) = 0.
\end{aligned}$$

The second integrand is

$$\begin{aligned}
&\langle \bar{G}_{\mathbf{Y}}(\mathbf{Y}_s), \big(F(Y_s), F^{\#}(Y_s^{\#})\big)\rangle_{\mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}} \\
&\quad = F(Y_s)(Y_s^{\#} + I) + (Y_s + I)F^{\#}(Y_s^{\#}) \\
&\quad = \frac{1}{2}(Y_s + I)D(Y_s^{\#} + I) + \frac{1}{2}(Y_s + I)D^{\#}(Y_s^{\#} + I) \\
&\quad = \frac{1}{2}(Y_s + I)(D + D^{\#})(Y_s^{\#} + I) \\
&\quad = (Y_s + I)D(Y_s^{\#} + I),
\end{aligned}$$

where we have used the fact that $D = D^{\#}$, since D is a diagonal matrix with all real entries.

The third integrand is

$$
\begin{aligned}
&\frac{1}{2}\mathrm{Tr}_{\mathfrak{sp}_{\mathrm{HS}}\times\mathfrak{sp}_{\mathrm{HS}}}\\
&\left[\bar{G}_{\mathbf{YY}}(\mathbf{Y}_s)\left(B(Y_s)Q^{1/2},B^{\#}(Y_s^{\#})Q^{1/2}\right)\left(B(Y_s)Q^{1/2},B^{\#}(Y_s^{\#})Q^{1/2}\right)^*\right]\\
&=\frac{1}{2}\mathrm{Tr}_{\mathfrak{sp}_{\mathrm{HS}}}\left[\left(B(Y_s)Q^{1/2},B^{\#}(Y_s^{\#})Q^{1/2}\right)^*\bar{G}_{\mathbf{YY}}(\mathbf{Y}_s)\left(B(Y_s)Q^{1/2},B^{\#}(Y_s^{\#})Q^{1/2}\right)\right]\\
&=\frac{1}{2}\sum_{\xi\in\mathscr{B}_{\mathfrak{sp}_{\mathrm{HS}}}}G_{\mathbf{YY}}(\mathbf{Y}_s)\Bigg(\left(B(Y_s)Q^{1/2}\xi,B^{\#}(Y_s^{\#})Q^{1/2}\xi\right)\\
&\qquad\qquad\qquad\qquad\otimes\left(B(Y_s)Q^{1/2}\xi,B^{\#}(Y_s^{\#})Q^{1/2}\xi\right)\Bigg)\\
&=\sum_{\xi\in\mathscr{B}_{\mathfrak{sp}_{\mathrm{HS}}}}\left(B(Y_s)Q^{1/2}\xi\right)\left(B^{\#}(Y_s^{\#})Q^{1/2}\xi\right)\\
&=\sum_{\xi\in\mathscr{B}_{\mathfrak{sp}_{\mathrm{HS}}}}(Y_s+I)\left(\left(Q^{1/2}\xi\right)\left(Q^{1/2}\xi\right)^{\#}\right)(Y_s^{\#}+I)\\
&=-(Y_s+I)D(Y_s^{\#}+I),
\end{aligned}
$$

where the second equality follows from Lemma 2.6.15, and the last equality follows from Lemma 2.6.16.

The above calculations show that the stochastic differential of G is zero. So $G(\mathbf{Y}_t)=G(\mathbf{Y}_0)=\Lambda(I)$ for any $t>0$ and any nonzero linear real bounded functional Λ on $\mathfrak{sp}_{\mathrm{HS}}\times\mathfrak{sp}_{\mathrm{HS}}$. This means $(Y_t+I)(Y_t^{\#}+I)=I$ almost surely for any $t>0$. Similarly we can show $(Y_t^{\#}+I)(Y_t+I)=I$ almost surely for any $t>0$. Therefore $Y_t+I\in\mathrm{Sp}(\infty)$ almost surely for any $t>0$.

$\square$

Chapter 3

Geometric Analysis of Infinite-dimensional Symplectic Group $\mathbf{Sp}(\infty)$

3.1 Introduction

For finite dimensional manifolds, it is well known that the behavior of the Brownian motion is closely related to the geometric properties of the manifolds. In particular, Ricci curvature plays an important role. For example, one can construct an example of a manifold whose Ricci curvature grows fast enough to negative infinity with the distance from an origin, and on such a manifold the Brownian motion has explosion [5]. In [15], A. Grigor'yan summarized the relationship between recurrence and explosion/non-explosion properties of Brownian motion on the one hand, and geometric properties of the manifold on the other hand.

In my research, I am dealing with infinite-dimensional groups $\mathrm{Diff}(S^1)$ and $\mathrm{Sp}(\infty)$. As infinite-dimensional manifolds, I expect geometric properties of $\mathrm{Diff}(S^1)$ and $\mathrm{Sp}(\infty)$ play similar roles as in finite-dimensional cases. But $\mathrm{Diff}(S^1)$ and $\mathrm{Sp}(\infty)$ are not merely infinite-dimensional manifolds. They are infinite-dimensional Lie groups. Therefore, all geometric properties should be the same around every element of the groups. As a consequence, one cannot make the Ricci curvature grows fast enough to negative infinity to construct a Brownian motion that has explosion as in [5]. In fact, in Chapter 1 Theorem (1.3.19), I proved that the Brownian motion I constructed on $\mathrm{Diff}(S^1)$ is non-explosive. Nevertheless, geometric analysis of infinite-dimensional groups is still important.

In [13], M. Gordina studied the geometric properties of the group $\mathrm{Diff}(S^1)/S^1$, in particular, she computed the Ricci curvature of $\mathrm{Diff}(S^1)/S^1$. In [11], using the same method, Gordina computed the Ricci curvatures of several Hilbert-Schmidt groups which can be represented as infinite-dimensional matrix groups. In this chapter, following Gordina's method, I will compute the Ricci curvature of the infinite-dimensional symplectic group $\mathrm{Sp}(\infty)$.

Let G be a finite dimensional Lie group, and $\mathfrak{g}$ its Lie algebra. Let $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ be an inner product on $\mathfrak{g}$. Then $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ defines a unique left-invariant metric on the Lie group G compactible with the Lie group structure. In [23], J. Milnor studied the Riemannian geometry of Lie groups. For $x,y,z \in \mathfrak{g}$, the *Levi-Civita connection* ∇_x is given by

$$\langle \nabla_x y, z \rangle_{\mathfrak{g}} = \frac{1}{2}(\langle [x,y],z \rangle_{\mathfrak{g}} - \langle [y,z],x \rangle_{\mathfrak{g}} + \langle [z,x],y \rangle_{\mathfrak{g}}) \tag{3.1.1}$$

The *Riemann curvature tensor* R_{xy} is given by

$$R_{xy} = \nabla_{[x,y]} - \nabla_x \nabla_y + \nabla_y \nabla_x \tag{3.1.2}$$

For any orthogonal $x, y \in \mathfrak{g}$, the *sectional curvature* $K(x,y)$ is given by

$$K(x,y) = \langle R_{xy}(x), y \rangle_{\mathfrak{g}} \tag{3.1.3}$$

Let us choose an orthonormal basis $\{\xi_i\}_{i=1}^N$ of $\mathfrak{g}$, where N is the dimension of the Lie group G. Let $x \in \mathfrak{g}$. Then the *Ricci curvature* $\mathrm{Ric}(x)$ is given by

$$\mathrm{Ric}(x) = \sum_{i=1}^{N} K(x, \xi_i) = \sum_{i=1}^{N} \langle R_{x\xi_i}(x), \xi_i \rangle_{\mathfrak{g}} \tag{3.1.4}$$

3.2 Ricci curvature of Sp(∞)

In this section, we apply the Ricci curvature theory to the infinite-dimensional symplectic group $\mathrm{Sp}(\infty)$ and its Lie algebra $\mathfrak{sp}(\infty)$. The group $\mathrm{Sp}(\infty)$ and its Lie algebra $\mathfrak{sp}(\infty)$ are defined in Definition 2.3.10 and Definition 2.6.1. Basically, elements in both the Lie group $\mathrm{Sp}(\infty)$ and the Lie algebra $\mathfrak{sp}(\infty)$ are block matrices of the form:

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$$

where each of the blocks is an infinite-dimensional matrix. The blocks a and $\bar{a}$ are complex conjugate with each other. The blocks b and $\bar{b}$ are also complex conjugate with each other, are required to be a Hilbert-Schmidt matrices. For a block matrix to be an element of $\mathrm{Sp}(\infty)$, it is also required that the matrix is invertible, and preserve a certain symplectic form. For a block matrix to be an element of $\mathfrak{sp}(\infty)$, it is required that $a + a^\dagger = 0$ or $a^T + \bar{a} = 0$, which means the block a is conjugate skew-symmetric, and $b = b^T$, which means the block b is symmetric.

To write the block matrix explicitly, we index the matrix by $\mathbb{Z}\backslash\{0\} \times \mathbb{Z}\backslash\{0\}$, so the matrix is written as $\{A_{mn}\}_{m,n \in \mathbb{Z}\backslash\{0\}}$. An entry in block a has $m,n > 0$; an entry in block $\bar{a}$ has $m,n < 0$; an entry in block b has $m > 0, n < 0$; an entry in block $\bar{b}$ has $m < 0, n > 0$. The condition that blocks a and b are conjugate to blocks $\bar{a}$ and $\bar{b}$ can be expressed as $A_{m,n} = \overline{A_{-m,-n}}$. The condition $a + a^\dagger = 0$ or $a^T + \bar{a} = 0$ can be expressed as $A_{nm} + \overline{A_{mn}} = 0$ where $m,n > 0$ or $m,n < 0$. The condition $b = b^T$ can be expressed as $A_{m,n} = A_{-n,-m}$ where $m > 0, n < 0$. These are summarized in Proposition 2.3.3 and Proposition 2.6.3.

To find Ricci curvature, we need to choose a metric for the Lie algebra $\mathfrak{sp}(\infty)$. Let us define a sequence of positive numbers

$$\{\lambda_i \in \mathbb{R}_+ | \lambda_i = \lambda_{-i}, i \in \mathbb{Z}\backslash\{0\}\}$$

The sequence $\{\lambda_i\}$ will serve as parameters to fine tune the metric that we are going to choose.

