LEFT-INVARIANT GEOMETRIES ON SU(2) ARE UNIFORMLY DOUBLING

NATHANIEL ELDREDGE [†], MARIA GORDINA[††], AND LAURENT SALOFF-COSTE[‡]

Abstract. A classical aspect of Riemannian geometry is the study of estimates that hold uniformly over some class of metrics. The best known examples are eigenvalue bounds under curvature assumptions. In this paper, we study the family of all left-invariant geometries on SU(2). We show that left-invariant geometries on SU(2) are uniformly doubling and give a detailed estimate of the volume of balls that is valid for any of these geometries and any radius. We discuss a number of consequences concerning the spectrum of the associated Laplacians and the corresponding heat kernels.

Contents

1991 *Mathematics Subject Classification.* Primary 53C21; Secondary 35K08, 53C17, 58J35, 58J60, 22C05, 22E30.

Key words and phrases. volume doubling, compact Lie group, special unitary group, heat kernel, Poincaré inequality.

[†] Research was supported by a grant from the Simons Foundation (#355659, Nathaniel Eldredge).

†† Research was supported in part by NSF grant DMS-1007496 and the Simons Fellowship.

‡ Research was supported in part by NSF grant DMS-1404435 and NSF grant DMS-1707589.

1. Introduction

1.1. **A conjecture and the main result.**

This work is devoted to the uniform analysis of the family of all left-invariant Riemannian metrics on the Lie group SU(2). This is the simplest case of a natural problem we now describe.

Let K be a connected real compact Lie group, and let $\mathfrak{L}(K)$ denote the family of all left-invariant Riemannian metrics g on K. We conjecture that for each group K, many aspects of spectral analysis of the corresponding Laplace-Beltrami operator Δ_g with $g \in \mathfrak{L}(K)$, as well as the analysis of the associated heat equation $\partial_t - \Delta_g = 0$, can be controlled *uniformly* over $\mathfrak{L}(K)$. Recall that the operator $-\Delta_g$ has non-negative discrete spectrum with finite multiplicity, and so we can consider the lowest non-zero eigenvalue denoted by λ_g.

It was shown in [34] (see also [28]) that on any compact homogeneous manifold, one has the lower bound

$$\lambda_g \geqslant \frac{\pi^2}{4\,\mathrm{diam}_g^2}. \tag{1.1}$$

We conjecture that a matching upper bound holds uniformly over $g \in \mathfrak{L}(K)$, so that

$$\lambda_g \leqslant \frac{C_K}{\mathrm{diam}_g^2} \tag{1.2}$$

where the constant C_K may depend on K but not on g.

In terms of the heat equation, we conjecture that there are constants $c_i = c_i(K) \in (0, \infty)$, $i = 1, \ldots, 4$ such that the fundamental solution (heat kernel) $(t, x, y) \mapsto p_t^g(x, y)$ of the heat equation on (K, g) satisfies

$$\frac{c_1}{V_g(\sqrt{t})} \exp(-c_2 d_g(x,y)^2/t) \leqslant p_t^g(x,y) \leqslant \frac{c_3}{V_g(\sqrt{t})} \exp(-c_4 d_g(x,y)^2/t). \tag{1.3}$$

Here $V_g(r)$ denotes the volume of the ball of radius r with respect to the Riemannian volume measure μ_g; $d_g(x, y)$ denotes the Riemannian distance between x and y; and diam_g denotes the diameter of K with respect to d_g.

One reason to believe that this conjecture might be true is that it can be reduced to a simpler question. Let (X, d, μ) be a metric measure space, that is, (X, d) is a metric space and μ is a Borel measure on X. By $B_r(x)$ we denote the ball centered at $x \in X$ of radius $r > 0$ with respect to the distance d. The metric measure space (X, d, μ) is **volume doubling** if

$$D\,(M, d, \mu) := \sup_{x \in X, r > 0} \frac{\mu(B\,(x, 2r))}{\mu(B\,(x, r))} < \infty. \tag{1.4}$$

The focus of this paper is the particular case where $(X, d, \mu) = (K, d_g, \mu_g)$ with the **volume doubling constant** denoted by $D_g := D(K, d_g, \mu_g)$.

Then, in the context of compact connected Lie groups, the two-sided spectral and heat kernel bounds in (1.2) and (1.3) would follow from the following conjecture.

Conjecture 1.1. *Let K be a connected real compact Lie group. There is a constant $D(K)$ such that*

$$D_g \leqslant D(K) \text{ for all } g \in \mathfrak{L}(K), \tag{1.5}$$

that is, K is uniformly doubling with constant $D(K)$.

As an illustration of the significance of this conjecture, the volume doubling constant also appears as the constant in the Poincaré inequality (see Section 8.1):

$$\int_{B_g(x,r)} |f - f_{x,r}|^2 d\mu_g \leqslant 2r^2 D_g \int_{B_g(x,2r)} |\nabla_g f|_g^2 d\mu_g \text{ for all } f \in \mathcal{C}^\infty(B_g(x, 2r)), \tag{1.6}$$

where $f_{x,r} := \fint_{B_g(x,r)} f d\mu_g$ denotes the mean of f over $B_g(x, r)$. Hence, the validity of (1.5) implies that the constant in the Poincaré inequality (1.6) is uniform over all metrics in $\mathfrak{L}(K)$. Together with known heat kernel estimates due to [21, 46, 51] this shows that the validity of Conjecture 1.1 implies that of the two-sided heat kernel bound (1.3). A simple test function argument shows that (1.5) also implies the spectral gap estimate in terms of the diameter as given in (1.2).

In this article, we prove that Conjecture 1.1 is valid for $K = \mathrm{SU}(2)$. Our main result is as follows.

Theorem 1.2. *There exists a constant D such that, for any left-invariant Riemannian metric g on $\mathrm{SU}(2)$, we have $D_g \leqslant D$.*

Since the underlying manifold of SU(2) is the 3-sphere S^3, this theorem provides uniform volume doubling for a large family of Riemannian metrics on S^3. This holds despite the fact that the geometries $g \in \mathfrak{L}(\mathrm{SU}(2))$ are not uniformly bounded in other senses; for instance, even after rescaling to constant diameter, there is no universal lower bound for the Ricci curvatures of metrics $g \in \mathfrak{L}(\mathrm{SU}(2))$ as we discuss in Section 7.

The proof of Theorem 1.2 is based on the following explicit description of the behavior of the volume growth function V_g. Each $g \in \mathfrak{L}(\mathrm{SU}(2))$ can be identified with an inner product on $\mathfrak{su}(2)$; let $0 < a_1 \leqslant a_2 \leqslant a_3 < \infty$ denote the square roots of its three eigenvalues, with respect to the standard Euclidean structure on $\mathfrak{su}(2)$ induced by the negative of the Killing form. We stress that these parameters depend on the metric g.

Theorem 1.3. *There are constants $b_1, b_2 \in (0, \infty)$ such that for all $g \in \mathfrak{L}(\mathrm{SU}(2))$, the function V_g satisfies*

$$b_1 \leqslant \frac{V_g(r)}{\overline{V_g}(r)} \leqslant b_2, \text{ where}$$

$$\overline{V_g}(r) = \begin{cases} r^3 & \text{if } 0 < r \leqslant a_1 a_2 / a_3, \\ (a_3/a_1 a_2)\, r^4 & \text{if } a_1 a_2/a_3 < r \leqslant a_1, \\ (a_1 a_3/a_2)\, r^2 & \text{if } a_1 < r \leqslant a_2, \\ a_1 a_2 a_3 & \text{if } a_2 < r < \infty. \end{cases}$$

We note that a_1 can be characterized as the length of the shortest closed geodesic for g, while a_2 can be replaced in the theorem above by the diameter $\mathrm{diam}_g(\mathrm{SU}(2))$ because the two are uniformly comparable (this is not entirely obvious, but will be proved in Section 7), and that a_3 is then uniformly comparable to the quantity $\mu_g(\mathrm{SU}(2))/a_1 \,\mathrm{diam}_g$.

As far as we know, the only other case when Conjecture 1.1 is known to hold is for $K = \mathbb{T}^n$, the n-dimensional torus, for any fixed n. This can be seen via lifting to the covering group, $\mathbb{R}^n$, on which all Euclidean metrics are isomorphic with the same doubling constant 2^n. But doubling passes to quotients. The key argument is given in [23, Lemma 1.1]; see also [18, (5.5), p.20]. Alternatively, this can be seen using curvature as explained in Section 1.2, since every left-invariant metric on a torus is flat and has zero Ricci curvature.

It is important to note that Theorem 1.2 implicitly includes two limit cases. In one case, the metric tends to infinity in one direction, and the manifold approaches a sub-Riemannian manifold, which itself is doubling. If the metric tends to zero in one direction, the 3-dimensional manifold SU(2) collapses to a 2-dimensional quotient, which is also doubling. Then in some sense, the question becomes whether the doubling constant varies continuously with respect to these limits. One of the difficulties is that both cases must be considered simultaneously.

Our approach for SU(2) is rather explicit and makes use of its specific structure, with the important benefit of providing a detailed estimate of the volume function as stated in Theorem 1.3. We show that the volume function exhibits different behavior at different scales: Euclidean behavior at very small scales, sub-Riemannian behavior at intermediate scales and "quotient geometry" behavior at relatively large scales, and this is done uniformly over all metrics in $\mathfrak{L}(K)$. This allows us to approximate the volume growth function of the metric g by the simple explicit function $\overline{V_g}$ which essentially "pieces together" the growth functions of those three spaces. We hope that the study of this special case will open the door to similar results for other compact groups.

1.2. **Curvature, or not.** In geometric analysis, ever since the pioneering work of S.-T. Yau in the 1970s, Ricci curvature has been the tool of choice to prove spectral bounds and other analytic estimates such as various forms of Harnack inequalities and heat kernel estimates, especially if one is interested in statements that are uniform over large families of Riemannian manifolds. In particular, the celebrated Bishop-Gromov volume comparison theorem implies that for any complete Riemannian manifold (M, g) of dimension at most n with a non-negative Ricci curvature, the doubling constant $D(M, d_g, \mu_g)$ is bounded by 2^n, the doubling constant of Euclidean space $\mathbb{R}^n$. If the curvature condition is relaxed to a Ricci curvature lower bound, say, $\mathrm{Ric}_g \geqslant -\kappa g$, while keeping the restriction that the dimension is at most n, one still has a uniform bound on the doubling constant $D(M, d_g, \mu_g)$ as long as one imposes a fixed upper bound on the diameter $\mathrm{diam}_g(M)$. In these contexts, the Poincaré inequality (1.6) is not a direct consequence of the doubling property, but it follows from the dimension and curvature assumptions (and an upper bound on the diameter in the case of $\mathrm{Ric}_g \geqslant -\kappa g$). In fact, fix an $\epsilon > 0$ and the dimension n. For Riemannian manifolds of that fixed dimension, the curvature-diameter assumption

$$\mathrm{Ric}_g \geqslant -\epsilon \,\mathrm{diam}_g^{-2}\, g$$

implies that (M, g) is doubling and satisfies the Poincaré inequality (1.6) with constant depending only on n and ϵ. Note, however, that this curvature-diameter assumption is not invariant under multiplication of the metric by a positive scalar. See the Bishop-Gromov comparison theorem and the result of P. Buser in [12] and also [51, Section 5.6.3].

In this spirit, Conjecture 1.1 is very much modeled on the non-negative Ricci curvature result described above. Even so, except in the commutative case of the flat tori, it is well known that no uniform Ricci lower bound can hold over the entire family $\mathfrak{L}(K)$ of left-invariant metric on a group K. In fact, the very nature of Conjecture 1.1 implies that it not only covers left-invariant Riemannian geometries but also left-invariant sub-Riemannian geometries which can be described, in some rather obvious ways, as limits of left-invariant Riemannian geometries. This is made explicit for SU(2) in Section 9.

Recently there have been interesting attempts to extend curvature techniques in the context of sub-Riemannian geometries e.g. [1, 4, 7, 13, 25]. However, even in the case of left-invariant geometries on SU(2), it seems that these curvature techniques (old and new) do not yield a proof of Theorem 1.2.

Other works have obtained geometric inequalities, including volume doubling and the stronger measure contraction property $MCP(k, n)$ introduced by [41], that hold uniformly over a one-parameter family of Riemannian geometries approximating a sub-Riemannian geometry [2, 5, 29, 32, 33, 45]. However, these works use very different techniques, and all known results appear to rely on assumptions of horizontal curvature bounds or additional symmetry, such as Sasakian structure. To the best of our knowledge, these assumptions are not satisfied uniformly over all left-invariant sub-Riemannian geometries on SU(2), and thus those results likewise do not imply Theorem 1.2.

2. Preliminaries

2.1. The group $G = \mathrm{SU}(2)$ and left-invariant metrics on G.

The compact Lie group SU(2) is the group of 2×2 complex matrices which are unitary and have determinant 1. The group identity of SU(2) is the identity matrix I, which we shall also denote by e when emphasizing the group structure. The corresponding Lie algebra $\mathfrak{su}(2)$, identified with the tangent space $T_e \mathrm{SU}(2)$, is the space of 2×2 complex matrices which are skew-Hermitian and have trace 0. We note that a left-invariant metric g on SU(2) is uniquely defined by its action on $\mathfrak{su}(2)$, the tangent space at the identity.

Since SU(2) is compact, the Killing form $B(v, w) = \frac{1}{2} \operatorname{tr}(\operatorname{ad}_v \operatorname{ad}_w)$ is negative definite, and so $-B$ is an inner product on $\mathfrak{su}(2)$ which is invariant. It induces a bi-invariant Riemannian metric on SU(2), which we will call the **canonical bi-invariant metric**; it is unique up to scaling because SU(2) is simple [36, Lemma 7.6]. In this canonical metric, SU(2) is isometric to a round sphere.

As SU(2) is compact, by [36, Lemma 7.2] a left-invariant metric g on SU(2) is bi-invariant if and only if ad_x is skew-adjoint with respect to g for every $x \in \mathfrak{su}(2)$. More detail (based mostly on [36]) can be found in [16, Chapter 1.4].

2.2. Standard Milnor bases.

A key property of SU(2) is that any left-invariant metric g can be diagonalized by a basis for $\mathfrak{su}(2)$ for which the structure constants have a very simple form. Such bases were studied by Milnor in [36].

Throughout this section, $\{i, j, k\}$ will be taken to range over all cyclic permutations of the indices $\{1, 2, 3\}$.

Definition 2.1. *We shall say that an ordered basis $\{e_1, e_2, e_3\}$ for $\mathfrak{su}(2)$ is a* ***standard Milnor basis*** *if it satisfies the relations*

$$[e_1, e_2] = e_3, \qquad [e_2, e_3] = e_1, \qquad [e_3, e_1] = e_2,$$

or for short

$$[e_i, e_j] = e_k.$$

Example 2.2. The Pauli matrices

$$\widehat{e}_1 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \widehat{e}_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \widehat{e}_3 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \tag{2.1}$$

are a standard Milnor basis.

Example 2.3. If $\{e_1, e_2, e_3\}$ is a standard Milnor basis, then so are

(1) the cyclic permutations $\{e_2, e_3, e_1\}$ and $\{e_3, e_1, e_2\}$;
(2) the ordered basis $\{-e_1, e_3, e_2\}$. As such, any permutation of a standard Milnor basis may itself be made into a standard Milnor basis by possibly negating one element;
(3) the basis

$$\{\cos(\theta)e_1 + \sin(\theta)e_2, -\sin(\theta)e_1 + \cos(\theta)e_2, e_3\}, \qquad \theta \in \mathbb{R}. \tag{2.2}$$

Remark 2.4. Definition 2.1 is slightly different from a more common notion of Milnor frames, in which one begins with a metric g, and in addition to the commutation relations one assumes that $\{e_1, e_2, e_3\}$ are orthogonal with respect to g.

The next lemma is a consequence of the fact that all Lie algebra automorphisms of $\mathfrak{su}(2)$ are inner, and therefore the set of all standard Milnor bases for $\mathfrak{su}(2)$ coincides with the orbit of Ad starting at any standard Milnor basis. Note that this is not so for $\mathrm{SU}(n), n \geqslant 3$. As always for a matrix Lie group G we use the fact that $\mathrm{Ad}_g X = gXg^{-1}$ for $g \in G$ and $X \in \mathfrak{g}$, the Lie algebra of G, where on the right we have the products of matrices.