Remark 3.2.1. Let us first consider the space HS of Hilbert-Schmidt matrices which we defined in Definition 2.6.4. The Hilbert space HS, if viewed as a complex Hilbert space, has a canonical inner product given by:

$$\langle A, B\rangle = \mathrm{Tr}(AB^\dagger) = \mathrm{Tr}(A\bar{B}^T), \quad A, B \in \mathrm{HS}$$

If viewed as *real* Hilbert space, HS has a canonical inner product given by: for $A, B \in$ HS, writing $A = A_1 + iA_2$ and $B = B_1 + iB_2$, where A_1, A_2, B_1, B_2 are matrices with real value entries, then

$$\langle A, B\rangle = \mathrm{Tr}(A_1 B_1^T) + \mathrm{Tr}(A_2 B_2^T)$$

Let e_{ab} be the infinite-dimensional matrix with 1 in the entry (a, b), and 0 in all other entries, where a, b are indices of the matrix such that $a, b \in \mathbb{Z}\backslash\{0\}$. Then the above canonical inner product on HS viewed as *real* Hilbert space is equivalent to choosing the set

$$\{e_{ab}, ie_{ab} | a, b \in \mathbb{Z}\backslash\{0\}\}$$

as an orthonormal basis.

Definition 3.2.2. Let

$$\xi_{ab} = 2\lambda_a \lambda_b e_{ab} \tag{3.2.1}$$

We define an inner product $\langle\cdot,\cdot\rangle_{\mathrm{HS}}$ on HS by choosing the following set

$$\{\xi_{ab}, i\xi_{ab} | a, b \in \mathbb{Z}\backslash\{0\}\} \tag{3.2.2}$$

as an orthonormal basis for the *real* Hilbert space HS.

Remark 3.2.3. If we set the parameter $\lambda_i = 1/\sqrt{2}$ for all $i \in \mathbb{Z}\backslash\{0\}$ in Definition 3.2.2, we can recover the canonical inner product of HS (remark 3.2.1) as a *real* Hilbert space.

The Lie algebra $\mathfrak{sp}(\infty)$ may contain unbounded opertors. For simplicity, we consider the subspace $\mathfrak{sp}_{\mathrm{HS}} = \mathfrak{sp}(\infty) \cap \mathrm{HS}$. Now we can choose orthonormal set of the space $\mathfrak{sp}_{\mathrm{HS}}$ according to the symmety of matrices in the Lie algebra $\mathfrak{sp}(\infty)$.

Definition 3.2.4. Let

$$\begin{aligned}
\mu_{ab}^{Re} &= \lambda_a\lambda_b(e_{a,b} - e_{b,a} + e_{-a,-b} - e_{-b,-a}), \quad a > b > 0 \\
\mu_{ab}^{Im} &= \lambda_a\lambda_b(ie_{a,b} - ie_{b,a} + ie_{-a,-b} - ie_{-b,-a}), \quad a \geq b > 0 \\
\nu_{ab}^{Re} &= \lambda_a\lambda_b(e_{a,b} + e_{-b,-a} + e_{-a,-b} + e_{b,a}), \quad a \geq -b > 0 \\
\nu_{ab}^{Im} &= \lambda_a\lambda_b(ie_{a,b} + ie_{-b,-a} - ie_{-a,-b} - ie_{b,a}), \quad a \geq -b > 0
\end{aligned}$$

Let $A^{Re} = \{\mu_{ab}^{Re} | a > b > 0\}$, $A^{Im} = \{\mu_{ab}^{Im} | a \geq b > 0\}$, $B^{Re} = \{\nu_{ab}^{Re} | a \geq -b > 0\}$, $B^{Im} = \{\nu_{ab}^{Im} | a \geq -b > 0\}$, and $\mathscr{B}_\lambda = A^{Re} \cup A^{Im} \cup B^{Re} \cup B^{Im}$.

Remark 3.2.5. It is easy to verify that matrices in the set $\mathscr{B}_\lambda$ all belong to the space $\mathfrak{sp}_{\mathrm{HS}}$. So $\mathscr{B}_\lambda$ is a subset of $\mathfrak{sp}_{\mathrm{HS}}$ and $\mathfrak{sp}(\infty)$. Also, by definition of ξ_{ab} (equation 3.2.1), it is easy to verify

$$\begin{aligned}
\mu_{ab}^{Re} &= \frac{1}{2}(\xi_{a,b} - \xi_{b,a} + \xi_{-a,-b} - \xi_{-b,-a}) \\
\mu_{ab}^{Im} &= \frac{1}{2}(i\xi_{a,b} - i\xi_{b,a} + i\xi_{-a,-b} - i\xi_{-b,-a}) \\
\nu_{ab}^{Re} &= \frac{1}{2}(\xi_{a,b} + \xi_{-b,-a} + \xi_{-a,-b} + \xi_{b,a}) \\
\nu_{ab}^{Im} &= \frac{1}{2}(i\xi_{a,b} + i\xi_{-b,-a} - i\xi_{-a,-b} - i\xi_{b,a})
\end{aligned} \tag{3.2.3}$$

Definition 3.2.6. We define an inner product $\langle \cdot, \cdot \rangle_{\mathfrak{sp}}$ on both $\mathfrak{sp}_{\mathrm{HS}}$ and $\mathfrak{sp}(\infty)$ by choosing the set $\mathscr{B}_\lambda$ as an orthonormal set.

Remark 3.2.7. We note that the inner product on $\mathfrak{sp}_{\mathrm{HS}}$ and $\mathfrak{sp}(\infty)$ is equivalent to the subspace inner product induced from the inner product on HS defined in Definition (3.2.2). Therefore, for $x, y \in \mathfrak{sp}_{\mathrm{HS}}$, $\langle x, y \rangle_{\mathrm{HS}} = \langle x, y \rangle_{\mathfrak{sp}}$.

Remark 3.2.8. For μ_{ab}^{Re}, the indices satisfy $a > b > 0$, which means the entry is in the strict upper triangular block. For μ_{ab}^{Im}, the indices satisfy $a \geq b > 0$, which means the entry is in the upper triangular block including the diagonal. For ν_{ab}^{Re}, the indices satisfy $a \geq -b > 0$, which means the entry is in the other upper triangular block including the diagonal. For ν_{ab}^{Im}, the indices satisfy $a \geq -b > 0$, which means the entry is in the other upper triangular block including the diagonal.

Definition 3.2.9. Using Ricci curvature formula (Equation 3.1.4) for $\mathrm{Sp}(\infty)$ and $\mathfrak{sp}(\infty)$, we define, for $x \in \mathfrak{sp}(\infty)$,

$$\mathrm{Ric}(x) = \sum_{\xi \in \mathscr{B}_\lambda} K(x, \xi) = \sum_{\xi \in \mathscr{B}_\lambda} \langle R_{x\xi}(x), \xi \rangle_{\mathfrak{sp}} \tag{3.2.4}$$

By definition of $\mathscr{B}_\lambda$, the above sum will break into four parts:

$$\mathrm{Ric}(x) = \sum_{a>b>0} K(x, \mu_{ab}^{Re}) + \sum_{a\geq b>0} K(x, \mu_{ab}^{Im}) + \sum_{a\geq -b>0} K(x, \nu_{ab}^{Re}) + \sum_{a\geq -b>0} K(x, \nu_{ab}^{Im}) \tag{3.2.5}$$

For computational reason, we define the following *truncated Ricci curvature*:

$$\begin{aligned}
\mathrm{Ric}^N(x) = \sum_{N\geq a>b>0} K(x, \mu_{ab}^{Re}) &+ \sum_{N\geq a\geq b>0} K(x, \mu_{ab}^{Im}) \\
&+ \sum_{N\geq a\geq -b>0} K(x, \nu_{ab}^{Re}) + \sum_{N\geq a\geq -b>0} K(x, \nu_{ab}^{Im})
\end{aligned} \tag{3.2.6}$$

We have $\mathrm{Ric}(x) = \lim_{N\to\infty} \mathrm{Ric}^N(x)$.

In the rest of the section, we will compute the following Ricci curvatures via the corresponding truncated Ricci curvatures:

$$\mathrm{Ric}(\mu_{ab}^{Re}), \mathrm{Ric}(\mu_{ab}^{Im}), \mathrm{Ric}(\nu_{ab}^{Re}), \mathrm{Ric}(\nu_{ab}^{Im})$$

All of these computations boil down to matrix multiplications. The following lemma is an important tool to the computation of Ricci curvature.

Lemma 3.2.10. *We have the following Levi-Civita connection formula, where δ is the Kronecker delta:*

$$\begin{aligned}
\nabla_{\xi_{ab}}\xi_{cd} &= \delta_{bc}\lambda_c^2\xi_{ad} - \delta_{da}\lambda_a^2\xi_{cb} - \delta_{ca}\lambda_d^2\xi_{db} + \delta_{db}\lambda_c^2\xi_{ac} + \delta_{bd}\lambda_a^2\xi_{ca} - \delta_{ac}\lambda_b^2\xi_{bd} \\
\nabla_{i\xi_{ab}}i\xi_{cd} &= -\delta_{bc}\lambda_c^2\xi_{ad} + \delta_{da}\lambda_a^2\xi_{cb} - \delta_{ca}\lambda_d^2\xi_{db} + \delta_{db}\lambda_c^2\xi_{ac} + \delta_{bd}\lambda_a^2\xi_{ca} - \delta_{ac}\lambda_b^2\xi_{bd} \\
\nabla_{\xi_{ab}}i\xi_{cd} &= \delta_{bc}\lambda_c^2 i\xi_{ad} - \delta_{da}\lambda_a^2 i\xi_{cb} + \delta_{ca}\lambda_d^2 i\xi_{db} - \delta_{db}\lambda_c^2 i\xi_{ac} + \delta_{bd}\lambda_a^2 i\xi_{ca} - \delta_{ac}\lambda_b^2 i\xi_{bd} \\
\nabla_{i\xi_{ab}}\xi_{cd} &= \delta_{bc}\lambda_c^2 i\xi_{ad} - \delta_{da}\lambda_a^2 i\xi_{cb} - \delta_{ca}\lambda_d^2 i\xi_{db} + \delta_{db}\lambda_c^2 i\xi_{ac} - \delta_{bd}\lambda_a^2 i\xi_{ca} + \delta_{ac}\lambda_b^2 i\xi_{bd}
\end{aligned}$$

Proof. We have

$$\xi_{ab}\xi_{cd} = 2\lambda_b^2\delta_{cb}\xi_{ad}$$

So

$$[\xi_{ab}, \xi_{cd}] = \xi_{ab}\xi_{cd} - \xi_{cd}\xi_{ab} = 2\lambda_c^2\delta_{cb}\xi_{ad} - 2\lambda_a^2\delta_{ad}\xi_{cb}$$