Lemma 2.5. *Suppose $\{e_1, e_2, e_3\}$ is a standard Milnor basis. Then $\{e_1', e_2', e_3'\} \subseteq \mathfrak{su}(2)$ is a standard Milnor basis if and only if there exists $y \in \mathrm{SU}(2)$ such that $\mathrm{Ad}_y e_i = ye_iy^{-1} = e_i'$ for $i = 1, 2, 3$.*

Proof. For any $y \in \mathrm{SU}(2)$, the map $v \mapsto yvy^{-1}$ is a Lie algebra automorphism of $\mathfrak{su}(2)$, so it is clear that $e_i' = ye_iy^{-1}$ produces a standard Milnor basis. Conversely, suppose $\{e_1', e_2', e_3'\}$ is a standard Milnor basis. Since $\{e_1, e_2, e_3\}$ and $\{e_1', e_2', e_3'\}$ are both bases, there is a unique linear automorphism T of the vector space $\mathfrak{su}(2)$ satisfying $Te_i = e_i'$, $i = 1, 2, 3$. Then if (i, j, k) is any cyclic permutation of the indices $(1, 2, 3)$, we have

$$[Te_i, Te_j] = [e_i', e_j'] = e_k' = Te_k = T[e_i, e_j].$$

It follows that $[Tu, Tv] = T[u, v]$ for any $u, v \in \{e_1, e_2, e_3\}$, and by linearity the same holds for any $u, v \in \mathfrak{su}(2)$. So T is a Lie algebra automorphism of $\mathfrak{su}(2)$. It is well-known that every Lie algebra automorphism of $\mathfrak{su}(2)$ is inner (i.e. the outer automorphism group is trivial) as pointed out in [59, Proposition 5.1]. Thus $T = \mathrm{Ad}_y$ for some $y \in \mathrm{SU}(2)$. □

Lemma 2.6. *Suppose $\{e_1, e_2, e_3\}$ is a basis for $\mathfrak{su}(2)$ satisfying $[e_i, e_j] = \lambda_k e_k$ where $\lambda_i, \lambda_j, \lambda_k \in \{\pm 1\}$. Then $\lambda_1 = \lambda_2 = \lambda_3$. In particular, either $\{e_1, e_2, e_3\}$ or $\{-e_1, e_2, e_3\}$ is a standard Milnor basis.*

Proof. Let $B(v, w) = \frac{1}{2}\operatorname{tr}(\mathrm{ad}_v \mathrm{ad}_w)$ be the Killing form of $\mathfrak{su}(2)$, which is negative definite since $\mathrm{SU}(2)$ is compact. Then a simple computation shows $B(e_i, e_i) = -\lambda_j\lambda_k$. Since this must be negative for each i, it follows that $\lambda_1, \lambda_2, \lambda_3$ are all $+1$

or all -1. In the former case, $\{e_1, e_2, e_3\}$ is already a standard Milnor basis, and in the latter case, it is easy to check that $\{-e_1, e_2, e_3\}$ is. $\square$

Lemma 2.7. *For any standard Milnor basis* $\{e_1, e_2, e_3\}$*, we have the following identities in the matrix algebra* $M^{2\times 2}(\mathbb{C})$

$$e_i^2 = -\frac{1}{4}I, \qquad e_ie_j = \frac{1}{2}e_k, \qquad e_ie_j + e_je_i = 0, \tag{2.3}$$

where (i, j, k) *is, as before, any cyclic permutation of the indices* $(1, 2, 3)$ *and* $i \neq j$.

Proof. Note that by Lemma 2.5, it is enough to verify identities (2.3) for one standard Milnor basis since $\operatorname{Ad}_h I = I$ and $\operatorname{Ad}_h 0 = 0$ for all $h \in \mathrm{SU}(2)$. A simple calculation proves the first two identities for Pauli matrices, while the last identity can be shown by appealing to Definition 2.1 and the second identity as follows

$$e_ie_j + e_je_i = 2e_ie_j - e_k = e_k - e_k = 0.$$

$\square$

2.3. Left-invariant Riemannian metrics on SU(2).

Lemma 2.8. *Let g be any left-invariant metric on* SU(2)*. There exists a standard Milnor basis* $\{e_1, e_2, e_3\}$ *which is orthogonal in the metric g and satisfies* $g(e_1, e_1) \leqslant g(e_2, e_2) \leqslant g(e_3, e_3)$.

Proof. Following [36], we define a cross product $\times$ on the 3-dimensional inner product space $(\mathfrak{su}(2), g)$, unique up to a choice of orientation. To see it another way, one can identify $(\mathfrak{su}(2), g)$ with $(\mathbb{R}^3, \cdot)$, uniquely up to a choice of orientation, and pull back the cross product from $\mathbb{R}^3$. As shown in [36, Lemma 4.1], there is a unique linear map L on $\mathfrak{su}(2)$ satisfying $L(u \times v) = [u, v]$, and it is self-adjoint with respect to g. Let $\{w_1, w_2, w_3\}$ be a g-orthonormal basis of eigenvectors for L, with eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Reordering this basis if necessary, we can assume it is positively oriented, so that $w_i \times w_j = w_k$. Then

$$[w_i, w_j] = L(w_i \times w_j) = L(w_k) = \lambda_k w_k.$$

Setting $e_i = |\lambda_j\lambda_k|^{-1/2} w_i$, we can verify that $[e_i, e_j] = \pm e_k$ for some choice of signs, and that $\{e_1, e_2, e_3\}$ is still g-orthogonal. Finally we can re-index this basis as needed so that $g(e_1, e_1) \leqslant g(e_2, e_2) \leqslant g(e_3, e_3)$. By Lemma 2.6, either $\{e_1, e_2, e_3\}$ or $\{-e_1, e_2, e_3\}$ is the desired standard Milnor basis. $\square$

Notation 2.9. For any left-invariant Riemannian metric g on SU(2) let $a_1 \leqslant a_2 \leqslant a_3$ be the (ordered) square roots of the eigenvalues of the metric g with respect to the canonical Euclidean form defined by the negative of the Killing form $B(v, w) = \frac{1}{2}\operatorname{tr}(\operatorname{ad}_v \operatorname{ad}_w)$. We call a_1, a_2, a_3 the **parameters associated to the metric** g. For any $0 < a_1 \leqslant a_2 \leqslant a_3 < \infty$, let $g_{(a_1,a_2,a_3)}$ denote the unique left-invariant Riemannian metric on SU(2) for which

$$g_{(a_1,a_2,a_3)}(\widehat{e}_i, \widehat{e}_j) = a_i^2\delta_{ij}, \qquad i = 1, 2, 3,$$

where $\widehat{e}_i$ are the Pauli matrices defined in Example 2.2. Since $B(e_i, e_j) = -\delta_{ij}$ for any standard Milnor basis, the parameters associated to $g_{(a_1,a_2,a_3)}$ are indeed a_1, a_2, a_3. Note that $g_{(1,1,1)}$ is the canonical bi-invariant metric.

Corollary 2.10. *Let g be any left-invariant metric on* SU(2)*, and let a_1, a_2, a_3 be its parameters. Then* $(\mathrm{SU}(2), g)$ *is isometrically isomorphic to* $(\mathrm{SU}(2), g_{(a_1,a_2,a_3)})$*, where $g_{(a_1,a_2,a_3)}$ is as defined in Notation 2.9.*

Proof. Choose a standard Milnor basis $\{e_1, e_2, e_3\}$ which diagonalizes g as in Lemma 2.8. Since $\{e_1, e_2, e_3\}$ is orthonormal with respect to $-B$, we have $g(e_i, e_i) = a_i^2$. The linear map $\varphi : \mathfrak{su}(2) \to \mathfrak{su}(2)$ defined by $\varphi(e_i) = \widehat{e_i}$ is a Lie algebra automorphism, since both bases have the same structure constants. Since SU(2) is simply connected, φ induces a Lie group automorphism of SU(2) whose differential at the identity is φ, which by construction is an isometry of the left-invariant metrics g and $g_{(a_1,a_2,a_3)}$. □

Remark 2.11. By Corollary 2.10, for each left-invariant Riemannian metric with parameters (a_1, a_2, a_3), there is a group isomorphism providing an isometry between $g_{(a_1,a_2,a_3)}$ and that metric. Hence it suffices to consider $g_{(a_1,a_2,a_3)}$. In what follows, we abuse notation and use $\{e_1, e_2, e_3\}$ to denote both a general Milnor basis or the particular Milnor basis formed by the Pauli matrices.

2.4. **Exponential identities.** Recall that we use I for the identity matrix when we treat it as an element of the matrix space $M^{2\times 2}(\mathbb{C})$. Whenever we want to emphasize the role of I as the identity in the group SU(2) we use e.

Lemma 2.12. *For any $A \in \mathfrak{su}(2)$, we have*

$$A^2 = -\det(A)I.$$

Proof. One can verify this by observing that a general matrix $A \in \mathfrak{su}(2)$ is of the form $A = \begin{pmatrix} ai & b+ci \\ -b+ci & -ai \end{pmatrix}$, $a, b, c \in \mathbb{R}$ and computing directly. □

Lemma 2.13. *For $A \in \mathfrak{su}(2)$, we have*

$$\exp(A) = (\cos\rho)I + \frac{\sin\rho}{\rho}A, \tag{2.4}$$

where $\rho^2 = \det A$.

Remark 2.14. First observe that this identity can be used also for $\rho = 0$, since then $A = 0$ and $\exp(A) = I$. This can be seen by using any standard Milnor basis and writing $A = ae_1 + be_2 + ce_3, a, b, c \in \mathbb{R}$. Then $\rho^2 = \frac{1}{4}\left(a^2 + b^2 + c^2\right) = \det A$. In particular, if $\rho = 0$, then $a = b = c = 0$.

Proof. Consider the expansion $\exp A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$. Grouping even and odd terms we can write $\exp A = \sum_{k=0}^{\infty} \frac{A^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{A^{2k+1}}{(2k+1)!}$. By Lemma 2.12

$$A^{2k} = (-\rho^2 I)^k = (-1)^k \rho^{2k} I,$$
$$A^{2k+1} = (-1)^k \rho^{2k} A = \frac{(-1)^k \rho^{2k+1}}{\rho} A,$$

so the first sum equals $(\cos\rho)I$ and the second equals $\frac{\sin\rho}{\rho}A$. □

Lemma 2.15. *For any $x \in \mathrm{SU}(2) \setminus \{-I\}$, we have $x = \exp(A)$, where*

$$A = \frac{\rho}{\sin\rho}(x - (\cos\rho)I)$$

and

$$\rho = \arccos\left(\frac{\operatorname{tr} x}{2}\right).$$

Remark 2.16. Similarly to Remark 2.14, if $\rho = 0$, so that $x = I$, we take $A = 0$ which is consistent with this identity. For $\rho = \pi$ we have $x = -I$ and can take $A = 2\pi e_1$, for instance.

Proof. Let A, ρ be as given. Since $\cos\rho = \frac{\operatorname{tr} x}{2}$, it is apparent that $\operatorname{tr} A = 0$. To see that A is skew-Hermitian, note that since x is unitary with $\det x = 1$, Cayley-Hamilton gives

$$x^* = x^{-1} = -x + (\operatorname{tr} x)I = -x + (2\cos\rho)I.$$

As such,

$$A + A^* = \frac{\rho}{\sin\rho}(x + x^* - (2\cos\rho)I) = 0.$$

Hence $A \in \mathfrak{su}(2)$.

We now verify that $\det A = \rho^2$; then the result follows immediately from Lemma 2.13. Using Lemma 2.12 and the fact that $A^* = -A$ we have

$$\det(A)I = A^*A = \frac{\rho^2}{\sin^2\rho}((1 + \cos^2\rho)I - \cos\rho(x + x^*))$$

since $xx^* = I$. Taking traces and noting that $\operatorname{tr} x = \operatorname{tr} x^* = 2\cos\rho$, we have

$$2\det(A) = \frac{\rho^2}{\sin^2\rho}(2 + 2\cos^2\rho - 4\cos^2\rho) = 2\rho^2$$

as desired. □

Lemma 2.17. *Suppose* $\{e_1, e_2, e_3\}$ *is a standard Milnor basis for* $\mathfrak{su}(2)$. *Then*

$$e^{se_1}e^{te_2}e^{-se_1} = \exp\left(t\left(\cos s\right)e_2 + t\left(\sin s\right)e_3\right), s, t \in \mathbb{R}.$$

Remark 2.18. The proof given below does not use $\mathfrak{su}(2)$ specifically, only the commutation relations for the Milnor basis. For $\mathfrak{su}(2)$ this result can also be shown directly by using Lemma 2.13 on both sides.

Proof. Let

$$\begin{aligned} f(s) &:= e^{se_1}e_2e^{-se_1}, \\ g(s) &:= (\cos s)\, e_2 + (\sin s)\, e_3. \end{aligned}$$

Then we see that

$$\begin{aligned} &f'(s) = e^{se_1}[e_1, e_2]e^{-se_1}, \\ &f''(s) = e^{se_1}[e_1, [e_1, e_2]]e^{-se_1} = -e^{se_1}e_2e^{-se_1} = -f(s); \\ &f(0) = e_2, f'(0) = [e_1, e_2] = e_3; \end{aligned}$$

$$\begin{aligned} &g'(s) = -(\sin s)\, e_2 + (\cos s)\, e_3, \\ &g''(s) = -(\cos s)\, e_2 - (\sin s)\, e_3 = -g(s); \\ &g(0) = e_2, g'(0) = e_3, \end{aligned}$$

so by uniqueness of the initial value problem for ODEs these two functions coincide, that is,

$$e^{se_1}e_2e^{-se_1} = (\cos s)\, e_2 + (\sin s)\, e_3. \tag{2.5}$$

Finally,

$$
\begin{aligned}
e^{se_1}e^{te_2}e^{-se_1} &= \sum_{n=0}^{\infty}\frac{t^n}{n!}e^{se_1}e_2^n e^{-se_1} \\
&= \sum_{n=0}^{\infty}\frac{t^n}{n!}\left(e^{se_1}e_2e^{-se_1}\right)^n = \sum_{n=0}^{\infty}\frac{t^n}{n!}\left((\cos s)\,e_2 + (\sin s)\,e_3\right)^n \\
&= \exp\left(t\left((\cos s)\,e_2 + (\sin s)\,e_3\right)\right).
\end{aligned}
$$

□

Remark 2.19. By applying Lemma 2.17 to the standard Milnor basis

$$\{e_1, \cos(\theta)e_2 + \sin(\theta)e_3, -\sin(\theta)e_2 + \cos(\theta)e_3\}$$

for $\theta \in \mathbb{R}$, as in (2.2), we obtain the more general identity

$$
\begin{aligned}
&\exp(se_1)\exp(t(\cos(\theta)e_1 + \sin(\theta)e_2))\exp(-se_1) \\
&\qquad = \exp(t(\cos(\theta+s)e_2 + \sin(\theta+s)e_3)).
\end{aligned}
\tag{2.6}
$$

Corollary 2.20. *Let $\{e_1, e_2, e_3\}$ be a standard Milnor basis, and $A = (t\cos s)\,e_2 + (t\sin s)\,e_3$. Letting $\rho^2 = \det A$ as in Lemma 2.13, for such A we have*

$$\rho = \frac{t}{2}$$

as noted in Remark 2.14. Then by Lemma 2.13

$$
\begin{aligned}
e^{se_1}e^{te_2}e^{-se_1} &= \exp A = \cos\rho I + \frac{\sin\rho}{\rho}A \\
&= \left(\cos\frac{t}{2}\right)I + 2\left(\sin\frac{t}{2}\cos s\right)e_2 + 2\left(\sin\frac{t}{2}\sin s\right)e_3.
\end{aligned}
\tag{2.7}
$$

2.5. **The volume function.** In what follows we take $0 < a_1 \leqslant a_2 \leqslant a_3 < \infty$. Recall that by Corollary 2.10 it is enough to consider the left-invariant Riemannian metric $g_{(a_1,a_2,a_3)}$ on SU(2) defined in Notation 2.9.

Notation 2.21. For the metric $g_{(a_1,a_2,a_3)}$ we denote by $d_{(a_1,a_2,a_3)}$ the corresponding Riemannian distance; by $B_{(a_1,a_2,a_3)}(x,r)$ we denote the open ball in the distance $d_{(a_1,a_2,a_3)}$ centered at x of radius r; by $\mu_{(a_1,a_2,a_3)}$ we denote the Riemannian volume measure corresponding to $g_{(a_1,a_2,a_3)}$.

Notation 2.22. By μ_0 we denote the bi-invariant Haar probability measure on SU(2).

Then the Riemannian volume measure $\mu_{(a_1,a_2,a_3)}$ is a constant multiple of μ_0. Specifically, we have $\mu_{(a_1,a_2,a_3)} = (16\pi^2 a_1a_2a_3)\mu_0$. The constant can be found by observing that in the bi-invariant metric $g_{(1,1,1)}$, the group SU(2) is a round sphere whose circumference is 4π as follows, for instance, from Lemma 2.13.

Notation 2.23. Let $V_{(a_1,a_2,a_3)}(r) = \mu_0(B_{(a_1,a_2,a_3)}(e,r))$ be the volume with respect to the measure μ_0 of the ball in the metric $g_{(a_1,a_2,a_3)}$.

Note that this is different from our previous notation V_g used in Section 1.1, since we are using the probability measure μ_0 instead of the Riemannian volume measure $\mu_{(a_1,a_2,a_3)}$. But this only makes a difference of a factor of $(16\pi^2 a_1a_2a_3)^{-1}$, which for the purposes of studying volume doubling is irrelevant; and it is slightly more convenient for our purposes.