In the following, $\langle\cdot,\cdot\rangle$ stands for $\langle\cdot,\cdot\rangle_{\mathrm{HS}}$. Using orthonormality,

$$\begin{aligned}
&2\langle\nabla_{\xi_{ab}}\xi_{cd}, \xi_{ef}\rangle \\
&\quad = \langle[\xi_{ab}, \xi_{cd}], \xi_{ef}\rangle - \langle[\xi_{cd}, \xi_{ef}], \xi_{ab}\rangle + \langle[\xi_{ef}, \xi_{ab}], \xi_{cd}\rangle \\
&\quad = 2\delta_{bc}\delta_{ae}\delta_{df}\lambda_c^2 - 2\delta_{da}\delta_{ce}\delta_{bf}\lambda_a^2 - 2\delta_{de}\delta_{ca}\delta_{fb}\lambda_e^2 \\
&\qquad\quad +2\delta_{fc}\delta_{ea}\delta_{db}\lambda_c^2 + 2\delta_{fa}\delta_{ec}\delta_{bd}\lambda_a^2 - 2\delta_{be}\delta_{ac}\delta_{fd}\lambda_e^2
\end{aligned}$$

and

$$2\langle\nabla_{\xi_{ab}}\xi_{cd}, i\xi_{ef}\rangle = \langle[\xi_{ab}, \xi_{cd}], i\xi_{ef}\rangle - \langle[\xi_{cd}, i\xi_{ef}], \xi_{ab}\rangle + \langle[i\xi_{ef}, \xi_{ab}], \xi_{cd}\rangle = 0$$

Therefore,

$$\nabla_{\xi_{ab}}\xi_{cd} = \delta_{bc}\lambda_c^2\xi_{ad} - \delta_{da}\lambda_a^2\xi_{cb} - \delta_{ca}\lambda_d^2\xi_{db} + \delta_{db}\lambda_c^2\xi_{ac} + \delta_{bd}\lambda_a^2\xi_{ca} - \delta_{ac}\lambda_b^2\xi_{bd}$$

Similarly,

$$\begin{aligned}
&2\langle\nabla_{i\xi_{ab}}i\xi_{cd}, \xi_{ef}\rangle \\
&\quad = \langle[i\xi_{ab}, i\xi_{cd}], \xi_{ef}\rangle - \langle[i\xi_{cd}, \xi_{ef}], i\xi_{ab}\rangle + \langle[\xi_{ef}, i\xi_{ab}], i\xi_{cd}\rangle \\
&\quad = -\delta_{bc}\delta_{ae}\delta_{df}\lambda_c^2 + \delta_{da}\delta_{ce}\delta_{bf}\lambda_a^2 - \delta_{de}\delta_{ca}\delta_{fb}\lambda_e^2 \\
&\qquad +\delta_{fc}\delta_{ea}\delta_{db}\lambda_c^2 + \delta_{fa}\delta_{ec}\delta_{bd}\lambda_a^2 - \delta_{be}\delta_{ac}\delta_{fd}\lambda_e^2
\end{aligned}$$

and

$$2\langle\nabla_{i\xi_{ab}}i\xi_{cd}, i\xi_{ef}\rangle = \langle[i\xi_{ab}, i\xi_{cd}], i\xi_{ef}\rangle - \langle[i\xi_{cd}, i\xi_{ef}], i\xi_{ab}\rangle + \langle[i\xi_{ef}, i\xi_{ab}], i\xi_{cd}\rangle = 0$$

Therefore,

$$\nabla_{i\xi_{ab}} i\xi_{cd} = -\delta_{bc}\lambda_c^2\xi_{ad} + \delta_{da}\lambda_a^2\xi_{cb} - \delta_{ca}\lambda_d^2\xi_{db} + \delta_{db}\lambda_c^2\xi_{ac} + \delta_{bd}\lambda_a^2\xi_{ca} - \delta_{ac}\lambda_b^2\xi_{bd}$$

Similarly,

$$\begin{aligned} &2\langle \nabla_{\xi_{ab}} i\xi_{cd}, i\xi_{ef}\rangle \\ &\quad = \langle [\xi_{ab}, i\xi_{cd}], i\xi_{ef}\rangle - \langle [i\xi_{cd}, i\xi_{ef}], \xi_{ab}\rangle + \langle [i\xi_{ef}, \xi_{ab}], i\xi_{cd}\rangle \\ &\quad = \delta_{bc}\delta_{ae}\delta_{df}\lambda_c^2 - \delta_{da}\delta_{ce}\delta_{bf}\lambda_a^2 + \delta_{de}\delta_{ca}\delta_{fb}\lambda_e^2 \\ &\qquad -\delta_{fc}\delta_{ea}\delta_{db}\lambda_c^2 + \delta_{fa}\delta_{ec}\delta_{bd}\lambda_a^2 - \delta_{be}\delta_{ac}\delta_{fd}\lambda_e^2 \end{aligned}$$

and

$$2\langle \nabla_{\xi_{ab}} i\xi_{cd}, \xi_{ef}\rangle = \langle [\xi_{ab}, i\xi_{cd}], \xi_{ef}\rangle - \langle [i\xi_{cd}, \xi_{ef}], \xi_{ab}\rangle + \langle [\xi_{ef}, \xi_{ab}], i\xi_{cd}\rangle = 0$$

Therefore,

$$\nabla_{\xi_{ab}} i\xi_{cd} = \delta_{bc}\lambda_c^2 i\xi_{ad} - \delta_{da}\lambda_a^2 i\xi_{cb} + \delta_{ca}\lambda_d^2 i\xi_{db} - \delta_{db}\lambda_c^2 i\xi_{ac} + \delta_{bd}\lambda_a^2 i\xi_{ca} - \delta_{ac}\lambda_b^2 i\xi_{bd}$$

Similarly,

$$\begin{aligned} &2\langle \nabla_{i\xi_{ab}} \xi_{cd}, i\xi_{ef}\rangle \\ &\quad = \langle [i\xi_{ab}, \xi_{cd}], i\xi_{ef}\rangle - \langle [\xi_{cd}, i\xi_{ef}], i\xi_{ab}\rangle + \langle [i\xi_{ef}, i\xi_{ab}], \xi_{cd}\rangle \\ &\quad = \delta_{bc}\delta_{ae}\delta_{df}\lambda_c^2 - \delta_{da}\delta_{ce}\delta_{bf}\lambda_a^2 - \delta_{de}\delta_{ca}\delta_{fb}\lambda_e^2 \\ &\qquad +\delta_{fc}\delta_{ea}\delta_{db}\lambda_c^2 - \delta_{fa}\delta_{ec}\delta_{bd}\lambda_a^2 + \delta_{be}\delta_{ac}\delta_{fd}\lambda_e^2 \end{aligned}$$

and

$$2\langle \nabla_{i\xi_{ab}} \xi_{cd}, \xi_{ef}\rangle = \langle [i\xi_{ab}, \xi_{cd}], \xi_{ef}\rangle - \langle [\xi_{cd}, \xi_{ef}], i\xi_{ab}\rangle + \langle [\xi_{ef}, i\xi_{ab}], \xi_{cd}\rangle = 0$$

Therefore,

$$\nabla_{i\xi_{ab}} \xi_{cd} = \delta_{bc}\lambda_c^2 i\xi_{ad} - \delta_{da}\lambda_a^2 i\xi_{cb} - \delta_{ca}\lambda_d^2 i\xi_{db} + \delta_{db}\lambda_c^2 i\xi_{ac} - \delta_{bd}\lambda_a^2 i\xi_{ca} + \delta_{ac}\lambda_b^2 i\xi_{bd}$$

□

Remark 3.2.11. Once we have the above lemma, we can use equation (3.2.3) to change the basis elements of $\mathfrak{sp}(\infty)$ into the basis elements of HS, and then use formula (3.1.1), (3.1.2), (3.1.3) and (3.1.4) to compute the Ricci curvature of $\mathrm{Sp}(\infty)$. But since each basis μ_{ab}^{Re}, μ_{ab}^{Im}, ν_{ab}^{Re}, and ν_{ab}^{Im} has four terms, and each connection formula in the above lemma has six terms, the combination will be huge. For example, the sectional curvature

$$\begin{aligned} K(\mu_{ab}^{Re}, \mu_{cd}^{Re}) &= \langle R_{\mu_{ab}^{Re}\mu_{cd}^{Re}}(\mu_{ab}^{Re}), \mu_{cd}^{Re}\rangle \\ &= \langle \nabla_{[\mu_{ab}^{Re}, \mu_{cd}^{Re}]}(\mu_{ab}^{Re}) - \nabla_{\mu_{ab}^{Re}}\nabla_{\mu_{cd}^{Re}}(\mu_{ab}^{Re}) + \nabla_{\mu_{cd}^{Re}}\nabla_{\mu_{ab}^{Re}}(\mu_{ab}^{Re}), \mu_{cd}^{Re}\rangle \end{aligned}$$

will have $21,504$ terms. Thereore, I use a computer program to facilitate the computation.

Theorem 3.2.12. *Let $a,b \in \mathbb{Z}\backslash\{0\}$.*

For $a > b > 0$,

$$Ric^N(\mu^{Re}_{ab}) = \frac{1}{16}\Big[-24\lambda_a^4 - 24\lambda_b^4 + 48\lambda_a^2\lambda_b^2 - 12\lambda_a^2\sum_{d=1}^{a-1}\lambda_d^2 + 8\lambda_a^2\sum_{d=1}^{b-1}\lambda_d^2 + 8\lambda_b^2\sum_{d=1}^{a-1}\lambda_d^2$$
$$-12\lambda_b^2\sum_{d=1}^{b-1}\lambda_d^2 + 8\sum_{d=1}^{a-1}\lambda_d^4 + 8\sum_{d=1}^{b-1}\lambda_d^4 - 16N\lambda_a^4 - 16N\lambda_b^4 - 12\lambda_a^2\sum_{c=a+1}^{N}\lambda_c^2$$
$$+8\lambda_a^2\sum_{c=b+1}^{N}\lambda_c^2 + 8\lambda_b^2\sum_{c=a+1}^{N}\lambda_c^2 - 12\lambda_b^2\sum_{c=b+1}^{N}\lambda_c^2 + 8\sum_{c=a+1}^{N}\lambda_c^4 + 8\sum_{c=b+1}^{N}\lambda_c^4\Big].$$