Remark 2.24. For any $c > 0$, we have the scaling

$$d_{(ca_1,ca_2,ca_3)}(x,y) = c d_{(a_1,a_2,a_3)}(x,y),$$

and so $B_{(ca_1,ca_2,ca_3)}(x,r) = B_{(a_1,a_2,a_3)}(x,r/c)$. As such, $g_{(a_1,a_2,a_3)}$ and $g_{(ca_1,ca_2,ca_3)}$ have the same volume doubling constant. So for our purposes, we can suppose without loss of generality that $a_2 = 1$. We show in Proposition 7.1 that a_2 is comparable to the diameter of $g_{(a_1,a_2,a_3)}$, so the effect of this is rescaling of the metric to a roughly constant diameter. The results in the remainder of the paper are written for general a_2, but in the proofs we generally work only with the case $a_2 = 1$.

Notation 2.25. Let $\Phi, \Psi : \mathbb{R}^3 \to \mathrm{SU}(2)$ be, respectively, the coordinates of the first and second kind (used in [39]), defined by

$$\Phi(x_1,x_2,x_3) = \exp\left(x_1e_1 + x_2e_2 + x_3e_3\right),$$
$$\Psi(y_1,y_2,y_3) = \exp(y_1e_1)\exp(y_2e_2)\exp(y_3e_3).$$

We note that Φ, Ψ are both smooth maps, and that their differentials are isomorphisms at $(0,0,0)$.

Notation 2.26. Suppose $U \subset \mathbb{R}^3$ is open and $F : U \to \mathrm{SU}(2)$ is a diffeomorphism onto its image. When we speak of the Jacobian $J : U \to (0,\infty)$ of F, we mean the normalization such that $\mu_0(F(K)) = \int_K J\,dm$ for measurable $K \subset U$. Here m is the Lebesgue measure on $\mathbb{R}^3$.

Remark 2.27. Let $\Phi, \Psi : \mathbb{R}^3 \to \mathrm{SU}(2)$ be coordinates of the first and second kind introduced in Notation 2.25. Since both $d\Phi(0,0,0)$ and $d\Psi(0,0,0)$ are nonsingular, then by the inverse function theorem, on some small box $(-\eta,\eta)^3$, both Φ and Ψ are diffeomorphisms onto their images. In particular, taking η smaller if needed, their Jacobian determinants (with the normalization defined in Notation 2.26) are bounded away from 0 on $[-\eta,\eta]^3$. Therefore there is some universal constant c such that for any measurable $K \subset (-\eta,\eta)^3$ we have

$$\mu_0(\Phi(K)) \geqslant c m(K), \quad \mu_0(\Psi(K)) \geqslant c m(K), \tag{2.8}$$

where m is the Lebesgue measure on $\mathbb{R}^3$ as before.

3. Euclidean regime

At sufficiently small scales, the Riemannian manifold $(\mathrm{SU}(2), g_{(a_1,a_2,a_3)})$ (with $0 < a_1 \leqslant a_2 \leqslant a_3 < \infty$) looks like Euclidean space, so we expect the volume of a ball of radius r to scale like r^3. We need to determine, in terms of a_1, a_2, a_3, how small the scale has to be to ensure this happens with a uniform constant.

Proposition 3.1. *There are constants c, C such that, uniformly in $a_1 \leqslant a_2 \leqslant a_3$, we have*

$$c(a_1a_2a_3)^{-1}r^3 \leqslant V_{(a_1,a_2,a_3)}(r) \leqslant C(a_1a_2a_3)^{-1}r^3 \quad \text{for } 0 \leqslant r \leqslant a_1a_2/a_3. \tag{3.1}$$

An upper bound can be obtained from the form of the Bishop–Gromov comparison theorem, and a direct computation of the Ricci curvature of $g_{(a_1,a_2,a_3)}$.

Lemma 3.2. *Let (M,g) be a 3-dimensional complete Riemannian manifold with $\mathrm{Ric}_g \geqslant -\kappa g$. Then for any $0 < s \leqslant r < \infty$ we have*

$$\frac{V_g(r)}{V_g(s)} \leqslant \left(\frac{r}{s}\right)^3 e^{\sqrt{2\kappa}r}.$$

Proof. By the Bishop–Gromov comparison theorem (see [19, Corollary 5.6] or [42, Lemma 36]), we have

$$\frac{V_g(r)}{V_g(s)} \leqslant \frac{V_\kappa(r)}{V_\kappa(s)}, \tag{3.2}$$

where $V_\kappa(r)$ is the volume of a ball of radius r in the 3-dimensional hyperbolic space of constant sectional curvature $-\kappa/2$ (which has constant Ricci curvature $-\kappa$). The volume $V_\kappa(r)$ is given by [58]

$$V_\kappa(r) = \pi(\kappa/2)^{-3/2}\left(\sinh(\sqrt{2\kappa}r) - \sqrt{2\kappa}r\right)$$

and the desired result follows by observing that

$$\frac{x^3}{6} \leqslant \sinh(x) - x \leqslant \frac{x^3 e^x}{6}, \qquad x \geqslant 0$$

which can be seen, for instance, by inspecting the Taylor series. □

Proof of Proposition 3.1. It is enough to bound the Ricci tensor of the metric $g_{(a_1,a_2,a_3)}$. In the basis $\{e_1, e_2, e_3\}$, Ric is diagonal, and we find

$$\operatorname{Ric}(e_i, e_i) = \frac{\left(a_i^4 - (a_j^2 - a_k^2)^2\right)}{2(a_j a_k)^2},$$

where (i,j,k) is any permutation of $(1,2,3)$ (note that the expression is symmetric in a_j and a_k, so it is not necessary to restrict to positive permutations). Now we need to find the smallest of the ratios

$$\frac{\operatorname{Ric}(e_i, e_i)}{g(e_i, e_i)} = \frac{\left(a_i^4 - (a_j^2 - a_k^2)^2\right)}{2(a_i a_j a_k)^2}.$$

Recall that $a_1 \leqslant a_2 \leqslant a_3$, and therefore $|a_j^2 - a_k^2| \leqslant a_3^2$, and so we have $a_i^4 - (a_j^2 - a_k^2)^2 \geqslant -a_3^4$. This yields the bound

$$\frac{\operatorname{Ric}(e_i, e_i)}{g(e_i, e_i)} \geqslant -\frac{a_3^4}{2(a_i a_j a_k)^2} = -\frac{1}{2}\left(\frac{a_3}{a_1 a_2}\right)^2 \tag{3.3}$$

which is sharp when $i = 3$ and $a_1 = a_2$. Let us denote by $\kappa := \frac{1}{2}\left(\frac{a_3}{a_1 a_2}\right)^2$ the quantity on the right side of (3.3). If $r \leqslant \frac{a_1 a_2}{a_3}$, then we have $\sqrt{2\kappa}r \leqslant 1$, and Lemma 3.2 gives

$$V_g(r) \leqslant e r^3 s^{-3} V_g(s), \qquad 0 < s \leqslant r \leqslant \frac{a_1 a_2}{a_3}. \tag{3.4}$$

Letting $s \to 0$, we have $V_g(s) \sim \frac{4}{3}\pi s^3$ (since a Riemannian manifold is locally Euclidean), so that (3.4) reads $V_g \leqslant C r^3$ where $C = \frac{4}{3}\pi e$. Rewriting this in terms of $V_{(a_1,a_2,a_3)}$ using Notation 2.23, we have the upper bound

$$V_{(a_1,a_2,a_3)}(r) \leqslant C(a_1 a_2 a_3)^{-1} r^3, \qquad 0 \leqslant r \leqslant \frac{a_1 a_2}{a_3}$$

absorbing $1/16\pi^2$ into the constant C.

Now we turn to the lower bound. Let $\Psi : \mathbb{R}^3 \to \mathrm{SU}(2)$ be coordinates of the second kind introduced in Notation 2.25. By Remark 2.27, there exist $\eta > 0$ and a constant c such that for any measurable $K \subset (-\eta, \eta)^3$ we have $\mu_0(\Psi(K)) \geqslant c m(K)$, where m is the Lebesgue measure on $\mathbb{R}^3$. Suppose that $t \leqslant \eta a_1$ and consider the box

$$K_t = \left[-\frac{t}{a_1}, \frac{t}{a_1}\right] \times \left[-\frac{t}{a_2}, \frac{t}{a_2}\right] \times \left[-\frac{t}{a_3}, \frac{t}{a_3}\right] \subset [-\eta, \eta]^3.$$

On the one hand, we have $\mu_0(\Psi(K_t)) \geqslant cm(K_t) = 8c(a_1a_2a_3)^{-1}t^3$. On the other hand, for any $(x, y, z) \in K_t$, we have

$$d_{(a_1,a_2,a_3)}(e, \Psi(x, y, z)) \leqslant a_1|x| + a_2|y| + a_3|z| \leqslant 3t.$$

That is, $\Psi(K_t) \subset B_{(a_1,a_2,a_3)}(3t)$, so we have

$$V_{(a_1,a_2,a_3)}(3t) = \mu_0(B_{(a_1,a_2,a_3)}(3t)) \geqslant \mu_0(\Psi(K_t)) \geqslant 8c(a_1a_2a_3)^{-1}t^3$$

or, letting $r = 3t$,

$$V_{(a_1,a_2,a_3)}(r) \geqslant c'(a_1a_2a_3)^{-1}r^3, \quad 0 \leqslant r \leqslant \frac{\eta}{3}a_1, \tag{3.5}$$

where $c' = \frac{8}{27}c$. To complete the proof for all $0 \leqslant r \leqslant a_1$, note that for any $\frac{\eta}{3}a_1 \leqslant r \leqslant a_1$ we have by the monotonicity of V that

$$V_{(a_1,a_2,a_3)}(r) \geqslant V_{(a_1,a_2,a_3)}\left(\frac{\eta}{3}a_1\right) \geqslant c'\frac{\eta^3}{27}\frac{a_1^2}{a_2a_3} \geqslant c'\frac{\eta^3}{27}(a_1a_2a_3)^{-1}r^3$$

so taking $c'' = \min\{1, \frac{\eta^3}{27}\}c'$ we have the desired

$$V_{(a_1,a_2,a_3)}(r) \geqslant c''(a_1a_2a_3)^{-1}r^3, \quad 0 \leqslant r \leqslant a_1 \tag{3.6}$$

and in particular for $0 \leqslant r \leqslant \frac{a_1a_2}{a_3}$, since $a_2 \leqslant a_3$. □

4. Heisenberg regime

For $r \geqslant a_1a_2/a_3$, the Euclidean behavior breaks down. The growth of a ball in the e_3 direction is now affected by the relation $[e_1, e_2] = e_3$; paths can make more efficient progress in the e_3 direction by making a loop in the e_1 and e_2 directions. This is well approximated by the sub-Riemannian geometry of the 3-dimensional Heisenberg group, in which one cannot move tangent to the vertical direction e_3 at all. The sub-Riemannian Heisenberg group has Hausdorff dimension 4, which accounts for the r^4 volume scaling that appears in this regime.

Lemma 4.1. *Define $H : \mathbb{R}^2 \to \mathrm{SU}(2)$ by*

$$H(u, v) = \exp(-ue_1)\exp(-ve_2)\exp(ue_1)\exp(ve_2). \tag{4.1}$$

Then in some neighborhood U of $(0, 0)$ in $\mathbb{R}^2$ we can write

$$H(u, v) = \exp(uvh(u, v)) \tag{4.2}$$

where $h : \mathbb{R}^2 \to \mathfrak{su}(2)$ is C^∞ and $h(0, 0) = e_3$.

Proof. We give two different arguments. Applying the Campbell–Baker–Dynkin–Hausdorff formula gives a power series for $\log H(u, v)$, convergent in a neighborhood U of $(0, 0)$. The first-order terms in u, v cancel out, while the second-order term is uve_3. Each higher-order term consists of a combinatorial coefficient multiplied by some iterated Lie bracket of the vectors ue_1 and ve_2. In any term that does not vanish, the innermost bracket must be of the form $\pm[ue_1, ve_2] = \pm uve_3$; so such an iterated bracket must equal $\pm u^a v^b e_i$ with $a, b \geqslant 1$. Thus we can factor out uv from every term of this power series, writing $\log H(u, v) = uvh(u, v)$ where h is given by a convergent power series and thus is real analytic in U.

Alternatively, one can give a more direct proof by using (2.7), (2.4), (2.3) and the double angle formula to write

$$\begin{aligned} H(u,v) = \frac{1}{2}\left((1+\cos u)+\cos v\,(1-\cos u)\right) I - \left((1-\cos v)\sin u\right) e_1 \\ + \sin v\,(1-\cos u)\, e_2 + (\sin v \sin u)\, e_3. \end{aligned} \tag{4.3}$$

Then by using Lemma 2.15, one obtains a formula for $\log(H(u,v))$. It can then be seen by inspection that $h(u,v) = \log(H(u,v))/(uv)$ has a removable singularity at $(0,0)$, where the limit equals e_3. □

Lemma 4.2. *Define* $F : \mathbb{R}^4 \to \mathrm{SU}(2)$ *by*

$$F(s_1,s_2,s_3,\delta) = \exp(s_1 e_1)\exp(s_2 e_2) H\left(\operatorname{sgn}(s_3)\sqrt{|s_3|}, \delta\sqrt{|s_3|}\right).$$

Then there is a neighborhood V *of* $(0,0,0) \in \mathbb{R}^3$ *such that on* $V \times [0,1]$*, the partial derivatives of* F *with respect to* s_1, s_2, s_3 *exist and are jointly continuous, and we have*

$$\partial_{s_1} F(0,0,0,\delta) = e_1, \qquad \partial_{s_2} F(0,0,0,\delta) = e_2, \qquad \partial_{s_3} F(0,0,0,\delta) = \delta e_3.$$

Moreover, there is a jointly continuous $f : V \times [0,1] \to T\,\mathrm{SU}(2)$ *such that*

$$\partial_{s_3} F(s_1,s_2,s_3,\delta) = \delta f(s_1,s_2,s_3,\delta)$$

and $f(0,0,0,\delta) = e_3$.

Proof. Observe that $s \mapsto \operatorname{sgn}(s)\sqrt{|s|}$ is continuous on $\mathbb{R}$, and H is smooth, with $H(0,0) = I$. Thus the desired statements about $\partial_{s_1} F$, $\partial_{s_2} F$ are clear.

For convenience, let $G(s,\delta) = H\left(\operatorname{sgn}(s)\sqrt{|s|}, \delta\sqrt{|s|}\right)$, so that $F(s_1,s_2,s_3,\delta) = \exp(s_1 e_1)\exp(s_2 e_2) G(s_3,\delta)$. Let us write $H(u,v) = \exp(uv\, h(u,v))$ as in the previous lemma. Then for s in some interval $(-\epsilon,\epsilon)$ we can write

$$G(s,\delta) = \exp(\delta k(s,\delta))$$

where

$$k(s,\delta) = s \cdot h\left(\operatorname{sgn}(s)\sqrt{|s|}, \delta\sqrt{|s|}\right).$$

For $s \neq 0$, we compute

$$\begin{aligned} \partial_s k(s,\delta) &= h\left(\operatorname{sgn}(s)\sqrt{|s|}, \delta\sqrt{|s|}\right) \\ &\quad + \frac{1}{2}\operatorname{sgn}(s)\sqrt{|s|}\,\partial_u h\left(\operatorname{sgn}(s)\sqrt{|s|}, \delta\sqrt{|s|}\right) \\ &\quad + \frac{\delta}{2}\sqrt{|s|}\,\partial_v h\left(\operatorname{sgn}(s)\sqrt{|s|}, \delta\sqrt{|s|}\right). \end{aligned}$$

As $s \to 0$, the right side approaches $h(0,0) = e_3$, uniformly in $\delta \in [0,1]$. Since k is continuous, it follows (by L'Hôpital's rule) that $\partial_s k(0,\delta)$ exists and equals e_3; moreover, $\partial_s k$ is jointly continuous on $(-\epsilon,\epsilon) \times [0,1]$.