For $a > b > 0$,

$$Ric^N(\mu^{Im}_{ab}) = \frac{1}{16}\Big[-40\lambda_a^4 - 40\lambda_b^4 - 32\lambda_a^2\lambda_b^2 - 12\lambda_a^2\sum_{d=1}^{a-1}\lambda_d^2 - 8\lambda_a^2\sum_{d=1}^{b-1}\lambda_d^2 - 8\lambda_b^2\sum_{d=1}^{a-1}\lambda_d^2$$
$$-12\lambda_b^2\sum_{d=1}^{b-1}\lambda_d^2 + 8\sum_{d=1}^{a-1}\lambda_d^4 + 8\sum_{d=1}^{b-1}\lambda_d^4 - 16N\lambda_a^4 - 16N\lambda_b^4 - 12\lambda_a^2\sum_{c=a+1}^{N}\lambda_c^2$$
$$-8\lambda_a^2\sum_{c=b+1}^{N}\lambda_c^2 - 8\lambda_b^2\sum_{c=a+1}^{N}\lambda_c^2 - 12\lambda_b^2\sum_{c=b+1}^{N}\lambda_c^2 + 8\sum_{c=a+1}^{N}\lambda_c^4 + 8\sum_{c=b+1}^{N}\lambda_c^4\Big].$$

For $a = b > 0$,

$$Ric^N(\mu^{Im}_{ab}) = 0.$$

For $a > -b > 0$,

$$Ric^N(\nu^{Re}_{ab}) = \frac{1}{16}\Big[-40\lambda_a^4 - 40\lambda_b^4 - 48\lambda_a^2\lambda_b^2 - 12\lambda_a^2\sum_{d=1}^{a-1}\lambda_d^2 - 8\lambda_a^2\sum_{d=1}^{b-1}\lambda_d^2 - 8\lambda_b^2\sum_{d=1}^{a-1}\lambda_d^2$$
$$-12\lambda_b^2\sum_{d=1}^{b-1}\lambda_d^2 + 8\sum_{d=1}^{a-1}\lambda_d^4 + 8\sum_{d=1}^{b-1}\lambda_d^4 - 16N\lambda_a^4 - 16N\lambda_b^4 - 12\lambda_a^2\sum_{c=a+1}^{N}\lambda_c^2$$
$$-8\lambda_a^2\sum_{c=b+1}^{N}\lambda_c^2 - 8\lambda_b^2\sum_{c=a+1}^{N}\lambda_c^2 - 12\lambda_b^2\sum_{c=b+1}^{N}\lambda_c^2 + 8\sum_{c=a+1}^{N}\lambda_c^4 + 8\sum_{c=b+1}^{N}\lambda_c^4\Big].$$

For $a = -b > 0$,

$$Ric^N(\nu^{Re}_{ab}) = \frac{1}{16}\Big[-192\lambda_a^4 - 32\sum_{d=1}^{a-1}\lambda_d^4 - 192N\lambda_a^4 - 32\sum_{c=a+1}^{N}\lambda_c^4\Big].$$

For $a > -b > 0$,

$$R^N(\nu_{ab}^{Im}) = \frac{1}{16}\Big[-40\lambda_a^4 - 40\lambda_b^4 - 32\lambda_a^2\lambda_b^2 - 12\lambda_a^2\sum_{d=1}^{a-1}\lambda_d^2 - 8\lambda_a^2\sum_{d=1}^{b-1}\lambda_d^2 - 8\lambda_b^2\sum_{d=1}^{a-1}\lambda_d^2$$
$$- 12\lambda_b^2\sum_{d=1}^{b-1}\lambda_d^2 + 8\sum_{d=1}^{a-1}\lambda_d^4 + 8\sum_{d=1}^{b-1}\lambda_d^4 - 16N\lambda_a^4 - 16N\lambda_b^4 - 12\lambda_a^2\sum_{c=a+1}^{N}\lambda_c^2$$
$$- 8\lambda_a^2\sum_{c=b+1}^{N}\lambda_c^2 - 8\lambda_b^2\sum_{c=a+1}^{N}\lambda_c^2 - 12\lambda_b^2\sum_{c=b+1}^{N}\lambda_c^2 + 8\sum_{c=a+1}^{N}\lambda_c^4 + 8\sum_{c=b+1}^{N}\lambda_c^4\Big].$$

For $a = -b > 0$,

$$Ric^N(\nu_{ab}^{Im}) = 0.$$

Corollary 3.2.13. *If we set the parameter $\lambda_i = 1/\sqrt{2}$, for all $i \in \mathbb{Z}\backslash\{0\}$, then we recover the canonical inner product on the space HS (remark 3.2.3). In this case, we have*

$$\begin{aligned}
&Ric^N(\mu_{ab}^{Re}) = -\frac{3}{8}N - \frac{1}{8}, && \textit{for } a > b > 0;\\
&Ric^N(\mu_{ab}^{Im}) = -\frac{7}{8}N - \frac{11}{8}, && \textit{for } a > b > 0;\\
&Ric^N(\mu_{ab}^{Im}) = 0, && \textit{for } a = b > 0;\\
&Ric^N(\nu_{ab}^{Re}) = -\frac{7}{8}N - \frac{13}{8}, && \textit{for } a > -b > 0;\\
&Ric^N(\nu_{ab}^{Re}) = -\frac{7}{2}N - \frac{5}{2}, && \textit{for } a = -b > 0;\\
&Ric^N(\nu_{ab}^{Im}) = -\frac{7}{8}N - \frac{11}{8}, && \textit{for } a > -b > 0;\\
&Ric^N(\nu_{ab}^{Im}) = 0, && \textit{for } a = -b > 0.
\end{aligned}$$

Remark 3.2.14. By the above corollary, we see that for most of the basis element $\xi \in \mathscr{B}_\lambda$, we have $\mathrm{Ric}(\xi) = \lim_{N\to\infty}\mathrm{Ric}^N(\xi) = -\infty$.

Proof. (of the theorem.)

The method of computing Ricci curvature and truncated Ricci curvature is stated in Definition 3.2.9. Ricci curvature is defined in terms of sectional curvature, which can be expressed in terms of Riemann tensor and the inner product of the Lie algebra. Riemann tensor is defined in terms of Levi-Civita connection. The formula of Levi-Civita connection is the content of Lemma 3.2.10. So the method of computing Ricci curvature is straightforward. But there are huge number of terms. Therefore, I used a computer program to facilitate the computation.

$$
\begin{aligned}
&\mathrm{Ric}^N(\mu_{ab}^{Re}) \\
&= \sum_{N\geq c>d>0} K(\mu_{ab}^{Re},\mu_{cd}^{Re}) + \sum_{N\geq c\geq d>0} K(\mu_{ab}^{Re},\mu_{cd}^{Im}) + \sum_{N\geq c\geq -d>0} K(\mu_{ab}^{Re},\nu_{cd}^{Re}) + \sum_{N\geq c\geq -d>0} K(\mu_{ab}^{Re},\nu_{cd}^{Im}) \\
&= \sum_{N\geq c>d>0} K(\mu_{ab}^{Re},\mu_{cd}^{Re}) + \sum_{N\geq c\geq d>0} K(\mu_{ab}^{Re},\mu_{cd}^{Im}) + \sum_{N\geq c\geq d>0} K(\mu_{ab}^{Re},\nu_{c,-d}^{Re}) + \sum_{N\geq c\geq d>0} K(\mu_{ab}^{Re},\nu_{c,-d}^{Im}) \\
&= \sum_{N\geq c>d>0} \Big[K(\mu_{ab}^{Re},\mu_{cd}^{Re}) + K(\mu_{ab}^{Re},\mu_{cd}^{Im}) + K(\mu_{ab}^{Re},\nu_{c,-d}^{Re}) + K(\mu_{ab}^{Re},\nu_{c,-d}^{Im})\Big] \\
&\qquad + \sum_{N\geq c=d>0} \Big[K(\mu_{ab}^{Re},\mu_{cd}^{Re}) + K(\mu_{ab}^{Re},\mu_{cd}^{Im}) + K(\mu_{ab}^{Re},\nu_{c,-d}^{Re}) + K(\mu_{ab}^{Re},\nu_{c,-d}^{Im})\Big] \\
&:= \sum_{N\geq c>d>0} A_{upper} + \sum_{N\geq c=d>0} A_{diagonal}
\end{aligned}
$$

We have

$$
\begin{aligned}
A_{upper} = \frac{1}{16}\Big[&-16\delta_{a,c}\lambda_a^4 - 24\delta_{a,c}\delta_{a,d}\lambda_a^4 + 24\delta_{a,c}\delta_{b,d}\lambda_a^4 - 16\delta_{a,d}\lambda_a^4 \\
&+24\delta_{a,d}\delta_{b,c}\lambda_a^4 + 8\delta_{a,c}\delta_{a,d}\lambda_a^2\lambda_b^2 + 8\delta_{b,c}\delta_{b,d}\lambda_a^2\lambda_b^2 - 12\delta_{a,d}\lambda_a^2\lambda_c^2 \\
&+8\delta_{b,d}\lambda_a^2\lambda_c^2 - 12\delta_{a,c}\lambda_a^2\lambda_d^2 + 8\delta_{b,c}\lambda_a^2\lambda_d^2 + 24\delta_{a,c}\delta_{b,d}\lambda_b^4 + 24\delta_{a,d}\delta_{b,c}\lambda_b^4 \\
&-16\delta_{b,c}\lambda_b^4 - 24\delta_{b,c}\delta_{b,d}\lambda_b^4 - 16\delta_{b,d}\lambda_b^4 + 8\delta_{a,d}\lambda_b^2\lambda_c^2 - 12\delta_{b,d}\lambda_b^2\lambda_c^2 \\
&+8\delta_{a,c}\lambda_b^2\lambda_d^2 - 12\delta_{b,c}\lambda_b^2\lambda_d^2 + 8\delta_{a,d}\lambda_c^4 + 8\delta_{b,d}\lambda_c^4 + 8\delta_{a,c}\lambda_d^4 + 8\delta_{b,c}\lambda_d^4\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
A_{diagonal} = \frac{1}{16}\Big[&-12\delta_{a,c}\lambda_a^4 - 16\delta_{a,c}\delta_{a,d}\lambda_a^4 + 18\delta_{a,c}\delta_{b,d}\lambda_a^4 - 12\delta_{a,d}\lambda_a^4 \\
&+18\delta_{a,d}\delta_{b,c}\lambda_a^4 + 12\delta_{a,c}\delta_{b,d}\lambda_a^2\lambda_b^2 + 12\delta_{a,d}\delta_{b,c}\lambda_a^2\lambda_b^2 - 18\delta_{a,d}\lambda_a^2\lambda_c^2 \\
&+12\delta_{b,d}\lambda_a^2\lambda_c^2 - 18\delta_{a,c}\lambda_a^2\lambda_d^2 + 12\delta_{b,c}\lambda_a^2\lambda_d^2 + 18\delta_{a,c}\delta_{b,d}\lambda_b^4 + 18\delta_{a,d}\delta_{b,c}\lambda_b^4 \\
&-12\delta_{b,c}\lambda_b^4 - 16\delta_{b,c}\delta_{b,d}\lambda_b^4 - 12\delta_{b,d}\lambda_b^4 + 12\delta_{a,d}\lambda_b^2\lambda_c^2 - 18\delta_{b,d}\lambda_b^2\lambda_c^2 \\
&+12\delta_{a,c}\lambda_b^2\lambda_d^2 - 18\delta_{b,c}\lambda_b^2\lambda_d^2 + 6\delta_{a,d}\lambda_c^4 + 6\delta_{b,d}\lambda_c^4 + 6\delta_{a,c}\lambda_d^4 + 6\delta_{b,c}\lambda_d^4\Big]
\end{aligned}
$$