Now from the chain rule, since exp is smooth, we conclude that $\partial_s G(s,\delta)$ exists on $(-\epsilon,\epsilon) \times [0,1]$ and is given by

$$\partial_s G(s,\delta) = d\exp_{\delta k(s,\delta)}\left[\partial_s[\delta k(s,\delta)]\right] = \delta \cdot d\exp_{\delta k(s,\delta)}\left[\partial_s k(s,\delta)\right]$$

where $d\exp_{\delta k(s,\delta)}\left[\partial_s k(s,\delta)\right]$ is a jointly continuous function of s and δ. It is also clear from this that $\partial_s G(0,\delta) = \delta e_3$. The desired statements about $\partial_{s_3} F$ follow. □

Lemma 4.3. *For $\delta \in [0,1]$, consider $F^\delta = F(\cdot,\cdot,\cdot,\delta)$ as a map from $\mathbb{R}^3$ to $\mathrm{SU}(2)$. Let J^δ be its Jacobian determinant as in Notation 2.26. Then there is a neighborhood W of $(0,0,0) \in \mathbb{R}^3$ and a constant $c > 0$, independent of δ, such that $J^\delta \geqslant c\delta$ on W. In particular, for any measurable $K \subset W$, we have $\mu_0(F^\delta(K)) \geqslant c\delta m(K)$.*

Proof. Let ω be the Riemannian volume form on SU(2) associated to the bi-invariant metric $g_{(1,1,1)}$. Then we have

$$J^\delta = \frac{1}{16\pi^2}\omega(\partial_{s_1}F^\delta, \partial_{s_2}F^\delta, \partial_{s_3}F^\delta).$$

If we set

$$j(s_1,s_2,s_3,\delta) = \omega(\partial_{s_1}F^\delta(s_1,s_2,s_3), \partial_{s_2}F^\delta(s_1,s_2,s_3), f(s_1,s_2,s_3,\delta))$$

where f is as in Lemma 4.2, then $J^\delta = \frac{\delta}{16\pi^2}j$. Moreover, j is jointly continuous on $V\times[0,1]$, and we have $j(0,0,0,\delta) = \omega(e_1,e_2,e_3) = 1$ for all δ. As such, by continuity and the compactness of $[0,1]$, there is a neighborhood $W \subset V$ of $(0,0,0) \in \mathbb{R}^3$ such that $j \geqslant \frac{1}{2}$ on $W \times [0,1]$, which implies $J \geqslant \frac{1}{32\pi^2}\delta$. □

Proposition 4.4 (Heisenberg type lower bound)**.** *There is a constant c such that, uniformly in $a_1 \leqslant a_2 \leqslant a_3$,*

$$V_{(a_1,a_2,a_3)}(r) \geqslant c(a_1a_2)^{-2}r^4 \quad \textit{for } 0 \leqslant r \leqslant a_1.$$

Note that for $r \simeq a_1a_2/a_3$ this lower bound matches the result provided by Proposition 3.1.

Proof. Since the right side is consistent with the scaling described in Remark 2.24, we suppose without loss of generality that $a_2 = 1$.

Let F^δ be as in Lemma 4.2 and W as in Lemma 4.3. Choose $\eta > 0$ so small that $[-\eta,\eta]^3 \subset W$. We note that

$$d_{(a_1,1,a_3)}(e, F^\delta(s_1,s_2,s_3)) \leqslant s_1a_1 + s_2 + 2a_1\sqrt{|s_3|} + 2\delta\sqrt{|s_3|}.$$

Now let us take $\delta = a_1 \in [0,1]$, so that this becomes

$$d_{(a_1,1,a_3)}(e, F^\delta(s_1,s_2,s_3)) \leqslant s_1a_1 + s_2 + 4a_1\sqrt{|s_3|}.$$

Suppose $r \leqslant a_1\eta \leqslant \eta$ and let

$$K_r = \left[-\frac{r}{a_1}, \frac{r}{a_1}\right] \times \left[-r, r\right] \times \left[-\frac{r^2}{a_1^2}, \frac{r^2}{a_1^2}\right]$$

so that $K_r \subset [-\eta,\eta]^3 \subset W$. We then have $m(K_r) = 8a_1^{-3}r^4$ and $F^{a_1}(K_r) \subset B_{(a_1,1,a_3)}(6r)$. By Lemma 4.3, we have

$$V_{(a_1,1,a_3)}(6r) = \mu_0(B_{(a_1,1,a_3)}(6r)) \geqslant \mu_0(F^{a_1}(K_r)) \geqslant ca_1m(K_r) = 8ca_1^{-2}r^4.$$

or

$$V_{(a_1,1,a_3)}(r) \geqslant c'a_1^{-2}r^4, \quad 0 \leqslant r \leqslant 6\eta a_1$$

where $c' = 8c/6^4$. If it happens that $6\eta \geqslant 1$ then we are finished; if not, we can drop the 6η in the upper limit on r as in the proof of Proposition 3.1, replacing c' by $(6\eta)^4c'$. □

Proposition 4.5 (Heisenberg type upper bound). *There exists $\eta \in (0,1)$ and a constant $C < \infty$ such that, uniformly in $a_1 \leqslant a_2 \leqslant a_3$,*

$$V_{(a_1,a_2,a_3)}(r) \leqslant C\left((a_1a_2a_3)^{-1}r^3 + (a_1a_2)^{-2}r^4\right) \quad \text{for } 0 \leqslant r \leqslant \eta a_1.$$

In particular, we have

$$V_{(a_1,a_2,a_3)}(r) \leqslant 2C(a_1a_2)^{-2}r^4 \quad \text{for } a_1a_2/a_3 \leqslant r \leqslant \eta a_1.$$

Proof. Again, we assume $a_2 = 1$. Suppose $r \leqslant \eta a_1$, where η is to be chosen later, and let $g \in B_{(a_1,1,a_3)}(r)$. This means that there is a smooth path $\gamma : [0,1] \to \mathrm{SU}(2)$ with $\gamma(0) = e$, $\gamma(1) = g$, and length $\ell_{(a_1,1,a_3)}[\gamma] < r$. Reparametrizing γ by constant speed (with respect to $g_{(a_1,1,a_3)}$), we can write $\dot\gamma(t) = \sum_{i=1}^3 \lambda_i(t)\widetilde{e_i}(\gamma(t))$, where $\widetilde{e_i}$ is the left-invariant vector field which equals e_i at the identity, and $\sum_{i=1}^3 |a_i\lambda_i(t)|^2 \leqslant r^2$ for all $t \in [0,1]$. In particular, $|\lambda_i(t)| \leqslant r/a_i$.

We now invoke a result of R. Strichartz [52] which extends the Baker–Campbell–Hausdorff–Dynkin formula by giving an exact expression for the exponential coordinates of g in terms of λ_i. The Strichartz (or Chen–Strichartz) formula says that $g = \exp z$, where

$$z = \sum_{n=1}^\infty \sum_{I\in\{1,2,3\}^n} \left(\sum_{\sigma\in S_n} \left(\frac{(-1)^{e(\sigma)}}{n^2\binom{n-1}{e(\sigma)}} \right) \int_{\Delta^n} \prod_{m=1}^n \lambda_{i_m}(s_{\sigma(m)})ds \right) e_I \in \mathfrak{su}(2). \tag{4.4}$$

Here $I = (i_1, \dots, i_n)$, and e_I is the n-fold iterated bracket

$$e_I = [[\dots[e_{i_1}, e_{i_2}], \dots], e_{i_n}].$$

Note that since $\{e_i\}$ is a standard Milnor basis, each e_I equals either 0 or some $\pm e_k$. S_n is the set of permutations of $\{1, \dots, n\}$, and following Strichartz's notation, $e(\sigma) = |\{m < n : \sigma(m+1) < \sigma(m)\}|$ denotes the number of "errors" (also called "descents") of the permutation σ; for our purposes, we need only note that $e(\sigma)$ is an integer between 0 and $n-1$. Finally, $\Delta^n \subset [0,1]^n$ is the standard n-simplex $\{0 \leqslant s_1 \leqslant \dots \leqslant s_n \leqslant 1\}$, whose volume is $1/n!$.

Let us write $z = \sum_{i=1}^3 z_ie_i$. We shall bound each of the $|z_i|$, which will show that g is contained in the image under the coordinates Φ (see Notation 2.25) of some box in $\mathbb{R}^3$ of bounded size. This fact, combined with Remark 2.27 on the Jacobian determinant of Φ, will give us an upper volume estimate for $B_{(a_1,1,a_3)}(r)$.

We begin with z_1; the analysis of z_2, z_3 will be similar. Let $\zeta_{i,n}$ be the coefficient of e_i in the n term of the sum in (4.4), so that $z_1 = \sum_{n=1}^\infty \zeta_{1,n}$. We must consider which values of I give $e_I = \pm e_1$. For $n = 1$ we have only $I = (1)$, and for $n = 2$ we have $I = (2,3)$ and $I = (3,2)$. So we have

$$\zeta_{1,1} = \int_0^1 \lambda_1(s)\,ds$$

$$\zeta_{1,2} = \frac{1}{4}\int_{0\leqslant s_1\leqslant s_2\leqslant 1} (\lambda_2(s_1)\lambda_3(s_2) - \lambda_3(s_1)\lambda_2(s_2))\,ds_1\,ds_2.$$

This trivially gives

$$|\zeta_{1,1}| \leqslant \frac{r}{a_1}, \qquad |\zeta_{1,2}| \leqslant \frac{1}{4}\frac{r^2}{a_3}. \tag{4.5}$$

For $n \geqslant 3$, in order to have $e_I = \pm e_1$ we note that i_1, i_2 cannot both equal 1 (else $e_I = 0$), and i_n cannot equal 1 either (since $[e_k, e_1] \neq \pm e_1$ for any $k = 1,2,3$). So at least two of the i_m are different from 1, meaning that the corresponding λ_{i_m}

are bounded by r. Since $|\lambda_i| \leqslant r/a_i$ and $a_1 \leqslant a_2 = 1 \leqslant a_3$, the remaining λ_{i_m} are bounded by r/a_1, and we conclude that $\left|\prod_{m=1}^n \lambda_{i_m}(s_{\sigma(m)})\right| \leqslant r^n/a_1^{n-2}$.

Now to estimate the value of the parenthesized sum over $\sigma \in S_n$ in (4.4), we note that Δ^n has a volume of $1/n!$, that $|S_n| = n!$, and that the combinatorial coefficient is at most 1. So this sum is bounded by r^n/a_1^{n-2} as well. Finally, the total number of $I \in \{1,2,3\}^n$ is 3^n, even though most of these do not yield $e_I = \pm e_1$. So we conclude

$$|\zeta_{1,n}| \leqslant 3^n \frac{r^n}{a_1^{n-2}} = 9r^2 \left(3\frac{r}{a_1}\right)^{n-2}, \qquad n \geqslant 3. \tag{4.6}$$

By taking, say, $\eta < \frac{1}{6}$, so that $3\frac{r}{a_1} < \frac{1}{2}$, we can conclude

$$\sum_{n=3}^{\infty} |\zeta_{1,n}| < 9r^2 \sum_{n=3}^{\infty} \left(\frac{1}{2}\right)^{n-2} = 9r^2. \tag{4.7}$$

Combining (4.5) and (4.7), we see that $\frac{r}{a_1}$ dominates, and we have

$$|z_1| \leqslant \sum_{n=1}^{\infty} |\zeta_{1,n}| \leqslant c\frac{r}{a_1} \tag{4.8}$$

for some universal constant c ($c = 11$ would do).

By similar arguments, we can obtain

$$|\zeta_{2,1}| \leqslant r, \quad |\zeta_{2,2}| \leqslant \frac{1}{4}\frac{r^2}{a_1 a_3}. \tag{4.9}$$

Since $\frac{r}{a_1} \leqslant \eta \leqslant 1$ and $a_3 \geqslant 1$, both terms are dominated by r. To estimate $\zeta_{2,n}$ for $n \geqslant 3$, we use the cruder fact that in order to get $e_I \neq 0$, we must have either i_1 or i_2 different from 1. This leads to the estimate

$$|\zeta_{2,n}| \leqslant 3^n \frac{r^n}{a_1^{n-1}} = 9\frac{r^2}{a_1}\left(3\frac{r}{a_1}\right)^{n-2}$$

and thus, stil with $\eta < \frac{1}{6}$,

$$\sum_{n=3}^{\infty} |\zeta_{2,n}| < 9\frac{r^2}{a_1}$$

which again is dominated by r. So we have shown

$$|z_2| \leqslant cr \tag{4.10}$$

increasing the value of the universal constant c as needed.

For z_3, we obtain

$$|\zeta_{3,1}| \leqslant \frac{r}{a_3}, \quad |\zeta_{3,2}| \leqslant \frac{1}{4}\frac{r^2}{a_1}$$

and as before

$$\sum_{n=3}^{\infty} |\zeta_{3,n}| < 9\frac{r^2}{a_1}.$$

We conclude

$$|z_3| \leqslant c\left(\frac{r}{a_3} + \frac{r^2}{a_1}\right), \tag{4.11}$$

where the first term dominates when $r \ll a_1/a_3$.

As such, if we let

$$K_r = \left[-c\frac{r}{a_1}, c\frac{r}{a_1}\right] \times \left[-cr, cr\right] \times \left[-c\left(\frac{r}{a_3}+\frac{r^2}{a_1}\right), c\left(\frac{r}{a_3}+\frac{r^2}{a_1}\right)\right]$$

so that in particular we have $K_r \subset [-c\eta, c\eta]^3$, we have that $B_{(a_1,1,a_3)} \subset \Phi(K_r)$. Letting M be the maximum of the Jacobian determinant of Φ over $[-c\eta, c\eta]^3$, we have

$$V_{(a_1,1,a_3)}(r) \leqslant Mm(K_r) = 8Mc^3\left(\frac{r^3}{a_1a_3}+\frac{r^4}{a_1^2}\right)$$

which is the desired bound. □

5. After Heisenberg

When r exceeds a_1, the global geometry of SU(2) becomes important. Our "budget" r is now large enough to let us travel all the way around the sphere $\mathrm{SU}(2) \cong S^3$ in the "cheap" e_1 direction, and nothing is gained by traveling around the sphere more than once. So travel in the e_1 direction has negligible cost, and the volume growth is comparable to what happens if we actually set $a_1 = 0$. The group SU(2) would collapse to a coset space mod the subgroup $S = \{\exp(se_1) : s \in \mathbb{R}\}$ which is homeomorphic to the 2-dimensional sphere S^2. For this reason, the volume in this regime grows as r^2.

Proposition 5.1. *There is a constant c such that, uniformly in $a_1 \leqslant a_2 \leqslant a_3$,*

$$V_{(a_1,a_2,a_3)}(r) \geqslant ca_2^{-2}r^2 \quad \text{for } a_1 \leqslant r \leqslant a_2.$$

Proof. As usual, it suffices to take $a_2 = 1$ (see Remark 2.24). We proceed along the lines similar to the proof of Proposition 4.4. For $\eta \in [0,1]$, let $F^\eta : \mathbb{R}^3 \to \mathrm{SU}(2)$ be defined by

$$F^\eta(s_1,s_2,s_3) = \exp(s_1e_1)\exp(s_2e_2)H(\eta,s_3),$$

where H is as in (4.1). Let J^η be the Jacobian determinant of F^η, normalized as in Notation 2.26. Then by the same arguments as in Lemmas 4.2 and 4.3, there is a neighborhood U of $(0,0,0) \in \mathbb{R}^3$ and a jointly continuous $j : U \times [0,1] \to \mathbb{R}$ such that

$$J^\eta(s_1,s_2,s_3) = \eta j(s_1,s_2,s_3,\eta), \qquad (s_1,s_2,s_3) \in U, \quad \eta \in [0,1]. \tag{5.1}$$

We can also directly compute

$$\partial_{s_1}F^\eta(0,0,0) = e_1 \text{ and } \partial_{s_2}F^\eta(0,0,0) = e_2.$$

For the partial derivative with respect to s_3, we can use either Lemma 2.17 or (4.3) to compute

$$\partial_{s_3}F^\eta(0,0,0) = \partial_v H(\eta,0) = (1-\cos\eta)e_2 + (\sin\eta)e_3.$$

Thus, letting ω be the Riemannian volume form of the bi-invariant metric $g_{(1,1,1)}$, we have

$$J^\eta(0,0,0) = \frac{1}{16\pi^2}\omega(e_1,e_2,(1-\cos\eta)e_2+(\sin\eta)e_3) = \frac{1}{16\pi^2}\sin\eta.$$

In particular, from (5.1), we have $j(0,0,0,\eta) = \frac{1}{16\pi^2}\frac{\sin\eta}{\eta} > 0$ for all $\eta \in [0,1]$. We can thus find a neighborhood $W \subset U$ of $(0,0,0)$ such that j is bounded away from 0 on $W \times [0,1]$, which implies $J^\eta \geqslant c\eta$ for some constant c.

Now choose $\eta > 0$ sufficiently small so that $[-\eta,\eta]^3 \subset W$. Suppose $\eta a_1 \leqslant r \leqslant \eta$ and set

$$K = \Big[-\eta,\eta\Big] \times \Big[-r,r\Big] \times \Big[-r,r\Big].$$

Note that $K \subset [-\eta,\eta]^3 \subset W$. Hence, we have $\mu_0(F^\eta(K)) \geqslant c\eta \cdot m(K) = 8c\eta^2 r^2$.

Also, we have

$$\begin{aligned} d_{(a_1,1,a_3)}(e, F^\eta(s_1,s_2,s_3)) &\leqslant a_1 s_1 + s_2 + 2a_1\eta + 2s_3 \\ &\leqslant a_1\eta + r + 2a_1\eta + 2r \\ &\leqslant 6r \end{aligned}$$

recalling that $r \geqslant a_1\eta$. So $F^\eta(K) \subset B_{(a_1,1,a_3)}(6r)$. We have thus shown

$$V_{(a_1,1,a_3)}(6r) \geqslant 8c\eta^2 r^2, \qquad a_1\eta \leqslant r \leqslant \eta.$$

Repeating this argument with r, η replaced by $r/6, \eta/6$ (which is valid since we still have $[-\eta/6,\eta/6]^3 \subset W$), we have

$$V_{(a_1,1,a_3)}(r) \geqslant c\eta^2 r^2, \qquad a_1\eta \leqslant r \leqslant \eta$$

where a factor of $8/6^4$ has been absorbed into the constant c. This is the desired result for $a_1 \leqslant r \leqslant \eta$. For $\eta \leqslant r \leqslant 1$, simply note that

$$V_{(a_1,1,a_3)}(r) \geqslant V_{(a_1,1,a_3)}(\eta) \geqslant c\eta^4 \geqslant c\eta^4 r^2$$

and so we have the desired result for all $a_1 \leqslant r \leqslant 1$. □

For the corresponding upper bound, we show that the ball $B_{(a_1,1,a_3)}(r)$ is contained in a tubular neighborhood of the circle $S = \{\exp(se_1) : s \in \mathbb{R}\}$.