So

$$\sum_{N\geq c>d>0} A_{upper} = \frac{1}{16}\Big[-16(a-1)\lambda_a^4 + 24\lambda_a^4 - 16(N-a)\lambda_a^4 - 12\lambda_a^2\sum_{c=a+1}^{N}\lambda_c^2$$

$$+8\lambda_a^2\sum_{c=b+1}^{N}\lambda_c^2 - 12\lambda_a^2\sum_{d=1}^{a-1}\lambda_d^2 + 8\lambda_a^2\sum_{d=1}^{b-1}\lambda_d^2 + 24\lambda_b^4$$

$$-16(b-1)\lambda_b^4 - 16(N-b)\lambda_b^4 + 8\lambda_b^2\sum_{c=a+1}^{N}\lambda_c^2 - 12\lambda_b^2\sum_{c=b+1}^{N}\lambda_c^2 + 8\lambda_b^2\sum_{d=1}^{a-1}\lambda_d^2$$

$$-12\lambda_b^2\sum_{d=1}^{b-1}\lambda_d^2 + 8\sum_{c=a+1}^{N}\lambda_c^4 + 8\sum_{c=b+1}^{N}\lambda_c^4 + 8\sum_{d=1}^{a-1}\lambda_d^4 + 8\sum_{d=1}^{b-1}\lambda_d^4\Big]$$

and

$$\sum_{N\geq c=d>0} A_{diagonal} = \frac{1}{16}\Big[-12\lambda_a^4 - 16\lambda_a^4 - 12\lambda_a^4 - 18\lambda_a^4 + 12\lambda_a^2\lambda_b^2 - 18\lambda_a^4 + 12\lambda_a^2\lambda_b^2 - 12\lambda_b^4$$

$$-16\lambda_b^4 - 12\lambda_b^4 + 12\lambda_a^2\lambda_b^2 - 18\lambda_b^4 + 12\lambda_a^2\lambda_b^2 - 18\lambda_b^4 + 6\lambda_a^4 + 6\lambda_b^4 + 6\lambda_a^4 + 6\lambda_b^4\Big]$$

Therefore, for $a > b > 0$,

$$\mathrm{Ric}^N(\mu_{ab}^{Re}) = \sum_{N\geq c>d>0} A_{upper} + \sum_{N\geq c=d>0} A_{diagonal}$$

$$= \frac{1}{16}\Big[-24\lambda_a^4 - 24\lambda_b^4 + 48\lambda_a^2\lambda_b^2 - 12\lambda_a^2\sum_{d=1}^{a-1}\lambda_d^2 + 8\lambda_a^2\sum_{d=1}^{b-1}\lambda_d^2 + 8\lambda_b^2\sum_{d=1}^{a-1}\lambda_d^2$$

$$-12\lambda_b^2\sum_{d=1}^{b-1}\lambda_d^2 + 8\sum_{d=1}^{a-1}\lambda_d^4 + 8\sum_{d=1}^{b-1}\lambda_d^4 - 16N\lambda_a^4 - 16N\lambda_b^4 - 12\lambda_a^2\sum_{c=a+1}^{N}\lambda_c^2$$

$$+8\lambda_a^2\sum_{c=b+1}^{N}\lambda_c^2 + 8\lambda_b^2\sum_{c=a+1}^{N}\lambda_c^2 - 12\lambda_b^2\sum_{c=b+1}^{N}\lambda_c^2 + 8\sum_{c=a+1}^{N}\lambda_c^4 + 8\sum_{c=b+1}^{N}\lambda_c^4\Big]$$

Next,

$$
\begin{aligned}
&\mathrm{Ric}^N(\mu^{Im}_{ab})\\
&= \sum_{N\geq c>d>0} K(\mu^{Im}_{ab},\mu^{Re}_{cd}) + \sum_{N\geq c\geq d>0} K(\mu^{Im}_{ab},\mu^{Im}_{cd}) + \sum_{N\geq c\geq -d>0} K(\mu^{Im}_{ab},\nu^{Re}_{cd}) + \sum_{N\geq c\geq -d>0} K(\mu^{Im}_{ab},\nu^{Im}_{cd})\\
&= \sum_{N\geq c>d>0} K(\mu^{Im}_{ab},\mu^{Re}_{cd}) + \sum_{N\geq c\geq d>0} K(\mu^{Im}_{ab},\mu^{Im}_{cd}) + \sum_{N\geq c\geq d>0} K(\mu^{Im}_{ab},\nu^{Re}_{c,-d}) + \sum_{N\geq c\geq d>0} K(\mu^{Im}_{ab},\nu^{Im}_{c,-d})\\
&= \sum_{N\geq c>d>0} \Big[K(\mu^{Im}_{ab},\mu^{Re}_{cd}) + K(\mu^{Im}_{ab},\mu^{Im}_{cd}) + K(\mu^{Im}_{ab},\nu^{Re}_{c,-d}) + K(\mu^{Im}_{ab},\nu^{Im}_{c,-d})\Big]\\
&\qquad + \sum_{N\geq c=d>0} \Big[K(\mu^{Im}_{ab},\mu^{Re}_{cd}) + K(\mu^{Im}_{ab},\mu^{Im}_{cd}) + K(\mu^{Im}_{ab},\nu^{Re}_{c,-d}) + K(\mu^{Im}_{ab},\nu^{Im}_{c,-d})\Big]\\
&:= \sum_{N\geq c>d>0} B_{upper} + \sum_{N\geq c=d>0} B_{diagonal}
\end{aligned}
$$

We have

$$
\begin{aligned}
B_{upper} = \frac{1}{16}\Big[&+32\delta_{a,b}\delta_{a,c}\lambda_a^4 + 32\delta_{a,b}\delta_{a,d}\lambda_a^4 - 16\delta_{a,c}\lambda_a^4 - 24\delta_{a,c}\delta_{a,d}\lambda_a^4 + 8\delta_{a,c}\delta_{b,d}\lambda_a^4\\
&-16\delta_{a,d}\lambda_a^4 + 8\delta_{a,d}\delta_{b,c}\lambda_a^4 - 8\delta_{a,c}\delta_{a,d}\lambda_a^2\lambda_b^2 + 16\delta_{a,c}\delta_{b,d}\lambda_a^2\lambda_b^2 + 16\delta_{a,d}\delta_{b,c}\lambda_a^2\lambda_b^2\\
&-8\delta_{b,c}\delta_{b,d}\lambda_a^2\lambda_b^2 + 40\delta_{a,b}\delta_{a,d}\lambda_a^2\lambda_c^2 - 12\delta_{a,d}\lambda_a^2\lambda_c^2 - 8\delta_{b,d}\lambda_a^2\lambda_c^2 + 40\delta_{a,b}\delta_{a,c}\lambda_a^2\lambda_d^2\\
&-12\delta_{a,c}\lambda_a^2\lambda_d^2 - 8\delta_{b,c}\lambda_a^2\lambda_d^2 + 8\delta_{a,c}\delta_{b,d}\lambda_b^4 + 8\delta_{a,d}\delta_{b,c}\lambda_b^4 - 16\delta_{b,c}\lambda_b^4 - 24\delta_{b,c}\delta_{b,d}\lambda_b^4\\
&-16\delta_{b,d}\lambda_b^4 - 8\delta_{a,d}\lambda_b^2\lambda_c^2 - 12\delta_{b,d}\lambda_b^2\lambda_c^2 - 8\delta_{a,c}\lambda_b^2\lambda_d^2 - 12\delta_{b,c}\lambda_b^2\lambda_d^2\\
&-16\delta_{a,b}\delta_{a,d}\lambda_c^4 + 8\delta_{a,d}\lambda_c^4 + 8\delta_{b,d}\lambda_c^4 - 16\delta_{a,b}\delta_{a,c}\lambda_d^4 + 8\delta_{a,c}\lambda_d^4 + 8\delta_{b,c}\lambda_d^4\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
B_{diagonal} = \frac{1}{16}\Big[&+24\delta_{a,b}\delta_{a,c}\lambda_a^4 - 32\delta_{a,b}\delta_{a,c}\delta_{a,d}\lambda_a^4 + 24\delta_{a,b}\delta_{a,d}\lambda_a^4 - 12\delta_{a,c}\lambda_a^4 - 16\delta_{a,c}\delta_{a,d}\lambda_a^4\\
&+6\delta_{a,c}\delta_{b,d}\lambda_a^4 - 12\delta_{a,d}\lambda_a^4 + 6\delta_{a,d}\delta_{b,c}\lambda_a^4 + 20\delta_{a,c}\delta_{b,d}\lambda_a^2\lambda_b^2 + 20\delta_{a,d}\delta_{b,c}\lambda_a^2\lambda_b^2\\
&+60\delta_{a,b}\delta_{a,d}\lambda_a^2\lambda_c^2 - 18\delta_{a,d}\lambda_a^2\lambda_c^2 - 12\delta_{b,d}\lambda_a^2\lambda_c^2 + 60\delta_{a,b}\delta_{a,c}\lambda_a^2\lambda_d^2 - 18\delta_{a,c}\lambda_a^2\lambda_d^2\\
&-12\delta_{b,c}\lambda_a^2\lambda_d^2 + 6\delta_{a,c}\delta_{b,d}\lambda_b^4 + 6\delta_{a,d}\delta_{b,c}\lambda_b^4 - 12\delta_{b,c}\lambda_b^4 - 16\delta_{b,c}\delta_{b,d}\lambda_b^4\\
&-12\delta_{b,d}\lambda_b^4 - 12\delta_{a,d}\lambda_b^2\lambda_c^2 - 18\delta_{b,d}\lambda_b^2\lambda_c^2 - 12\delta_{a,c}\lambda_b^2\lambda_d^2 - 18\delta_{b,c}\lambda_b^2\lambda_d^2\\
&-12\delta_{a,b}\delta_{a,d}\lambda_c^4 + 6\delta_{a,d}\lambda_c^4 + 6\delta_{b,d}\lambda_c^4 - 12\delta_{a,b}\delta_{a,c}\lambda_d^4 + 6\delta_{a,c}\lambda_d^4 + 6\delta_{b,c}\lambda_d^4\Big]
\end{aligned}
$$