Lemma 5.2. *Let $0 < a_1 \leqslant 1 \leqslant a_3 < \infty$. For any $x \in \mathrm{SU}(2)$, we may write $x = \exp(se_1)y$ where $d_{(1,1,1)}(e,y) \leqslant d_{(a_1,1,a_3)}(e,x)$. In particular, $d_{(1,1,1)}(S,x) \leqslant d_{(a_1,1,a_3)}(e,x)$.*

Proof. Note first that without loss of generality we can assume $a_3 = 1$, since $d_{(a_1,1,1)}(e,x) \leqslant d_{(a_1,1,a_3)}(e,x)$.

Fix $\epsilon > 0$. Consider the smooth map $\Theta : \mathbb{R}^3 \to \mathrm{SU}(2)$ defined by

$$\Theta(z_1,z_2,z_3) = \exp(z_1 e_1/a_1)\exp(z_2 e_2 + z_3 e_3).$$

Then $d\Theta$ is an isomorphism at $(0,0,0)$, so that Θ is a diffeomorphism near $(0,0,0)$. If we equip $\mathbb{R}^3$ with the standard Euclidean metric and SU(2) with the $g_{(a_1,1,1)}$ metric, then $d\Theta_e^{-1} : T_e\,\mathrm{SU}(2) \to T_{(0,0,0)}\mathbb{R}^3$ is an isometry; in particular the operator norm is $\|d\Theta_e^{-1}\|_{(a_1,1,1)} = 1$. Hence we may find some neighborhood V of $e \in \mathrm{SU}(2)$ such that $\|d\Theta_x^{-1}\|_{(a_1,1,1)} \leqslant 1+\epsilon$ for all $x \in V$. Taking V smaller if necessary, we may also assume that V is a $g_{(a_1,1,1)}$-normal neighborhood of e; that is, for any $x \in V$ there is a $g_{(a_1,1,1)}$-minimizing geodesic from e to x contained in V. Then Θ^{-1} is a $(1+\epsilon)$-Lipschitz map from $(V, d_{(a_1,1,1)})$ into $\mathbb{R}^3$. So for $x \in V$, if we write $(z_1,z_2,z_3) = \Theta^{-1}(x)$, we have

$$\begin{aligned} x = \Theta(z_1,z_2,z_3) &= \exp(z_1 e_1/a_1)\exp(z_2 e_2 + z_3 e_3) \\ &= \exp(se_1)\exp(t(\cos(\theta)e_2 + \sin(\theta)e_3)) \end{aligned}$$

where we let $s = z_1/a_1$, $z_2 = t\cos\theta$, $z_3 = t\sin\theta$. Moreover,

$$|t| = \sqrt{z_2^2 + z_3^2} \leqslant |(z_1,z_2,z_3)| \leqslant (1+\epsilon)d_{(a_1,1,1)}(e,x).$$

Now let $x \in \mathrm{SU}(2)$ be arbitrary. Let $\gamma : [0,1] \to \mathrm{SU}(2)$ be a $g_{(a_1,1,1)}$-minimizing geodesic from e to x, parametrized by arc length. For an integer N to be chosen later, let $\tau_i = i/N$ and $x_i = \gamma(\tau_{i-1})^{-1}\gamma(\tau_i)$, $i = 0, \ldots, N$, so that $x = \prod_{i=1}^N x_i$. Note that by the left invariance of the metric,

$$d_{(a_1,1,1)}(e, x_i) = d_{(a_1,1,1)}(\gamma(\tau_{i-1}), \gamma(\tau_i)) = \frac{1}{N} d_{(a_1,1,1)}(e, x)$$

since γ was parametrized by arc length. We may now choose N so large that $x_i \in V$ for every i. Then, as above, each x_i may be written as

$$x_i = \exp(s_i e_1) \exp(t_i(\cos(\theta_i) e_2 + \sin(\theta_i) e_3))$$

where

$$|t_i| \leqslant (1+\epsilon) d_{(a_1,1,1)}(e, x_i) = \frac{1+\epsilon}{N} d_{(a_1,1,1)}(e, x).$$

By repeated application of (2.6), we may now write

$$\begin{aligned} x &= \prod_{i=1}^N \exp(s_i e_1) \exp(t_i(\cos(\theta_i) e_2 + \sin(\theta_i) e_3)) \\ &= \exp(s e_1) \prod_{i=1}^N \exp(t_i(\cos(\phi_i) e_2 + \sin(\phi_i) e_3)) \end{aligned}$$

where

$$s = s_1 + \cdots + s_N, \qquad \phi_i = \theta_i - s_{i+1} - \cdots - s_N.$$

Setting $y = \prod_{i=1}^N \exp(t_i(\cos(\phi_i) e_2 + \sin(\phi_i) e_3))$, we have by left-invariance of $d_{(1,1,1)}$ that

$$\begin{aligned} d_{(1,1,1)}(e, y) &\leqslant \sum_{i=1}^N d_{(1,1,1)}(e, \exp(t_i(\cos(\phi_i) e_2 + \sin(\phi_i) e_3))) \\ &\leqslant \sum_{i=1}^N |t_i| \\ &\leqslant (1+\epsilon) d_{(a_1,1,1)}(e, x). \end{aligned}$$

To remove the ϵ, we note that for each n, we can write $x = \exp(s_n e_1) y_n$ where, without loss of generality, $s_n \in [-2\pi, 2\pi]$, and $y_n \in \mathrm{SU}(2)$ with $d_{(1,1,1)}(e, y_n) \leqslant (1 + \frac{1}{n}) d_{(a_1,1,1)}(e, x)$. Since $[-2\pi, 2\pi]$ and $\mathrm{SU}(2)$ are compact, we can pass to a subsequence so that $s_n \to s$ and $y_n \to y$ for some s, y, which will then be as desired. □

Proposition 5.3. *There is a constant C such that, uniformly in $a_1 \leqslant a_2 \leqslant a_3$,*

$$V_{(a_1,a_2,a_3)}(r) \leqslant C a_2^{-2} r^2 \quad \text{for } 0 \leqslant r \leqslant a_2.$$

Proof. As usual we assume $a_2 = 1$. Let $K = B_{(1,1,1)}(S, r)$, so that by the previous lemma $B_{(a_1,1,a_3)}(r) \subset K$. It only remains to estimate the volume of K. Let $N = \lceil \frac{4\pi}{r} \rceil$, so that $\frac{4\pi}{r} \leqslant N \leqslant \frac{4\pi}{r} + 1 \leqslant (4\pi + 1)\frac{1}{r}$. Set $x_i = \exp(4\pi i e_1/N)$ for $0 \leqslant i \leqslant N$, so that $x_0 = x_N = e$ and $d_{(1,1,1)}(x_i, x_{i+1}) \leqslant \frac{4\pi}{N} \leqslant r$. As such, the balls $B_{(1,1,1)}(x_i, 2r)$, $1 \leqslant i \leqslant N$, cover K. Since $(\mathrm{SU}(2), g_{(1,1,1)})$ is a compact 3-dimensional Riemannian manifold and μ_0 is (up to a constant) its volume measure, there is a constant C such that $\mu_0(B_{(1,1,1)}(x, R)) \leqslant CR^3$ for any R. So we conclude

$$\mu_0(B_{(a_1,1,a_3)}(r)) \leqslant \mu_0(K) \leqslant CN(2r)^3 \leqslant 2^3(4\pi + 1)Cr^2.$$

□

6. Combining the cases

Combining the foregoing bounds yields the estimates on $V_g(r)$ of Theorem 1.3.

Proof of Theorem 1.3. Similarly to the $V_{(a_1,a_2,a_3)}$ notation, set

$$\overline{V}_{(a_1,a_2,a_3)}(r) = \begin{cases} (a_1a_2a_3)^{-1}r^3, & 0 \leqslant r \leqslant a_1a_2/a_3 \\ (a_1a_2)^{-2}r^4, & a_1a_2/a_3 \leqslant r \leqslant a_1 \\ a_2^{-2}r^2, & a_1 \leqslant r \leqslant a_2 \\ 1, & r \geqslant a_2. \end{cases} \tag{6.1}$$

We need to show that $b_1\overline{V}_{(a_1,a_2,a_3)}(r) \leqslant V_{(a_1,a_2,a_3)}(r) \leqslant b_2\overline{V}_{(a_1,a_2,a_3)}(r)$ for some constants b_1, b_2 not depending on a_1, a_2, a_3. This will establish Theorem 1.3 for metrics of the form $g = g_{(a_1,a_2,a_3)}$, recalling from Notation 2.23 that $V_g(r)$ differs from $V_{(a_1,a_2,a_3)}(r)$ by a factor of $(16\pi^2a_1a_2a_3)^{-1}$. The general case follows since, as noted in Corollary 2.10, every $g \in \mathfrak{L}(\mathrm{SU}(2))$ is isometric to some $g_{(a_1,a_2,a_3)}$.

The upper and lower bounds in the case $0 \leqslant r \leqslant a_1a_2/a_3$ are covered by Proposition 3.1.

For $a_1a_2/a_3 \leqslant r \leqslant a_1$, the lower bound is shown by Proposition 4.4. The upper bound is shown by Proposition 4.5 for $a_1a_2/a_3 \leqslant r \leqslant \eta a_1$, where η is a certain small constant, so it remains to handle the case $\eta a_1 \leqslant r \leqslant a_1$. In this case we can apply Proposition 5.3 to obtain

$$V_{(a_1,a_2,a_3)}(r) \leqslant Ca_2^{-2}r^2 \leqslant C\eta^{-2}(a_1a_2)^{-2}r^4.$$

For $a_1 \leqslant r \leqslant a_2$, the desired bounds are given by Propositions 5.1 and 5.3.

For $r \geqslant a_2$, the lower bound follows simply by noting

$$V_{(a_1,a_2,a_3)}(r) \geqslant V_{(a_1,a_2,a_3)}(a_2) \geqslant c$$

from the bound in Proposition 5.1. The upper bound $V_{(a_1,a_2,a_3)}(r) \leqslant 1$ is trivial because $V_{(a_1,a_2,a_3)}(r)$ is the volume with respect to the probability measure μ_0. □

To prove Theorem 1.2, it now suffices to show that the function $\overline{V_g}$, or equivalently $\overline{V}_{(a_1,a_2,a_3)}$ as in (6.1), satisfies a uniform volume doubling condition. This is an elementary calculation which we insert here for convenience.

Lemma 6.1. *For any $a_1 \leqslant a_2 \leqslant a_3$, any $r \geqslant 0$ and $k \geqslant 1$, we have*

$$\overline{V}_{(a_1,a_2,a_3)}(kr) \leqslant k^4\,\overline{V}_{(a_1,a_2,a_3)}(r).$$

Proof. We have ten cases depending on which of the four regions defined in (6.1) are occupied by r and kr.

If r, kr occupy the same region, then the result is immediate. For instance, when $0 \leqslant r \leqslant kr \leqslant a_1a_2/a_3$, then we have $\overline{V}(kr)/\overline{V}(r) = k^3$ (we suppress the subscripts). In the other similar cases, we get k^4, k^2 or 1; all are bounded by k^4.

The next cases are when they occupy consecutive regions.

- If $0 \leqslant r \leqslant a_1a_2/a_3 \leqslant kr \leqslant a_1$, then $\overline{V}(kr)/\overline{V}(r) = k^4(a_1a_2/a_3)^{-1}r \leqslant k^4$ because $r \leqslant a_1a_2/a_3$.
- If $a_1a_2/a_3 \leqslant r \leqslant a_1 \leqslant kr \leqslant a_2$, then $\overline{V}(kr)/\overline{V}(r) = k^2a_1^2r^{-2} \leqslant k^4$, using $a_1 \leqslant kr$.
- If $a_1 \leqslant r \leqslant a_2 \leqslant kr$, then $\overline{V}(kr)/\overline{V}(r) = a_2^2r^{-2} \leqslant k^2 \leqslant k^4$, using $a_2 \leqslant kr$.

The remaining cases follow by combining those already shown. For instance, if $0 \leqslant r \leqslant a_1a_2/a_3 \leqslant a_1 \leqslant kr \leqslant a_2$, choose $1 \leqslant k' \leqslant k$ so that $a_1a_2/a_3 \leqslant k'r \leqslant a_1$. Then by the previous cases we have

$$\overline{V}(kr) \leqslant \left(\frac{k}{k'}\right)^4 \overline{V}(k'r) \leqslant k^4\overline{V}(r).$$

The last two cases are similar. □

Combining Theorem 1.3 and Lemma 6.1 (with $k = 2$) establishes Theorem 1.2, with $D = 16b_2/b_1$.

7. Diameter bounds

In this brief section, we prove the remark following Theorem 1.3: for any metric $g \in \mathfrak{L}(\mathrm{SU}(2))$, the diameter $\mathrm{diam}_g(\mathrm{SU}(2))$ is uniformly comparable to a_2, the square root of the middle eigenvalue.

An interesting consequence is that, by inspection of (3.3), there is no uniform lower bound on the Ricci curvatures of the metrics $g \in \mathfrak{L}(\mathrm{SU}(2))$, even after rescaling to constant diameter; the metrics $g_{(1,1,a_3)}$, as $a_3 \to \infty$, have comparable diameters, but their Ricci curvatures in the e_3 direction tend to $-\infty$. As such, the uniform volume doubling bound of Theorem 1.2 cannot be obtained solely by Ricci curvature considerations as in Section 3.

Proposition 7.1. *For a left-invariant Riemannian metric $g \in \mathfrak{L}(\mathrm{SU}(2))$, let a_2 be the square root of the middle eigenvalue of the matrix A_g, as in Theorem 1.3. There are universal constants $0 < D_0 \leqslant D_\infty < +\infty$ such that*

$$D_0 a_2 \leqslant \mathrm{diam}_g(\mathrm{SU}(2)) \leqslant D_\infty a_2.$$

Proof. By Corollary 2.10, we can assume without loss of generality that $g = g_{(a_1,a_2,a_3)}$ for some $a_1 \leqslant a_2 \leqslant a_3$, and by scaling, we can assume $a_2 = 1$.

For an upper bound, we consider a sub-Riemannian metric on SU(2). Let $\mathcal{H} \subset T\,\mathrm{SU}(2)$ be the two-dimensional sub-bundle spanned at each point by the left translates of $\hat{e}_1, \hat{e}_2$, and let $g_{(1,1,\infty)}$ be the left-invariant sub-Riemannian metric on $\mathcal{H}$ making $\hat{e}_1, \hat{e}_2$ orthonormal. Then $(\mathrm{SU}(2), \mathcal{H}, g_{(1,1,\infty)})$ is a sub-Riemannian manifold. The sub-bundle $\mathcal{H}$ satisfies Hörmander's bracket-generating condition, since $[\hat{e}_1, \hat{e}_2] = \hat{e}_3$, and so by the Chow–Rashevskii theorem [38, p. 43], the sub-Riemannian (or Carnot–Carathéodory) distance $d_{(1,1,\infty)}$ is finite and induces the original manifold topology. Since SU(2) is compact, it has finite diameter under $d_{(1,1,\infty)}$. Let D_∞ be this diameter. It is clear that for any $v \in T\,\mathrm{SU}(2)$, we have $g_{(a_1,1,a_3)}(v,v) \leqslant g_{(1,1,\infty)}(v,v)$ (where for $v \notin \mathcal{H}$ we can take $g_{(1,1,\infty)}(v,v) = \infty$), so the same inequality holds for their distances, and we have shown that the diameter under $g_{(a_1,1,a_3)}$ is bounded above by D_∞.

For the lower bound, consider the pseudo-metric $g_{(0,1,1)}$ for which $g_{(0,1,1)}(\hat{e}_1, \hat{e}_1) = 0$ and $\hat{e}_2, \hat{e}_3$ are orthonormal. Then the pseudo-distance $d_{(0,1,1)}$ is symmetric and satisfies the triangle inequality, but is not positive definite. For instance,

$d_{(0,1,1)}(e, \exp(s\hat{e}_1)) = 0$ for any s. However, we claim $d_{(0,1,1)}$ is not identically zero, so that SU(2) has nonzero diameter under $d_{(0,1,1)}$. As above, $d_{(0,1,1)}$ is a lower bound for any $d_{(a_1,1,a_3)}$, so we may take D_0 to be the $d_{(0,1,1)}$-diameter of SU(2).