For $a > b > 0$,

$$\sum_{N\geq c>d>0} B_{upper} = \frac{1}{16}\Big[-16(a-1)\lambda_a^4 + 8\lambda_a^4 - 16(N-a)\lambda_a^4 + 16\lambda_a^2\lambda_b^2 - 12\lambda_a^2\sum_{c=a+1}^{N}\lambda_c^2$$
$$-8\lambda_a^2\sum_{c=b+1}^{N}\lambda_c^2 - 12\lambda_a^2\sum_{d=1}^{a-1}\lambda_d^2 - 8\lambda_a^2\sum_{d=1}^{b-1}\lambda_d^2 + 8\lambda_b^4 - 16(b-1)\lambda_b^4$$
$$-16(N-b)\lambda_b^4 - 8\lambda_b^2\sum_{c=a+1}^{N}\lambda_c^2 - 12\lambda_b^2\sum_{c=b+1}^{N}\lambda_c^2 - 8\lambda_b^2\sum_{d=1}^{a-1}\lambda_d^2 - 12\lambda_b^2\sum_{d=1}^{b-1}\lambda_d^2$$
$$+8\sum_{c=a+1}^{N}\lambda_c^4 + 8\sum_{c=b+1}^{N}\lambda_c^4 + 8\sum_{d=1}^{a-1}\lambda_d^4 + 8\sum_{d=1}^{b-1}\lambda_d^4\Big]$$

For $a = b > 0$,

$$\sum_{N\geq c>d>0} B_{upper} = \frac{1}{16}\Big[+32(a-1)\lambda_a^4 + 32(N-a)\lambda_a^4 - 16(a-1)\lambda_a^4 - 16(N-a)\lambda_a^4$$
$$+40\lambda_a^2\sum_{c=a+1}^{N}\lambda_c^2 - 12\lambda_a^2\sum_{c=a+1}^{N}\lambda_c^2 - 8\lambda_a^2\sum_{c=a+1}^{N}\lambda_c^2 + 40\lambda_a^2\sum_{d=1}^{a-1}\lambda_d^2 - 12\lambda_a^2\sum_{d=1}^{a-1}\lambda_d^2 - 8\lambda_a^2\sum_{d=1}^{a-1}\lambda_d^2$$
$$-16(a-1)\lambda_a^4 - 16(N-a)\lambda_a^4 - 8\lambda_a^2\sum_{c=a+1}^{N}\lambda_c^2 - 12\lambda_a^2\sum_{c=a+1}^{N}\lambda_c^2 - 8\lambda_a^2\sum_{d=1}^{a-1}\lambda_d^2$$
$$-12\lambda_a^2\sum_{d=1}^{a-1}\lambda_d^2 - 16\sum_{c=a+1}^{N}\lambda_c^4 + 8\sum_{c=a+1}^{N}\lambda_c^4 + 8\sum_{c=a+1}^{N}\lambda_c^4 - 16\sum_{d=1}^{a-1}\lambda_d^4 + 8\sum_{d=1}^{a-1}\lambda_d^4 + 8\sum_{d=1}^{a-1}\lambda_d^4\Big]$$

For $a > b > 0$,

$$\sum_{N\geq c=d>0} B_{diagonal} = \frac{1}{16}\Big[-12\lambda_a^4 - 16\lambda_a^4 - 12\lambda_a^4 - 18\lambda_a^4 - 12\lambda_a^2\lambda_b^2 - 18\lambda_a^4 - 12\lambda_a^2\lambda_b^2 - 12\lambda_b^4$$
$$-16\lambda_b^4 - 12\lambda_b^4 - 12\lambda_a^2\lambda_b^2 - 18\lambda_b^4 - 12\lambda_a^2\lambda_b^2 - 18\lambda_b^4 + 6\lambda_a^4 + 6\lambda_b^4 + 6\lambda_a^4 + 6\lambda_b^4\Big]$$

For $a = b > 0$,

$$\sum_{N\geq c=d>0} B_{diagonal} = 0$$

Therefore, for $a > b > 0$,

$$
\begin{aligned}
\mathrm{Ric}^N(\mu_{ab}^{Im}) &= \sum_{N\geq c>d>0} B_{upper} + \sum_{N\geq c=d>0} B_{diagonal} \\
&= \frac{1}{16}\Big[-40\lambda_a^4 - 40\lambda_b^4 - 32\lambda_a^2\lambda_b^2 - 12\lambda_a^2\sum_{d=1}^{a-1}\lambda_d^2 - 8\lambda_a^2\sum_{d=1}^{b-1}\lambda_d^2 - 8\lambda_b^2\sum_{d=1}^{a-1}\lambda_d^2 \\
&\quad -12\lambda_b^2\sum_{d=1}^{b-1}\lambda_d^2 + 8\sum_{d=1}^{a-1}\lambda_d^4 + 8\sum_{d=1}^{b-1}\lambda_d^4 - 16N\lambda_a^4 - 16N\lambda_b^4 - 12\lambda_a^2\sum_{c=a+1}^{N}\lambda_c^2 \\
&\quad -8\lambda_a^2\sum_{c=b+1}^{N}\lambda_c^2 - 8\lambda_b^2\sum_{c=a+1}^{N}\lambda_c^2 - 12\lambda_b^2\sum_{c=b+1}^{N}\lambda_c^2 + 8\sum_{c=a+1}^{N}\lambda_c^4 + 8\sum_{c=b+1}^{N}\lambda_c^4\Big]
\end{aligned}
$$

and for $a = b > 0$,

$$
R^N(\mu_{ab}^{Im}) = \sum_{N\geq c>d>0} B_{upper} + \sum_{N\geq c=d>0} B_{diagonal} = 0
$$

Next, we compute $R^N(\nu_{ab}^{Re})$ for $a \geq -b > 0$. Replacing b with $-b$, it's equivalent to computing $R^N(\nu_{a,-b}^{Re})$ for $a \geq b > 0$.

$$
\begin{aligned}
&\mathrm{Ric}^N(\nu_{a,-b}^{Re}) \\
&= \sum_{N\geq c>d>0} K(\nu_{a,-b}^{Re}, \mu_{cd}^{Re}) + \sum_{N\geq c\geq d>0} K(\nu_{a,-b}^{Re}, \mu_{cd}^{Im}) \\
&\qquad + \sum_{N\geq c\geq -d>0} K(\nu_{a,-b}^{Re}, \nu_{cd}^{Re}) + \sum_{N\geq c\geq -d>0} K(\nu_{a,-b}^{Re}, \nu_{cd}^{Im}) \\
&= \sum_{N\geq c>d>0} K(\nu_{a,-b}^{Re}, \mu_{cd}^{Re}) + \sum_{N\geq c\geq d>0} K(\nu_{a,-b}^{Re}, \mu_{cd}^{Im}) \\
&\qquad + \sum_{N\geq c\geq d>0} K(\nu_{a,-b}^{Re}, \nu_{c,-d}^{Re}) + \sum_{N\geq c\geq d>0} K(\nu_{a,-b}^{Re}, \nu_{c,-d}^{Im}) \\
&= \sum_{N\geq c>d>0} \Big[K(\nu_{a,-b}^{Re}, \mu_{cd}^{Re}) + K(\nu_{a,-b}^{Re}, \mu_{cd}^{Im}) + K(\nu_{a,-b}^{Re}, \nu_{c,-d}^{Re}) + K(\nu_{a,-b}^{Re}, \nu_{c,-d}^{Im})\Big] \\
&\qquad + \sum_{N\geq c=d>0} \Big[K(\nu_{a,-b}^{Re}, \mu_{cd}^{Re}) + K(\nu_{a,-b}^{Re}, \mu_{cd}^{Im}) + K(\nu_{a,-b}^{Re}, \nu_{c,-d}^{Re}) + K(\nu_{a,-b}^{Re}, \nu_{c,-d}^{Im})\Big] \\
&:= \sum_{N\geq c>d>0} C_{upper} + \sum_{N\geq c=d>0} C_{diagonal}
\end{aligned}
$$

We have

$$C_{upper} = \frac{1}{16}\Big[-160\delta_{a,b}\delta_{a,c}\lambda_a^4 + 480\delta_{a,b}\delta_{a,c}\delta_{a,d}\lambda_a^4 - 160\delta_{a,b}\delta_{a,d}\lambda_a^4 - 16\delta_{a,c}\lambda_a^4$$
$$-24\delta_{a,c}\delta_{a,d}\lambda_a^4 + 8\delta_{a,c}\delta_{b,d}\lambda_a^4 - 16\delta_{a,d}\lambda_a^4 + 8\delta_{a,d}\delta_{b,c}\lambda_a^4 - 8\delta_{a,c}\delta_{a,d}\lambda_a^2\lambda_b^2$$
$$-8\delta_{b,c}\delta_{b,d}\lambda_a^2\lambda_b^2 - 24\delta_{a,b}\delta_{a,d}\lambda_a^2\lambda_c^2 - 12\delta_{a,d}\lambda_a^2\lambda_c^2 - 8\delta_{b,d}\lambda_a^2\lambda_c^2$$
$$-24\delta_{a,b}\delta_{a,c}\lambda_a^2\lambda_d^2 - 12\delta_{a,c}\lambda_a^2\lambda_d^2 - 8\delta_{b,c}\lambda_a^2\lambda_d^2 + 8\delta_{a,c}\delta_{b,d}\lambda_b^4 + 8\delta_{a,d}\delta_{b,c}\lambda_b^4$$
$$-16\delta_{b,c}\lambda_b^4 - 24\delta_{b,c}\delta_{b,d}\lambda_b^4 - 16\delta_{b,d}\lambda_b^4 - 8\delta_{a,d}\lambda_b^2\lambda_c^2 - 12\delta_{b,d}\lambda_b^2\lambda_c^2 - 8\delta_{a,c}\lambda_b^2\lambda_d^2$$
$$-12\delta_{b,c}\lambda_b^2\lambda_d^2 + 16\delta_{a,b}\delta_{a,d}\lambda_c^4 + 8\delta_{a,d}\lambda_c^4 + 8\delta_{b,d}\lambda_c^4 + 16\delta_{a,b}\delta_{a,c}\lambda_d^4 + 8\delta_{a,c}\lambda_d^4 + 8\delta_{b,c}\lambda_d^4\Big]$$