Indeed, let $S = \{\exp(s\hat{e}_1) : s \in \mathbb{R}\}$ be the subgroup generated by $\hat{e}_1$. Suppose $d_{(0,1,1)}(e, x) = 0$; we claim that $x \in S$. For any $\epsilon > 0$, we can choose a so small that $d_{(a,1,1)}(e, x) < \epsilon$. By Lemma 5.2, we can write $x = \exp(se_1)y$ where $d_{(1,1,1)}(e, y) < \epsilon$. Thus $d_{(1,1,1)}(S, x) < \epsilon$. Since ϵ was arbitrary and S is closed, we conclude that $x \in S$. So for any $x \in \mathrm{SU}(2) \setminus S$, we have $d_{(0,1,1)}(e, x) > 0$. □

Remark 7.2. In effect, the pseudo-metric space $(\mathrm{SU}(2), d_{(0,1,1)})$ is the two-dimensional left coset space $\mathrm{SU}(2)/S$, which is homeomorphic to S^2. This statement is not so obvious as it might appear. For instance, suppose we instead consider the Heisenberg group $\mathbb{H}^3$ with the standard basis $\{X, Y, Z\}$ for $\mathfrak{h}^3$ satisfying $[X, Y] = Z$, $[X, Z] = [Y, Z] = 0$, and a left-invariant pseudo-metric g with $g(X, X) = 0$ and Y, Z orthonormal. Then the resulting pseudo-metric space is only one-dimensional, and in particular it does not equal the quotient of $\mathbb{H}^3$ by $\{\exp(tX) : t \in \mathbb{R}\}$. Indeed, by writing $\exp(s^2 Z) = \exp(s\epsilon^{-1}X)\exp(s\epsilon Y)\exp(-s\epsilon^{-1}X)\exp(-s\epsilon Y)$ where $\epsilon \to 0$, we see that we can reach the z-axis by paths of arbitrarily small length with respect to this metric, by making a rectangle that is very large in the X direction and very small in Y. However, compactness prevents this phenomenon in SU(2).

Remark 7.3. In a recent article [43], A. V. Podobryaev has computed the diameter of the metrics $g_{(a_1,a_2,a_3)}$ in the case where two of the three parameters a_1, a_2, a_3 are equal. This leads to the explicit values $D_0 = \pi$, $D_\infty = 2\pi$. The value $D_\infty = 2\pi$ also follows from a sub-Riemannian distance formula proved in [11].

8. Consequences of volume doubling

Let (M, g) be a Riemannian manifold, and Δ_g the (positive) Laplace–Beltrami operator associated with the metric g. The gradient ∇_g is determined by the metric g and we let

$$|\nabla_g f|_g^2 := g\left(\nabla_g f, \nabla_g f\right).$$

The connection between the Laplace–Beltrami operator and the gradient is given by

$$\int_M f \Delta_g f d\mu_g = \int_M |\nabla_g f|^2 d\mu_g,$$

where as before μ_g is the Riemannian volume measure. Finally the heat kernel is the fundamental solution to the heat equation with the Laplace–Beltrami operator Δ_g, which equivalently can be described as the kernel for the heat semigroup

$$P_t f(x) = e^{-t\Delta_g} f(x) = \int_M f(y)\, p_t^g(x, y)\, d\mu_g(y).$$

We concentrate on the case when M is a compact Lie group. Namely, let K be a connected compact group equipped with a left-invariant Riemannian metric $g \in \mathfrak{L}(K)$. In this case, the heat kernel $p_t^g(x, y)$ is a symmetric function of (x, y) and is invariant under left multiplication, that is, $p_t^g(x, y) = p_t^g(e, x^{-1}y) = p_t^g(e, y^{-1}x)$. Abusing notation, we write $p_t^g(z) := p_t^g(e, z)$. In addition, the heat kernel satisfies the Chapman–Kolmogorov equations

$$p_{s+t}^g(x) = \int_K p_s^g\left(y^{-1}x\right) p_t^g(y)\, d\mu_g, \quad s, t > 0, \tag{8.1}$$

which implies (using symmetry and multiplication invariance) that

$$p_{s+t}^{g}(e)=\int_{K}p_{s}^{g}(y)\,p_{t}^{g}(y)\,d\mu_{g}. \tag{8.2}$$

As mentioned before, the volume doubling constant is quantitatively related to many analytic properties of the Laplace–Beltrami operator Δ_g. Given a Riemannian metric g on a compact manifold M, let

$$0=\lambda_{g,0}<\lambda_{g}=\lambda_{g,1}\leqslant\cdots\leqslant\lambda_{g,i}\leqslant\cdots \tag{8.3}$$

be the eigenvalues of Δ_g, repeated according to multiplicity. In the case when M is a compact Lie group, we use will use repeatedly the following connection between the heat kernel p_t^g and the eigenvalues

$$\mu_{g}(K)p_{t}^{g}(e)=\mu_{g}(K)p_{t}^{g}(x,x)=\sum_{i=o}^{\infty}e^{-t\lambda_{g,i}}. \tag{8.4}$$

We discuss some of the properties of Δ_g and the volume doubling constant here, including a spectral gap, Weyl eigenvalue counting function, parabolic Harnack inequalities, and heat kernel bounds.

Definition 8.1. *Let K be a connected real Lie group. We say that K is* ***uniformly doubling with constant at most*** *D if there is a constant D such that*

$$D_g\leqslant D$$

for all left-invariant metrics Riemannian metrics $g\in\mathfrak{L}(K)$.

Observe that by Lemma 6.1 we see that on SU(2)

$$\frac{V_g(r)}{V_g(s)}\leqslant D\left(\frac{r}{s}\right)^4 \text{ for any } 0<s\leqslant r.$$

This can be compared with a more general statement as follows. Suppose (X,d,μ) is a metric measure space, then one can ask if there are constants $D'>0$ and $\delta>0$ such that for any $0<s\leqslant r$ and $x\in X$

$$\frac{V(x,r)}{V(x,s)}\leqslant D'\left(\frac{r}{s}\right)^{\delta}, \tag{8.5}$$

where $V(x,r):=\mu(B(x,r))$. If the metric measure space (X,d,μ) is doubling with constant at most D, then by [17, Section 4.2] and [51, Lemma 5.2.4] we see that (8.5) holds with $D'=D$ and $\delta=\frac{\ln D}{\ln 2}$. Indeed, if $\lfloor\cdot\rfloor$ denotes the integer part of a real number (floor function), we see that

$$V(x,r)\leqslant V\left(x,2^{\lfloor\frac{\ln\left(\frac{r}{s}\right)}{\ln 2}\rfloor+1}s\right)\leqslant D^{1+\frac{\ln\left(\frac{r}{s}\right)}{\ln 2}}V(x,s)=D\left(\frac{r}{s}\right)^{\frac{\ln D}{\ln 2}}V(x,s).$$

Below we state several interesting properties which would follow from Conjecture 1.1. First and foremost, we note that it implies a uniform version of the Poincaré inequality for metric balls stated in Corollary 8.3. This is the key to a host of other consequences. In particular, by Theorem 1.2 these properties hold on SU(2). In some instances, Theorem 1.3 provides a particularly explicit form of these statements.

8.1. **The Poincaré inequality on compact Lie groups.** The following theorem is proved in [51, Section 5.6.1]. The first instance of this type of inequality appeared in [57]; a discrete version of this inequality is one of the key elements of B. Kleiner's proof of Gromov's theorem on groups of polynomial growth [30].

Theorem 8.2. *Let K be a compact Lie group equipped with a left-invariant Riemannian metric g. On any ball $B_g(x,r)$, we have the Poincaré inequality*

(8.6)
$$\int_{B_g(x,r)} |f - f_{x,r}|^2 d\mu_g \leqslant 2r^2 D_g \int_{B_g(x,2r)} |\nabla_g f|_g^2 d\mu_g \text{ for all } f \in \mathcal{C}^\infty(B_g(x,2r)),$$

where $f_{x,r} := \int_{B_g(x,r)} f d\mu_g$ denotes the mean of f over $B_g(x,r)$, and D_g is the volume doubling constant of (K,g).

Corollary 8.3. *If K is uniformly volume doubling with constant at most D, then the Poincaré inequality* (8.6) *holds with the same constant D for every $g \in \mathfrak{L}(K)$. In particular, by Theorem 1.2 this is true for $K = \mathrm{SU}(2)$.*

The proof given in [51] also establishes, by a straightforward modification, the following L^p Poincaré inequality:

Theorem 8.4. *In the same notation as Theorem 8.2, for any $1 \leqslant p < \infty$, we have*

(8.7)
$$\int_{B_g(x,r)} |f - f_{x,r}|^p d\mu_g \leqslant (2r)^p D_g \int_{B_g(x,2r)} |\nabla_g f|_g^p d\mu_g \text{ for all } f \in \mathcal{C}^\infty(B_g(x,2r)).$$

(In the special case $p = 2$, one can improve the constant by a factor of $\frac{1}{2}$ to recover (8.6).)

Note that the weak Poincaré inequality (8.6) and volume doubling imply that the strong Poincaré inequality holds, that is, (8.6) with the same ball $B_g(x,r)$ on both sides (and the same for the L^p Poincaré inequality (8.7)). This is shown by a covering argument; see [26] and [51, Section 5.3.2]. In particular, this implies that on a uniformly doubling group K the lowest eigenvalue $\lambda_{N,g,r}$ of the Laplacian Δ_g with Neumann boundary condition on the ball $B_g(x,r)$ satisfies $cr^{-2} \leqslant \lambda_{N,g,r} \leqslant Cr^{-2}$ uniformly over all $g \in \mathfrak{L}(K)$ and $r \in (0, \mathrm{diam}_g]$.

8.2. **Spectral gap.** Let λ_g be the lowest non-zero eigenvalue for the Laplace-Beltrami operator Δ_g. We show that when K is uniformly doubling, we obtain the uniform upper bound (1.2) for λ_g, matching the lower bound (1.1) obtained in [34], up to a constant depending on the doubling constant.

Theorem 8.5. *Assume that the compact connected Lie group K is uniformly doubling with constant at most D. For any metric $g \in \mathfrak{L}(K)$, the lowest non-zero eigenvalue λ_g of the Laplacian Δ_g satisfies*

$$\frac{\pi^2}{4\,\mathrm{diam}_g^2} \leqslant \lambda_g \leqslant \frac{16D^2}{\mathrm{diam}_g^2}.$$

Proof. As mentioned earlier, the lower bound was proved in [34]. (An improved lower bound was recently obtained in [28].) To obtain an upper bound, we note that

$$\lambda_g = \min\left\{ \frac{\int |\nabla_g f|^2 d\mu_g}{\|f\|_2^2} : f \neq 0,\ \int_K f d\mu_g = 0,\ f \in \mathrm{Lip}(K) \right\}, \tag{8.8}$$

We construct an appropriate test function to use in (8.8). Let y be a point which realizes the diameter of K under g, i.e., $d_g(e,y) = \operatorname{diam}_g$. Let $R = \operatorname{diam}_g/2$. For any $z \in K$, let $f_{z,r}(x) = (r - d_g(z,x))_+$ be the tent function over the ball $B_g(z,r)$; observe that this is a Lipschitz function with gradient $|\nabla_g f_{z,r}| \leqslant 1$ (almost everywhere). As a test function, take $f_R = f_{e,R} - f_{y,R}$. By group invariance, $\int_K f_R d\mu_g = 0$ and $\int_K |\nabla_g f_R|^2 d\mu_g \leqslant \mu_g(K) = V_g(2R)$. To estimate the L^2-norm of f_R from below, observe that $|f_R|$ is at least $R/2$ on two disjoint balls of radius $R/2$. Hence we have $\|f_R\|_2^2 \geqslant (R/2)^2 V_g(R/2)$. Plugging this in the variational formula (8.8) for λ_g yields

$$\lambda_g \leqslant \frac{4V_g(2R)}{R^2 V_g(R/2)} \leqslant \frac{16D^2}{\operatorname{diam}_g^2}.$$

□

In the special case when $K = \mathrm{SU}(2)$, we have from Proposition 7.1 that the diameter $\operatorname{diam}_g(\mathrm{SU}(2))$ is uniformly comparable to the parameter a_2 of g (as defined in Notation 2.9), and hence we have the following statement.

Corollary 8.6. *There are positive constants $0 < c \leqslant C < \infty$ such that for all $g \in \mathfrak{L}(\mathrm{SU}(2))$ with parameters $0 < a_1 \leqslant a_2 \leqslant a_3$ as in Notation 2.9, we have*

$$\frac{c}{a_2^2} \leqslant \lambda_g \leqslant \frac{C}{a_2^2}. \tag{8.9}$$

Remark 8.7. In a very recent preprint [31], E. A. Lauret has given an exact expression for the smallest eigenvalue λ_g of SU(2) in terms of the parameters of the metric, which in our notation reads as follows:

$$\lambda_g = \min\left\{ \frac{1}{4}\left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \frac{1}{a_3^2}\right), \frac{1}{a_2^2} + \frac{1}{a_3^2} \right\}. \tag{8.10}$$

Indeed, (8.10) is consistent with (8.9).

In earlier work, as part of a more general construction, H. Urakawa [55] computed λ_g for a particular one-parameter family of metrics $g(t)$ on SU(2), which in our notation is

$$g(t) = \begin{cases} g_{(t/\sqrt{2},\, 1/\sqrt{2t},\, 1/\sqrt{2t})}, & 0 < t \leqslant 1 \\ g_{(1/\sqrt{2t},\, 1/\sqrt{2t},\, t/\sqrt{2})}, & 1 \leqslant t < \infty. \end{cases}$$

See [55, Theorem 5]. This family has the property that the volume $\mu_{g(t)}(\mathrm{SU}(2))$ is the same for all t, while $\lambda_{g(t)} \sim t$.

Urakawa's example answered, in the negative, a previous question of M. Berger [10]: whether we have $\lambda_g \leqslant C(M)\mu_g(M)^{-2/n}$ on any n-dimensional compact connected manifold M, with a constant $C(M)$ depending on M but not on the metric g. It is interesting to compare this with Theorem 8.5, which implies that, when M is a uniformly doubling group K and the metrics are left-invariant, the quantity $\mu_g(K)^{1/n}$ in Berger's statement ought to be replaced with diam_g.

8.3. **Heat kernel estimates.** In the section we would like to comment on the heat kernel estimates (1.3) for uniformly doubling compact Lie groups. Given a complete Riemannian manifold that satisfies the volume doubling property and the Poincaré inequality (8.6), there are several ways to obtain heat kernel upper bounds. One of the most direct and efficient is based on the notion of a Faber–Krahn inequality as developed in [14, 22] or the equivalent notion of local Sobolev inequality (see [51, Section 5.2]).

Assuming that doubling and the Poincaré inequality hold, these methods provide the heat kernel upper bound in terms of the volume

$$p_t(x,y) \leqslant \frac{C_1(\varepsilon)}{\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}} \exp\left(-\frac{d(x,y)^2}{4(1+\varepsilon)t}\right) \tag{8.11}$$

with a constant $C_1(\varepsilon)$ [51, Equation(5.2.17)] and [50] that depends only on $\varepsilon \in (0,1)$ and the constants involved in the doubling property and the Poincaré inequality. Here $V(x,r)$ denotes the volume of the ball of radius $r>0$ around the point x.

In fact, these arguments provide the more precise bound of the type

$$p_t(x,y) \leqslant \frac{C_1(1+d(x,y)^2/4t)^{\kappa}}{\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}} \exp\left(-\frac{d(x,y)^2}{4t}\right)$$

for some $\kappa > 0$. The best value of κ that can be obtained from these arguments is $\kappa = \delta/2$, where δ is as in (8.5), e.g. [51, Section 5.2.3], [53, Corollary 4.2] and variations on the arguments in [17]. Further, one also obtains the time derivative estimates such as in [54, Corollary 2.7])

$$|\partial_t^k p_t(x,y)| \leqslant \frac{C_k(1+d(x,y)^2/4t)^{k+\delta/2}}{t^k\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}} \exp\left(-\frac{d(x,y)^2}{4t}\right).$$

In addition, [17] provides assorted estimates for the heat kernel in complex time and pointers to further references.

The proofs of these estimates simplify, and a greater varieties of arguments can be employed, when the volume of balls is independent of the center, which is the case for left-invariant metrics on Lie groups.

Theorem 8.8. *Let K be a compact Lie group. If K is uniformly volume doubling with constant at most D, then for each integer $k=0,1,\dots$ there exists a constant C_k depending only on D and k such that for any $g \in \mathfrak{L}(K)$, and for all $x,y \in K$ and $t>0$*

$$|\partial_t^k p_t^g(x,y)| \leqslant \frac{C_k(1+d_g(x,y)^2/4t)^{k+\delta/2}}{t^k V_g(\sqrt{t})} \exp\left(-\frac{d_g(x,y)^2}{4t}\right),$$

where δ is as in (8.5).

Regarding a lower bound, the only directly applicable results are proved by a simple chaining argument using the parabolic Harnack inequality discussed in Section 8.4. Assuming that doubling and the Poincaré inequality hold, this line of reasoning provides the following heat kernel lower bound

$$p_t^g(x,y) \geqslant \frac{c_2}{V(x,\sqrt{t})} \exp\left(-C_2\frac{d(x,y)^2}{t}\right),$$

where $0 < c_2, C_2$ depends only on the constants involved in the doubling property and the Poincaré inequality. See, for instance, [51, Section 5.4.6] and [53, Corollary 4.10].