and

$$C_{diagonal} = \frac{1}{16}\Big[-120\delta_{a,b}\delta_{a,c}\lambda_a^4 + 480\delta_{a,b}\delta_{a,c}\delta_{a,d}\lambda_a^4 - 120\delta_{a,b}\delta_{a,d}\lambda_a^4 - 12\delta_{a,c}\lambda_a^4$$
$$-16\delta_{a,c}\delta_{a,d}\lambda_a^4 - 6\delta_{a,c}\delta_{b,d}\lambda_a^4 - 12\delta_{a,d}\lambda_a^4 - 6\delta_{a,d}\delta_{b,c}\lambda_a^4 + 4\delta_{a,c}\delta_{b,d}\lambda_a^2\lambda_b^2$$
$$+4\delta_{a,d}\delta_{b,c}\lambda_a^2\lambda_b^2 - 36\delta_{a,b}\delta_{a,d}\lambda_a^2\lambda_c^2 - 18\delta_{a,d}\lambda_a^2\lambda_c^2 - 12\delta_{b,d}\lambda_a^2\lambda_c^2 - 36\delta_{a,b}\delta_{a,c}\lambda_a^2\lambda_d^2$$
$$-18\delta_{a,c}\lambda_a^2\lambda_d^2 - 12\delta_{b,c}\lambda_a^2\lambda_d^2 - 6\delta_{a,c}\delta_{b,d}\lambda_b^4 - 6\delta_{a,d}\delta_{b,c}\lambda_b^4 - 12\delta_{b,c}\lambda_b^4 - 16\delta_{b,c}\delta_{b,d}\lambda_b^4$$
$$-12\delta_{b,d}\lambda_b^4 - 12\delta_{a,d}\lambda_b^2\lambda_c^2 - 18\delta_{b,d}\lambda_b^2\lambda_c^2 - 12\delta_{a,c}\lambda_b^2\lambda_d^2 - 18\delta_{b,c}\lambda_b^2\lambda_d^2$$
$$+12\delta_{a,b}\delta_{a,d}\lambda_c^4 + 6\delta_{a,d}\lambda_c^4 + 6\delta_{b,d}\lambda_c^4 + 12\delta_{a,b}\delta_{a,c}\lambda_d^4 + 6\delta_{a,c}\lambda_d^4 + 6\delta_{b,c}\lambda_d^4\Big]$$

For $a > b > 0$,

$$\sum_{N\geq c>d>0} C_{upper} = \frac{1}{16}\Big[-16(a-1)\lambda_a^4 + 8\lambda_a^4 - 16(N-a)\lambda_a^4 - 12\lambda_a^2\sum_{c=a+1}^{N}\lambda_c^2$$
$$-8\lambda_a^2\sum_{c=b+1}^{N}\lambda_c^2 - 12\lambda_a^2\sum_{d=1}^{a-1}\lambda_d^2 - 8\lambda_a^2\sum_{d=1}^{b-1}\lambda_d^2 + 8\lambda_b^4$$
$$-16(b-1)\lambda_b^4 - 16(N-b)\lambda_b^4 - 8\lambda_b^2\sum_{c=a+1}^{N}\lambda_c^2 - 12\lambda_b^2\sum_{c=b+1}^{N}\lambda_c^2 - 8\lambda_b^2\sum_{d=1}^{a-1}\lambda_d^2$$
$$-12\lambda_b^2\sum_{d=1}^{b-1}\lambda_d^2 + 8\sum_{c=a+1}^{N}\lambda_c^4 + 8\sum_{c=b+1}^{N}\lambda_c^4 + 8\sum_{d=1}^{a-1}\lambda_d^4 + 8\sum_{d=1}^{b-1}\lambda_d^4\Big]$$

For $a = b > 0$,

$$\sum_{N \geq c > d > 0} C_{upper} = \frac{1}{16}\Big[-160(a-1)\lambda_a^4 - 160(N-a)\lambda_a^4 - 16(a-1)\lambda_a^4 - 16(N-a)\lambda_a^4$$
$$-24\lambda_a^2 \sum_{c=a+1}^{N} \lambda_c^2 - 12\lambda_a^2 \sum_{c=a+1}^{N} \lambda_c^2 - 8\lambda_a^2 \sum_{c=a+1}^{N} \lambda_c^2 - 24\lambda_a^2 \sum_{d=1}^{a-1} \lambda_d^2 - 12\lambda_a^2 \sum_{d=1}^{a-1} \lambda_d^2$$
$$-8\lambda_a^2 \sum_{d=1}^{a-1} \lambda_d^2 - 16(a-1)\lambda_a^4 - 16(N-a)\lambda_a^4 - 8\lambda_a^2 \sum_{c=a+1}^{N} \lambda_c^2$$
$$-12\lambda_a^2 \sum_{c=a+1}^{N} \lambda_c^2 - 8\lambda_a^2 \sum_{d=1}^{a-1} \lambda_d^2 - 12\lambda_a^2 \sum_{d=1}^{a-1} \lambda_d^2 + 16 \sum_{c=a+1}^{N} \lambda_c^4 + 8 \sum_{c=a+1}^{N} \lambda_c^4$$
$$+8 \sum_{c=a+1}^{N} \lambda_c^4 + 16 \sum_{d=1}^{a-1} \lambda_d^4 + 8 \sum_{d=1}^{a-1} \lambda_d^4 + 8 \sum_{d=1}^{a-1} \lambda_d^4 \Big]$$

For $a > b > 0$,

$$\sum_{N \geq c = d > 0} C_{diagonal} = \frac{1}{16}\Big[-12\lambda_a^4 - 16\lambda_a^4 - 12\lambda_a^4 - 18\lambda_a^4 - 12\lambda_a^2\lambda_b^2 - 18\lambda_a^4 - 12\lambda_a^2\lambda_b^2 - 12\lambda_b^4$$
$$-16\lambda_b^4 - 12\lambda_b^4 - 12\lambda_a^2\lambda_b^2 - 18\lambda_b^4 - 12\lambda_a^2\lambda_b^2 - 18\lambda_b^4 + 6\lambda_a^4 + 6\lambda_b^4 + 6\lambda_a^4 + 6\lambda_b^4 \Big]$$

For $a = b > 0$,

$$\sum_{N \geq c = d > 0} C_{diagonal} = 0$$

Therefore, for $a > -b > 0$,

$$\mathrm{Ric}^N(v_{ab}^{Re}) = \sum_{N \geq c > d > 0} C_{upper} + \sum_{N \geq c = d > 0} C_{diagonal}$$
$$= \frac{1}{16}\Big[-40\lambda_a^4 - 40\lambda_b^4 - 48\lambda_a^2\lambda_b^2 - 12\lambda_a^2 \sum_{d=1}^{a-1} \lambda_d^2 - 8\lambda_a^2 \sum_{d=1}^{b-1} \lambda_d^2 - 8\lambda_b^2 \sum_{d=1}^{a-1} \lambda_d^2$$
$$-12\lambda_b^2 \sum_{d=1}^{b-1} \lambda_d^2 + 8 \sum_{d=1}^{a-1} \lambda_d^4 + 8 \sum_{d=1}^{b-1} \lambda_d^4 - 16N\lambda_a^4 - 16N\lambda_b^4 - 12\lambda_a^2 \sum_{c=a+1}^{N} \lambda_c^2$$
$$-8\lambda_a^2 \sum_{c=b+1}^{N} \lambda_c^2 - 8\lambda_b^2 \sum_{c=a+1}^{N} \lambda_c^2 - 12\lambda_b^2 \sum_{c=b+1}^{N} \lambda_c^2 + 8 \sum_{c=a+1}^{N} \lambda_c^4 + 8 \sum_{c=b+1}^{N} \lambda_c^4 \Big]$$

and, for $a = -b > 0$,

$$
\begin{aligned}
R^N(v^{Re}_{ab}) &= \sum_{N \geq c > d > 0} C_{upper} + \sum_{N \geq c = d > 0} C_{diagonal} \\
&= \frac{1}{16}\Big[-192\lambda_a^4 - 32\sum_{d=1}^{a-1}\lambda_d^4 - 192N\lambda_a^4 - 32\sum_{c=a+1}^{N}\lambda_c^4 \Big]
\end{aligned}
$$

Next, we compute $R^N(v^{Im}_{ab})$ for $a \geq -b > 0$. Replacing b with $-b$, it's equivalent to computing $R^N(v^{Im}_{a,-b})$ for $a \geq b > 0$.

$$
\begin{aligned}
&\mathrm{Ric}^N(v^{Im}_{a,-b}) \\
&= \sum_{N \geq c > d > 0} K(v^{Im}_{a,-b}, \mu^{Re}_{cd}) + \sum_{N \geq c \geq d > 0} K(v^{Im}_{a,-b}, \mu^{Im}_{cd}) \\
&\qquad + \sum_{N \geq c \geq -d > 0} K(v^{Im}_{a,-b}, v^{Re}_{cd}) + \sum_{N \geq c \geq -d > 0} K(v^{Im}_{a,-b}, v^{Im}_{cd}) \\
&= \sum_{N \geq c > d > 0} K(v^{Im}_{a,-b}, \mu^{Re}_{cd}) + \sum_{N \geq c \geq d > 0} K(v^{Im}_{a,-b}, \mu^{Im}_{cd}) \\
&\qquad + \sum_{N \geq c \geq d > 0} K(v^{Im}_{a,-b}, v^{Re}_{c,-d}) + \sum_{N \geq c \geq d > 0} K(v^{Im}_{a,-b}, v^{Im}_{c,-d}) \\
&= \sum_{N \geq c > d > 0} \Big[K(v^{Im}_{a,-b}, \mu^{Re}_{cd}) + K(v^{Im}_{a,-b}, \mu^{Im}_{cd}) + K(v^{Im}_{a,-b}, v^{Re}_{c,-d}) + K(v^{Im}_{a,-b}, v^{Im}_{c,-d}) \Big] \\
&\qquad + \sum_{N \geq c = d > 0} \Big[K(v^{Im}_{a,-b}, \mu^{Re}_{cd}) + K(v^{Im}_{a,-b}, \mu^{Im}_{cd}) + K(v^{Im}_{a,-b}, v^{Re}_{c,-d}) + K(v^{Im}_{a,-b}, v^{Im}_{c,-d}) \Big] \\
&:= \sum_{N \geq c > d > 0} D_{upper} + \sum_{N \geq c = d > 0} D_{diagonal}
\end{aligned}
$$