Theorem 8.9. *Let K be a compact Lie group. If K is uniformly volume doubling with constant at most D then there exist positive constants c and A depending only* *on D such that, for any $g \in \mathfrak{L}(K)$, for all $x, y \in K$ and $t > 0$*

$$p_t^g(x,y) \geqslant \frac{c}{V_g(\sqrt{t})} \exp\left(-A\frac{d_g(x,y)^2}{t}\right).$$

For SU(2), Lemma 6.1 shows that we can take $\delta = 4$ (uniformly over $\mathcal{L}(\mathrm{SU}(2))$) in Theorem 8.8, and this gives the following result.

Theorem 8.10. *There exist constants $0 < c, A$ and for each $k = 0, 1, \dots,$ a constant C_k, such that, for all $g \in \mathcal{L}(\mathrm{SU}(2))$ and all $x, y \in \mathrm{SU}(2)$, $t > 0$, we have*

$$|\partial_t^k p_t^g(x,y)| \leqslant \frac{C_k(1 + d_g(x,y)^2/4t)^{k+2}}{t^k V_g(\sqrt{t})} \exp\left(-\frac{d_g(x,y)^2}{4t}\right)$$

and

$$p_t^g(x,y) \geqslant \frac{c}{V_g(\sqrt{t})} \exp\left(-A\frac{d_g(x,y)^2}{t}\right).$$

Remark 8.11. The results in [56] imply that for each metric $g \in \mathfrak{L}(K)$ (in particular for SU(2)) and $\epsilon \in (0,1)$ there is a constant $c_{\varepsilon(g)} > 0$ such that, for all x, y, t

$$p_t^g(x,y) \geqslant \frac{c_{\varepsilon(g)}}{V_g(\sqrt{t})} \exp\left(-\frac{d_g(x,y)^2}{4(1-\varepsilon)t}\right).$$

However, it is not clear that the arguments in [56] are sufficient to yield a constant c_ε that is uniform in g, even if one assumes that the group K is uniformly doubling. This remains an open question, although we conjecture that this inequality holds uniformly.

Remark 8.12. Detailed asymptotics originally developed by S. Molchanov in [37] show that for the heat kernel on the n-sphere equipped with its canonical round metric and with x and y being antipodal points (e.g., the south and north poles)

$$p_t(x,y) \sim c_n t^{-n/2} \left(\frac{d(x,y)^2}{t}\right)^{(n-1)/2} \exp(-d(x,y)^2/4t)$$

as t tends to 0. This shows that one cannot dispense entirely with the factor $(1 + d_g(x,y)^2/t)^\kappa$ in heat kernel upper bounds, even on SU(2). For more on this, see [40].

8.4. **Harnack inequality.** Let the parabolic Harnack constant $H(M,g)$ be the infimum of all real H such that for any $x \in M$, $r > 0$ and any positive solution u of the heat equation on (M,g) in $(s, s+4r^2) \times B(x,2r)$, it holds that

$$\sup_{Q_-}\{u\} \leqslant H \inf_{Q_+}\{u\}, \tag{8.12}$$

where $Q_- = (s+r^2, s+2r^2) \times B(x,r)$ and $Q_+ = (s+3r^2, s+4r^2) \times B(x,r)$. In particular, for any connected compact real Lie group K equipped with $g \in \mathfrak{L}(K)$, we denote by $H(K,g)$ the best constant in the parabolic Harnack inequality (8.12). Then one can ask if the parabolic Harnack inequality is satisfied uniformly over all $g \in \mathfrak{L}(K)$.

Proposition 8.13 (See [21,46])**.** *Let K be a connected compact Lie group. Assume that K is uniformly doubling with constant at most D; then there is a $H(D)$ such that*

$$H(K,g) \leqslant H(D)$$

for all $g \in \mathfrak{L}(K)$.

In particular, Theorem 1.2 implies the following.

Corollary 8.14 (Uniform Harnack inequality for SU(2))**.** *The parabolic Harnack inequality is satisfied uniformly over all $g \in \mathfrak{L}(\mathrm{SU}(2))$.*

8.5. **Gradient inequalities.** In addition to the Harnack inequality (8.12) several related useful inequalities involve gradient estimates. For instance, one can consider the property that for any $x \in M$, $r > 0$ and any positive solution u of the heat equation on (M, g) in $(s, s+4r^2) \times B(x, 2r)$, it holds that

$$\sup_{Q_-}\{|\nabla_g u|_g\} \leqslant H_1 t^{-1/2} \inf_{Q_+}\{u\} \tag{8.13}$$

with Q_-, Q_+ defined as above. Or one may prefer the Li-Yau parabolic inequality for global positive solutions $u\,(t, x)$ of the heat equation on (M, g) in $(0, T) \times M$,

$$|\nabla_g \log u|_g^2 - \partial_t \log u \leqslant \frac{H_1'}{t}. \tag{8.14}$$

In this direction, we can only prove the following weaker result for the heat kernel $p_t^g\,(x)$.

Theorem 8.15. *Assume that K is uniformly doubling with constant at most D. Then there is a constant $C\,(D)$ such that*

$$|\nabla_g p_t^g\,(x)\,|_g \leqslant \frac{C\,(D)}{\sqrt{t} V_g\left(\sqrt{t}\right)} \left(1 + \frac{d_g^2(e,x)}{4t}\right)^{3\delta+1} \exp\left(-\frac{d_g^2(e,x)}{4t}\right), \tag{8.15}$$

$$\|\nabla_g p_t^g\|_{L^1} = \int_K |\nabla_g p_t^g\,(x)\,|_g d\mu_g(x) \leqslant C\,(D)\, t^{-1/2} \tag{8.16}$$

where $\delta = \delta\,(D)$ is as in (8.5).

Proof. Spectral theory easily gives

$$\|\Delta_g P_t^g\|_{L^2 \to L^2} = \|\partial_t P_t^g\|_{L^2 \to L^2} = \sup_{\lambda > 0}\left\{\lambda e^{-t\lambda}\right\} \leqslant (et)^{-1} \leqslant t^{-1}, \tag{8.17}$$

where we used the operator norm on $L^2(K, \mu_g)$ for P_t^g. Now observe that by (8.1)

$$\nabla_g p_t^g(x) = \int_K \nabla_g p_{t/2}^g(y^{-1}x) p_{t/2}^g(y) d\mu_g(y).$$

Hence

$$\begin{aligned} |\nabla_g p_t^g(x)|_g^2 &\leqslant \left(\int_K |\nabla_g p_{t/2}^g(y^{-1}x)|_g p_{t/2}^g(y) d\mu_g(y)\right)^2 \\ &\leqslant \int_K |\nabla_g p_{t/2}^g(y)|_g^2 d\mu_g(y) \int_K |p_{t/2}^g(y)|^2 d\mu_g(y). \end{aligned}$$

By (8.2) we have $\|p_{t/2}^g\|_2^2 = \int_K |p_{t/2}^g(y)|^2 d\mu_g(y) = p_t^g(e)$ and

$$\begin{aligned} \int_K |\nabla_g p_{t/2}^g(y)|_g^2 d\mu_g(y) &= \int -\Delta_g p_{t/2}^g(y) p_{t/2}^g(y) d\mu_g(y) \\ &\leqslant \|\Delta_g P_{t/4}^g p_{t/4}^g\|_2 \|p_{t/2}^g\|_2 \leqslant \frac{4}{t}\|p_{t/4}^g\|_2 \|p_{t/2}^g\|_2 \end{aligned}$$

Recall that by (1.3) if K is uniformly doubling with constant at most D, there is a constant $C(D)$ such that

$$p_t^g(e) \leqslant \frac{C(D)}{V_g(\sqrt{t})}. \tag{8.18}$$

Moreover, by (8.5), for any $0 < a < 1$

$$p_{at}^g(e) \leqslant \frac{C(D)}{Da^{\delta/2}V_g(\sqrt{t})}$$

This yields

$$|\nabla_g p_t^g(x)|_g \leqslant \frac{2}{\sqrt{t}}\left(p_t^g(e)\right)^{3/4}\left(p_{t/2}^g(e)\right)^{1/4} \leqslant \frac{C_1(D)}{t^{1/2}V_g(\sqrt{t})}.$$

From [49] or, more directly, [17, Theorem 4.11] (see also [3]) Equation 8.15 follows.

Inequality (8.16) follows by integration. See, e.g., [51, Lemma 5.2.13]. □

Remark 8.16. We do not know if it is possible to prove statements (8.13) and (8.14) with a uniform constant (H_1 or H_1') over all $g \in \mathfrak{L}(K)$ in the case when $M = K$ is a compact Lie group, solely from the validity of Conjecture 1.1. In particular, we do not know if these statements hold uniformly for all left-invariant metrics on SU(2). There seems to be no reasons why they should not hold but the known techniques to attack these problems usually involve curvature.

In this direction we note that the heat kernel lower bound in Theorem 8.9 and (8.15) imply that there exist $C > 0$ and $b > 1$ such that for all x, t

$$|\nabla_g p_t(x)|_g \leqslant Ct^{-1/2}p_{bt}(x). \tag{8.19}$$

This is (8.13) for the heat kernel $p_t^g(x)$. Note that given (1.3) Equation (8.19) is equivalent to

$$|\nabla_g p_t(x)| \leqslant \frac{C}{\sqrt{t}V_g(\sqrt{t})}\exp(-b'|x|_g^2/t).$$

All the constants depend only on D as follows from the proofs in [3].

Finally, Theorem 8.15 by [3] gives the following corollary regarding the Riesz transforms.

Corollary 8.17 (Uniform boundness of Riesz transforms)**.** *Assume K is uniformly doubling with constant at most D. Then for all $1 < p < \infty$ there are $c_p(D), C_p(D)$ such that*

$$c_p(D)\,\|\Delta_g^{-1/2}f\|_p \leqslant \||\nabla_g f|_g\|_p \leqslant C_p(D)\,\|\Delta_g^{-1/2}f\|_p.$$

8.6. **Weyl counting function.** For a compact Riemannian manifold (M, g), consider the Weyl spectral counting function

$$\mathfrak{W}_{M,g}(s) := \#\{i : \lambda_{g,i} < s\},$$

where $0 = \lambda_{g,0} < \lambda_g \leqslant \cdots \leqslant \lambda_{g,i} \leqslant \cdots$ are the eigenvalues of Δ_g as defined in (8.3). The asymptotic behavior of this function is described classically by Weyl's law (see [15, p. 155]) as follows.

$$\mathfrak{W}_{M,g}(s) \sim \frac{\omega_n}{(2\pi)^n}\mu_g(M)s^{n/2}, \tag{8.20}$$

where ω_n is the volume of the Euclidean n-ball. However, even when $M = K$ is a compact connected Lie group, these asymptotics do not hold uniformly over all left-invariant metrics g – not even when $K = \mathbb{T}^n$ is a torus.

When (M, g) is a compact homogeneous space, C. Judge and R. Lyons in [28] have recently obtained the following uniform upper bound.

$$\mathfrak{W}_{M,g}(s) \leqslant C \frac{\mu_g(M)}{V_g(s^{-1/2})}, \tag{8.21}$$

where C is a universal constant. If $M = K$ is a compact connected Lie group which is uniformly doubling, we obtain a matching lower bound, uniformly over all left-invariant metrics.

Proposition 8.18. *Let K be a connected compact real Lie group which is uniformly doubling with constant at most D. Then there is a constant $c(D) > 0$, depending only on D, such that for all $g \in \mathfrak{L}(K)$ we have*

$$\mathfrak{W}_{K,g}(s) \geqslant c(D) \frac{\mu_g(K)}{V_g(s^{-1/2})}. \tag{8.22}$$

For a proof, see [35, Théorème 7.1]; an explicit statement is also given in [48, Theorem 4.2]. The proof is based on the min-max characterization of eigenvalues and a covering argument. A matching upper bound is also proved in [35], but with a constant depending on D. Such a bound can be obtained another way using the trace of the heat kernel, via (8.4) and the heat kernel estimates of Section 8.3. In [20, Theorem 2] a similar statement is proved for individual sub-elliptic operators in $\mathbb{R}^n$, but without explicit control of the constants involved in terms of doubling.

The bound (8.22) is informative for $t \geqslant c\,\mathrm{diam}_g^{-2}$ as discussed in Section 8.2. Indeed, the spectral gap estimate in Theorem 8.5 implies that the step function $\mathfrak{W}_{K,g}(t)$ equals 1 on $[0, c\,\mathrm{diam}_g^{-2})$ for some $c = c(D)$, uniformly over all left-invariant metrics in $\mathfrak{L}(K)$.

When $K = \mathrm{SU}(2)$, Theorem 1.3 and Proposition 7.1 yield detailed explicit estimates for $W_{\mathrm{SU}(2),g}(s)$ as follows.

Corollary 8.19 (Weyl counting function for SU(2))**.** *There are constants $0 < C_0(D) \leqslant C_\infty < \infty$, with $C_0(D)$ depending only on D and C_∞ universal, such that for all $g =\in \mathcal{L}(\mathrm{SU}(2))$ we have*

$$C_0(D) f_{a_1,a_2,a_3}(t) \leqslant \mathfrak{W}_{\mathrm{SU}(2),g}(t) \leqslant C_\infty f_{a_1,a_2,a_3}(t), \tag{8.23}$$

where

$$f_{a_1,a_2,a_3}(t) = \begin{cases} 1 & \text{if } 0 < t < 1/a_2^2, \\ a_2^2 t & \text{if } 1/a_2^2 \leqslant t < 1/a_1^2 \\ a_1^2 a_2^2 t^2 & \text{if } 1/a_1^2 \leqslant t < a_3^2/a_1^2 a_2^2 \\ a_1 a_2 a_3 t^{3/2} & \text{if } a_3^2/a_1^2 a_2^2 \leqslant t < \infty. \end{cases}$$

Here a_1, a_2, a_3 are the parameters of g as in Notation 2.9.

8.7. **Heat kernel estimates: ergodicity.** Let $\mathbf{V}_g$ be the total Riemannian volume of the given group K under a Riemannian metric $g \in \mathfrak{L}(K)$, that is, $\mathbf{V}_g = \mu_g(K)$. It is well-known that the heat semigroup associated to any given $g \in \mathfrak{L}(K)$ is ergodic and that $p_t^g(x) \longrightarrow \mathbf{V}_g^{-1}$ as t tends to infinity. As before let λ_g by the lowest non-zero eigenvalue of the Laplacian Δ_g on K. We would like to describe this convergence to equilibrium in terms of the eigenvalue λ_g in the case when K

is a uniformly doubling compact Lie group. For relevant results we refer to [47,49]. In what follows we set $\|f\|_1 = \|f\|_{L^1(K,\mu_g)}$ and $\|f\|_2 = \|f\|_{L^2(K,\mu_g)}$.

Theorem 8.20. *Let K be a compact Lie group which is uniformly doubling with constant at most D. For any $\epsilon > 0$ there is a constant $C_\varepsilon(K) \in (0,\infty)$ such that for any metric $g \in \mathfrak{L}(K)$ we have*

$$\mathbf{V}_g \|p_t^g - \mathbf{V}_g^{-1}\|_1 \geqslant e^{-t\lambda_g} \quad \text{for all } t > 0,$$

and

$$\mathbf{V}_g \|p_t^g - \mathbf{V}_g^{-1}\|_2^2 \leqslant C_\varepsilon(K) e^{-2s\lambda_g} \text{ for } t \geqslant \varepsilon \operatorname{diam}_g^2 + s.$$

Moreover, there are constants $c_i \in (0,\infty)$, $1 \leqslant i \leqslant 4$, such that for any metric $g \in \mathfrak{L}(K)$ we have

$$\frac{c_1}{V_g(\sqrt{t})} e^{-2\lambda_g t} \leqslant \|p_t^g - \mathbf{V}_g^{-1}\|_2^2 \leqslant \frac{c_2}{V_g(\sqrt{t})} e^{-2\lambda_g t} \quad \text{for all } t > 0.$$

Remark 8.21. As we described in Section 8.2, under the hypothesis of this Theorem, λ_g is of order $\operatorname{diam}_g^{-2}$, uniformly over $\mathfrak{L}(K)$. Note also that, by definition, $\mathbf{V}_g = V_g(\operatorname{diam}_g/2)$. Further, for any function $f \in L^2(K,\mu_g)$, $\|f\|_1^2 \leqslant \mathbf{V}_g \|f\|_2^2$.