We have

$$
\begin{aligned}
D_{upper} = \frac{1}{16}\Big[&+32\delta_{a,b}\delta_{a,c}\lambda_a^4 + 32\delta_{a,b}\delta_{a,d}\lambda_a^4 - 16\delta_{a,c}\lambda_a^4 - 24\delta_{a,c}\delta_{a,d}\lambda_a^4 \\
&+8\delta_{a,c}\delta_{b,d}\lambda_a^4 - 16\delta_{a,d}\lambda_a^4 + 8\delta_{a,d}\delta_{b,c}\lambda_a^4 - 8\delta_{a,c}\delta_{a,d}\lambda_a^2\lambda_b^2 + 16\delta_{a,c}\delta_{b,d}\lambda_a^2\lambda_b^2 \\
&+16\delta_{a,d}\delta_{b,c}\lambda_a^2\lambda_b^2 - 8\delta_{b,c}\delta_{b,d}\lambda_a^2\lambda_b^2 + 40\delta_{a,b}\delta_{a,d}\lambda_a^2\lambda_c^2 - 12\delta_{a,d}\lambda_a^2\lambda_c^2 - 8\delta_{b,d}\lambda_a^2\lambda_c^2 \\
&+40\delta_{a,b}\delta_{a,c}\lambda_a^2\lambda_d^2 - 12\delta_{a,c}\lambda_a^2\lambda_d^2 - 8\delta_{b,c}\lambda_a^2\lambda_d^2 + 8\delta_{a,c}\delta_{b,d}\lambda_b^4 + 8\delta_{a,d}\delta_{b,c}\lambda_b^4 \\
&-16\delta_{b,c}\lambda_b^4 - 24\delta_{b,c}\delta_{b,d}\lambda_b^4 - 16\delta_{b,d}\lambda_b^4 - 8\delta_{a,d}\lambda_b^2\lambda_c^2 - 12\delta_{b,d}\lambda_b^2\lambda_c^2 - 8\delta_{a,c}\lambda_b^2\lambda_d^2 \\
&-12\delta_{b,c}\lambda_b^2\lambda_d^2 - 16\delta_{a,b}\delta_{a,d}\lambda_c^4 + 8\delta_{a,d}\lambda_c^4 + 8\delta_{b,d}\lambda_c^4 - 16\delta_{a,b}\delta_{a,c}\lambda_d^4 + 8\delta_{a,c}\lambda_d^4 + 8\delta_{b,c}\lambda_d^4 \Big]
\end{aligned}
$$

and

$$D_{diagonal} = \frac{1}{16}\Big[+24\delta_{a,b}\delta_{a,c}\lambda_a^4 - 32\delta_{a,b}\delta_{a,c}\delta_{a,d}\lambda_a^4 + 24\delta_{a,b}\delta_{a,d}\lambda_a^4 - 12\delta_{a,c}\lambda_a^4$$
$$-16\delta_{a,c}\delta_{a,d}\lambda_a^4 + 6\delta_{a,c}\delta_{b,d}\lambda_a^4 - 12\delta_{a,d}\lambda_a^4 + 6\delta_{a,d}\delta_{b,c}\lambda_a^4 + 20\delta_{a,c}\delta_{b,d}\lambda_a^2\lambda_b^2$$
$$+20\delta_{a,d}\delta_{b,c}\lambda_a^2\lambda_b^2 + 60\delta_{a,b}\delta_{a,d}\lambda_a^2\lambda_c^2 - 18\delta_{a,d}\lambda_a^2\lambda_c^2 - 12\delta_{b,d}\lambda_a^2\lambda_c^2 + 60\delta_{a,b}\delta_{a,c}\lambda_a^2\lambda_d^2$$
$$-18\delta_{a,c}\lambda_a^2\lambda_d^2 - 12\delta_{b,c}\lambda_a^2\lambda_d^2 + 6\delta_{a,c}\delta_{b,d}\lambda_b^4 + 6\delta_{a,d}\delta_{b,c}\lambda_b^4 - 12\delta_{b,c}\lambda_b^4$$
$$-16\delta_{b,c}\delta_{b,d}\lambda_b^4 - 12\delta_{b,d}\lambda_b^4 - 12\delta_{a,d}\lambda_b^2\lambda_c^2 - 18\delta_{b,d}\lambda_b^2\lambda_c^2 - 12\delta_{a,c}\lambda_b^2\lambda_d^2$$
$$-18\delta_{b,c}\lambda_b^2\lambda_d^2 - 12\delta_{a,b}\delta_{a,d}\lambda_c^4 + 6\delta_{a,d}\lambda_c^4 + 6\delta_{b,d}\lambda_c^4 - 12\delta_{a,b}\delta_{a,c}\lambda_d^4 + 6\delta_{a,c}\lambda_d^4 + 6\delta_{b,c}\lambda_d^4\Big]$$

For $a > b > 0$,

$$\sum_{N\geq c>d>0} D_{upper} = \frac{1}{16}\Big[-16(a-1)\lambda_a^4 + 8\lambda_a^4 - 16(N-a)\lambda_a^4 + 16\lambda_a^2\lambda_b^2$$
$$-12\lambda_a^2\sum_{c=a+1}^{N}\lambda_c^2 - 8\lambda_a^2\sum_{c=b+1}^{N}\lambda_c^2 - 12\lambda_a^2\sum_{d=1}^{a-1}\lambda_d^2 - 8\lambda_a^2\sum_{d=1}^{b-1}\lambda_d^2 + 8\lambda_b^4$$
$$-16(b-1)\lambda_b^4 - 16(N-b)\lambda_b^4 - 8\lambda_b^2\sum_{c=a+1}^{N}\lambda_c^2 - 12\lambda_b^2\sum_{c=b+1}^{N}\lambda_c^2$$
$$-8\lambda_b^2\sum_{d=1}^{a-1}\lambda_d^2 - 12\lambda_b^2\sum_{d=1}^{b-1}\lambda_d^2 + 8\sum_{c=a+1}^{N}\lambda_c^4 + 8\sum_{c=b+1}^{N}\lambda_c^4 + 8\sum_{d=1}^{a-1}\lambda_d^4 + 8\sum_{d=1}^{b-1}\lambda_d^4\Big]$$

For $a = b > 0$,

$$\sum_{N\geq c>d>0} D_{upper} = \frac{1}{16}\Big[+32(a-1)\lambda_a^4 + 32(N-a)\lambda_a^4 - 16(a-1)\lambda_a^4 - 16(N-a)\lambda_a^4$$
$$+40\lambda_a^2\sum_{c=a+1}^{N}\lambda_c^2 - 12\lambda_a^2\sum_{c=a+1}^{N}\lambda_c^2 - 8\lambda_a^2\sum_{c=a+1}^{N}\lambda_c^2 + 40\lambda_a^2\sum_{d=1}^{a-1}\lambda_d^2 - 12\lambda_a^2\sum_{d=1}^{a-1}\lambda_d^2$$
$$-8\lambda_a^2\sum_{d=1}^{a-1}\lambda_d^2 - 16(a-1)\lambda_a^4 - 16(N-a)\lambda_a^4 - 8\lambda_a^2\sum_{c=a+1}^{N}\lambda_c^2$$
$$-12\lambda_a^2\sum_{c=a+1}^{N}\lambda_c^2 - 8\lambda_a^2\sum_{d=1}^{a-1}\lambda_d^2 - 12\lambda_a^2\sum_{d=1}^{a-1}\lambda_d^2 - 16\sum_{c=a+1}^{N}\lambda_c^4$$
$$+8\sum_{c=a+1}^{N}\lambda_c^4 + 8\sum_{c=a+1}^{N}\lambda_c^4 - 16\sum_{d=1}^{a-1}\lambda_d^4 + 8\sum_{d=1}^{a-1}\lambda_d^4 + 8\sum_{d=1}^{a-1}\lambda_d^4\Big]$$

For $a > b > 0$,

$$\sum_{N \geq c=d>0} D_{diagonal} = \frac{1}{16}\Big[-12\lambda_a^4 - 16\lambda_a^4 - 12\lambda_a^4 - 18\lambda_a^4 - 12\lambda_a^2\lambda_b^2 - 18\lambda_a^4 - 12\lambda_a^2\lambda_b^2$$
$$-12\lambda_b^4 - 16\lambda_b^4 - 12\lambda_b^4 - 12\lambda_a^2\lambda_b^2 - 18\lambda_b^4 - 12\lambda_a^2\lambda_b^2 - 18\lambda_b^4 + 6\lambda_a^4 + 6\lambda_b^4 + 6\lambda_a^4 + 6\lambda_b^4\Big]$$

For $a = b > 0$,

$$\sum_{N \geq c=d>0} D_{diagonal} = 0$$

Therefore, for $a > -b > 0$,

$$\begin{aligned}\mathrm{Ric}^N(v_{ab}^{Im}) &= \sum_{N \geq c>d>0} D_{upper} + \sum_{N \geq c=d>0} D_{diagonal} \\ &= \frac{1}{16}\Big[-40\lambda_a^4 - 40\lambda_b^4 - 32\lambda_a^2\lambda_b^2 - 12\lambda_a^2\sum_{d=1}^{a-1}\lambda_d^2 - 8\lambda_a^2\sum_{d=1}^{b-1}\lambda_d^2 - 8\lambda_b^2\sum_{d=1}^{a-1}\lambda_d^2 \\ &\quad -12\lambda_b^2\sum_{d=1}^{b-1}\lambda_d^2 + 8\sum_{d=1}^{a-1}\lambda_d^4 + 8\sum_{d=1}^{b-1}\lambda_d^4 - 16N\lambda_a^4 - 16N\lambda_b^4 - 12\lambda_a^2\sum_{c=a+1}^{N}\lambda_c^2 \\ &\quad -8\lambda_a^2\sum_{c=b+1}^{N}\lambda_c^2 - 8\lambda_b^2\sum_{c=a+1}^{N}\lambda_c^2 - 12\lambda_b^2\sum_{c=b+1}^{N}\lambda_c^2 + 8\sum_{c=a+1}^{N}\lambda_c^4 + 8\sum_{c=b+1}^{N}\lambda_c^4\Big]\end{aligned}$$

and, for $a = -b > 0$,

$$\mathrm{Ric}^N(v_{ab}^{Im}) = \sum_{N \geq c>d>0} D_{upper} + \sum_{N \geq c=d>0} D_{diagonal} = 0$$

□

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