Proof. Let φ be an eigenfunction of Δ_g associated with the lowest non-zero eigenvalue $\lambda_{1,g}$ and normalized by $\max_x |\varphi(x)| = \varphi(e) = 1$ (such a normalization is always possible by translation in K and multiplication by a constant). Then the lower L^1-bound follows from

$$\|p_t^g - \mathbf{V}_g^{-1}\|_1 \geqslant \int \left(p_t^g(x) - \mathbf{V}_g^{-1}\right) \varphi(x) d\mu_g(x) = e^{-\lambda_g t},$$

where we used the fact that $\varphi \perp 1$ and

$$\int_K p_t^g(x)\varphi(x) d\mu_g(x) = (P_t\varphi)(e) = e^{-\lambda_g t}\varphi(e).$$

For the two-sided L^2-estimate, first observe that for any constant $C >$ by (8.1) we have

$$\int_K |p_t^g(x) - C|^2 d\mu_g(x) = p_{2t}^g(e) - 2C + C^2 \mathbf{V}_g,$$

and so by (8.4)

$$\mathbf{V}_g \|p_t^g - \mathbf{V}_g^{-1}\|_2^2 = \mathbf{V}_g p_{2t}^g(e) - 1 = \sum_{i=1}^{\infty} e^{-2t\lambda_{g,i}},$$

where $\lambda_{g,i}$ are eigenvalues of Δ_g as defined in (8.3). For the lower bound, noting that $e^{-2t\lambda_g} \leqslant 1$ we have

$$\begin{aligned}
\mathbf{V}_g p_{2t}^g(e) - 1 = e^{-2t\lambda_g} + \sum_{i=2}^{\infty} e^{-2t\lambda_{g,i}} &\geqslant \frac{1}{2} e^{-2t\lambda_g}\left(1 + e^{-2t\lambda_g}\right) + \sum_{i=2}^{\infty} e^{-2t\lambda_{g,i}} \\
&\geqslant \frac{1}{2} e^{-2t\lambda_g}\left(1 + e^{-2t\lambda_g}\right) + \frac{1}{2} e^{-2t\lambda_g} \sum_{i=2}^{\infty} e^{-2t\lambda_{g,i}} \\
&= \frac{1}{2} e^{-2t\lambda_g} \mathbf{V}_g p_{2t}^g(e).
\end{aligned}$$

By Theorem 8.9, this gives the desired lower bound.

For the upper bound, write

$$\begin{aligned}\mathbf{V}_g p_{2t}^g(e) - 1 = \sum_{i=1}^{\infty} e^{-2t\lambda_{g,i}} &= \sum_{\lambda_{g,i}\leq 2\lambda_g} e^{-2t\lambda_{g,i}} + \sum_{\lambda_{g,i}>2\lambda_g} e^{-2t\lambda_{g,i}} \\ &\leqslant e^{-2t\lambda_g}\left(\mathfrak{W}_g(2\lambda_g) + \sum_{\lambda_{g,i}>2\lambda_g} e^{-t\lambda_{g,i}}\right) \\ &\leqslant e^{-2t\lambda_g}\left(\mathfrak{W}_g(2\lambda_g) + \mathbf{V}_g p_t^g(e)\right).\end{aligned}$$

The upper bound in (8.22) and the upper bound $\lambda_g \leqslant C_1(D)\,\mathrm{diam}_g^{-2}$ in Theorem 8.5 yield $\mathfrak{W}_g(2\lambda_g) \leqslant C_2(D)$. This, together with (8.11), gives

$$\mathbf{V}_g p_{2t}^g(e) - 1 \leqslant \frac{C_3(D)\mathbf{V}_g}{V_g(\sqrt{t})} e^{-2t\lambda_g}$$

as desired. □

8.8. **Infinite products.** Let $\{K_i\}_{i=1}^{\infty}$ be a sequence of compact connected Lie groups, each equipped with the Haar probability measure μ_i. Consider the compact group

$$K^{\infty} := \prod_{i=1}^{\infty} K_i.$$

Note that this includes the case when $K_i = K$ for all i. Suppose each K_i is equipped with a Riemannian metric $g_i \in \mathfrak{L}(K_i)$; from now on by K_i we denote (K_i, g_i), and by $\mathbf{g}$ we denote the sequence of metrics $\{g_i\}_{i=1}^{\infty}$. Note that the Riemannian volume measure μ_{g_i} is just a rescaling of μ_i, so $D(K_i, d_{g_i}, \mu_{g_i}) = D(K_i, d_{g_i}, \mu_i)$. We endow K^{∞} with its Haar probability measure μ which is the product of the Haar measures μ_i. For background on this setting see [9, 24].

The space of *cylinder functions*, i.e. smooth functions depending on only finitely many coordinates, is dense in $L^2(K^{\infty}, \mu)$. For a cylinder function f, set

$$\mathcal{E}_{\mathbf{g}}(f,f) := \int_{K^{\infty}} \sum_{i=1} g_i(\nabla_{g_i} f, \nabla_{g_i} f) d\mu.$$

The quadratic form $\mathcal{E}_{\mathbf{g}}$ is closable and its closure is a strictly local regular Dirichlet form associated to a self-adjoint Markov semigroup $H_t^{\mathbf{g}}$. It is a convolution semigroup on K^{∞} associated with a convolution semigroup of symmetric measures $\nu_t^{\mathbf{g}}$, i.e.

$$H_t^{\mathbf{g}} f(x) = \int f(xy) d\nu_t^{\mathbf{g}}(y), \quad t > 0.$$

For each metric g_i, we let $\gamma_i := \lambda_{1,i}$ be the second smallest eigenvalue of the operator $-\Delta_i$, where Δ_i is the Laplace–Beltrami operator on K_i.

Denote by t_A to be the infimum of all times $t > 0$ at which the measure $\nu_t^{\mathbf{g}}$ is absolutely continuous with respect to the Haar measure μ. Note that if this property holds at time t, it also holds at all later times.

The following are special cases of more general open problems considered in [49, Section 2]. Is is true that $\nu_t^{\mathbf{g}}$ is singular with respect to the Haar measure μ for all time $t < t_A$? Is it true that for all $t > t_A$, the density $f_t^{\mathbf{g}}$ of the measure $\nu_t^{\mathbf{g}}$ with respect to μ is in $L^2(K^{\infty}, \mu)$? Is it true that if $t_A = 0$, then $f_t^{\mathbf{g}}$ has a continuous representative?

Proposition 8.22. *Assume there exists a constant D such that for any $i = 1, 2, \ldots.$ and any $g_i \in \mathfrak{L}(K_i)$ we have $D(K_i, d_{g_i}, \mu_i) \leqslant D$. Denote*

$$t_* := \inf\{t : \sum_1^\infty e^{-2t\gamma_i} < \infty\}.$$

Then the following properties hold:

- *The measure $\nu_t^{\mathbf{g}}$ is absolutely continuous with respect to the Haar measure μ for $t > t_*$ whereas $\nu_t^{\mathbf{g}}$ has no absolutely continuous part with respect to μ for $0 < t < t_*$;*
- *Furthermore, for all $t > t_*$, the density $\frac{d\nu_t^{\mathbf{g}}}{d\mu}$ is in $L^2(K^\infty, \mu)$. It is unbounded for $t_* < t < 2t_*$, and it is bounded and continuous for $t > 2t_*$;*
- *In particular, if $t_* = 0$, the semigroup $H_t^{\mathbf{g}}$ admits a continuous convolution kernel for all times $t > 0$.*

Proof. This follows from (1.2), (1.3) Proposition 8.13 and [8, Theorems 3.1, 4.1, 4.2]. □

In particular, by Theorem 1.2 and the similar result for tori, these properties hold when $K_i \in \{\mathrm{SU}(2), \mathbb{T}, \mathbb{T}^2, \ldots, \mathbb{T}^n\}$.

9. Connections to sub-Riemannian geometry

We have focused this paper on Riemannian geometry, but in fact our results carry over to sub-Riemannian geometry as well. In this section, we make those connections explicit. We briefly review the relevant definitions as they apply to Lie groups; we refer to [38] for a discussion of sub-Riemannian geometry in a general context.

On a connected Lie group K, a left-invariant sub-Riemannian geometry is determined by a choice of a linear subspace $H \subset \mathfrak{k}$ of the Lie algebra, and a Euclidean inner product g on H. Let $\mathfrak{L}_{\mathrm{sub}}(K)$ denote the set of all such pairs (H, g); by abuse of notation, we will refer to such a pair simply by g. It is also common to view g as an extended quadratic form on $\mathfrak{k}$, where $g(v, w) = \infty$ unless $v, w \in H$.

By left translation, H extends to a left-invariant distribution $\mathcal{H} \subset TK$ with $\mathcal{H}_e = H$, and g extends to a left-invariant sub-Riemannian metric, still called g, on $\mathcal{H}$ (or an extended quadratic form on TK).

The geometry (H, g) satisfies the Hörmander bracket generating condition iff H generates the Lie algebra $\mathfrak{k}$; let $\mathfrak{L}^*_{\mathrm{sub}}(K) \subset \mathfrak{L}_{\mathrm{sub}}(K)$ denote the set of such geometries. Note that for $K = \mathrm{SU}(2)$, this happens iff $\dim H \geqslant 2$, since the Lie algebra $\mathfrak{su}(2)$ is generated by any two linearly independent elements. When $H = \mathfrak{k}$ we recover the left-invariant Riemannian geometries $\mathfrak{L}(K)$.

To any $g \in \mathfrak{L}_{\mathrm{sub}}(K)$ is associated a length structure giving finite length to continuous piecewise smooth curves that stay tangent to $\mathcal{H}$ (these are called horizontal curves). The left-invariant Carnot–Carathéorody (pseudo)-distance $d_g(x, y)$ is defined as the infimum of the lengths of horizontal curves joining x to y in K, where $d_g(x, y) = \infty$ if no such curve exists. By the Chow–Rashevskii theorem [38, Theorems 2.1.2 and 2.1.3], if $g \in \mathfrak{L}^*_{\mathrm{sub}}(K)$ then $d_g(x, y)$ is finite for any pair $x, y \in K$, so that d_g is a genuine distance, and moreover the topology induced by d_g coincides with the manifold topology of K.

Each sub-Riemannian geometry $(H, g) \in \mathfrak{L}_{\text{sub}}(K)$ is also associated with a canonical left-invariant sub-Laplacian Δ_g, which may be defined by

$$\Delta_g = -\sum_{i=1}^{k} \widetilde{u_i}^2 \tag{9.1}$$

where $k = \dim H$, $\{u_i : 1 \leqslant i \leqslant k\}$ is a g-orthonormal basis for H, and $\{\widetilde{u_i}\}$ are the corresponding left-invariant vector fields. This definition is independent of the basis chosen. The operator Δ_g is hypoelliptic iff $g \in \mathfrak{L}^*_{\text{sub}}(K)$, and when g is Riemannian $(H = \mathfrak{k})$ we recover the Laplace–Beltrami operator.

Likewise, for $f \in C^\infty(K)$, we have the left-invariant sub-gradient $\nabla_g f$ which is a smooth section of $\mathcal{H}$ defined by

$$\nabla_g f = \sum_{i=1}^{k} (\widetilde{u_i} f)\widetilde{u_i}. \tag{9.2}$$

In particular, we have

$$|\nabla_g f|^2 := g(\nabla_g f, \nabla_g f) = \sum_{i=1}^{k} |\widetilde{u_i} f|^2. \tag{9.3}$$

When g is Riemannian this is the usual Riemannian gradient.

In the case $K = \text{SU}(2)$, a sub-Riemannian metric $g \in \mathfrak{L}_{\text{sub}}(\text{SU}(2))$ can be diagonalized by a standard Milnor basis, in the same way as in Lemma 2.8 for Riemannian metrics.

Proposition 9.1. *Let* $(H, g) \in \mathfrak{L}_{sub}(\text{SU}(2))$, *with* $\dim H = k$. *There exists a standard Milnor basis* $\{e_1, e_2, e_3\}$ *and an ordered triplet of extended non-negative reals* $0 < a_1 \leqslant a_2 \leqslant a_3 \leqslant \infty$ *such that* $H = \text{span}\{e_i : 1 \leqslant i \leqslant k\}$ *and* $g(e_i, e_j) = a_i^2 \delta_{ij}$ *for* $1 \leqslant i, j \leqslant k$. *We take* $a_i = \infty$ *for* $i > k$.

Proof. The case $k = 0$ is trivial (any standard Milnor basis will do), and $k = 3$ is Lemma 2.8.

For $k = 2$, let $\{v_1, v_2\}$ be a g-orthonormal basis for H, and set $v_3 = [v_1, v_2]$. Observe that $v_3 \notin H$; indeed, under the invariant inner product given by the negative Killing form, v_3 is orthogonal to both v_1, v_2. Let g' be the Euclidean inner product on $\mathfrak{su}(2)$ which makes v_1, v_2, v_3 orthonormal, and define $\times, L$ with respect to g' as in the proof of Lemma 2.8, choosing $\times$ so that $v_1 \times v_2 = v_3$. Note that v_3 is an eigenvector of L (with eigenvalue 1), since $L(v_3) = L(v_1 \times v_2) = [v_1, v_2] = v_3$. So if $\{w_1, w_2, w_3\}$ is a g'-orthonormal basis of eigenvectors for L, where we let $w_3 = v_3$, then necessarily $w_1, w_2 \in H$ and they are g-orthonormal. Proceeding as in Lemma 2.8, there is a standard Milnor basis $\{e_1, e_2, e_3\}$ where e_i is a scalar multiple of w_i, and in particular $e_1, e_2 \in H$ and they are g-orthogonal.

For $k = 1$, let v_1 span H, choose $v_2 \notin H$ arbitrarily, and proceed as in the previous case. We obtain a standard Milnor basis $\{e_1, e_2, e_3\}$ where $\text{span}\{e_1, e_2\} = \text{span}\{v_1, v_2\}$. In particular there is some $\theta \in \mathbb{R}$ such that v_1 is a scalar multiple of $\cos(\theta)e_1 + \sin(\theta)e_2$, and then

$$\{\cos(\theta)e_1 + \sin(\theta)e_2, \sin(\theta)e_1 - \cos(\theta)e_2, e_3\}$$

is the desired standard Milnor basis, as in Example 2.3. $\square$

Thus, as in Corollary 2.10, the left-invariant sub-Riemannian geometries $g \in \mathfrak{L}_{\mathrm{sub}}(\mathrm{SU}(2))$ are given, up to isometry, by the geometries $g_{(a_1,a_2,a_3)}$, where the a_i are allowed to take the value ∞. In fact, these geometries arise as the limits of the Riemannian geometries $g_{(a_1,a_2,a_3)}$ where the a_i are finite. The "standard" sub-Riemannian metric commonly encountered in the literature (e.g. [6, 7]) corresponds to $g_{(1,1,\infty)}$, but we stress that this is just one element of the infinite family $\mathfrak{L}^*_{\mathrm{sub}}(\mathrm{SU}(2))$.

Lemma 9.2. *Given $g = g_{(a_1,a_2,a_3)} \in \mathfrak{L}_{sub}(\mathrm{SU}(2))$, where $0 < a_1 \leqslant a_2 \leqslant a_3 \leqslant \infty$, and $\epsilon > 0$, let $a_{\epsilon,i} = \min(a_i, \epsilon^{-1})$, and set $g_\epsilon = g_{(a_{\epsilon,1},a_{\epsilon,2},a_{\epsilon,3})} \in \mathfrak{L}(\mathrm{SU}(2))$. Then for any $x, y \in \mathrm{SU}(2)$ we have $d_g(x, y) = \lim_{\epsilon\to 0} d_{g_\epsilon}(x, y)$.*

Proof. By left invariance, it suffices to consider $d_g(e, x)$ where $x \neq e$.

If $d_g(e, x) < \infty$, the result follows by the argument in [27, Proposition 3.1] for the distance α_L. In particular, this covers all cases when $a_2 < \infty$ (so that $\dim H$ is 2 or 3).

In the trivial case of $g_{(\infty,\infty,\infty)}$, where $\dim H = 0$, we have $d_g(e, x) = \infty$ for all $x \neq e$, and we simply note that $d_{g_\epsilon}(e, x) = d_{(\epsilon^{-1},\epsilon^{-1},\epsilon^{-1})}(e, x) = \epsilon^{-1} d_{(1,1,1)}(e, x) \to \infty$ as $\epsilon \to 0$.

The remaining case is where $g = g_{(a_1,\infty,\infty)}$, with $a_1 < \infty$ (so that $\dim H = 1$) and $d_g(e, x) = \infty$. Let $S = \{\exp(s\hat{e}_1) : s \in \mathbb{R}\}$ be the circle subgroup defined in the proof of Proposition 7.1. If $x \in S$, so that $x = \exp(T\hat{e}_1)$ for some T, then $\gamma(t) = \exp(t\hat{e}_1)$, $0 \leqslant t \leqslant T$ is a finite-length horizontal curve joining e to x, and thus $d_g(e, x) < \infty$. So suppose $x \notin S$. As shown in the proof of Proposition 7.1, we have $d_{(0,1,1)}(e, x) > 0$. Hence for all $\epsilon \leqslant \min(a_1^{-1}, 1)$ we have

$$0 < d_{(0,1,1)}(e, x) \leqslant d_{(\epsilon a_1,1,1)}(e, x) = \epsilon d_{g_\epsilon}(e, x)$$

which implies that $d_{g_\epsilon}(e, x) \to \infty$. □

Corollary 9.3. *The family of metric measure spaces*

$$\{(\mathrm{SU}(2), d_g, \mu_0) : g \in \mathfrak{L}^*_{sub}(\mathrm{SU}(2))\}$$

is uniformly volume doubling with the same constant D as in Theorem 1.2.

Proof. By the previous lemma, the closed ball $\bar{B}_g(r)$ equals the decreasing intersection $\bigcap_n B_{g_{1/n}}(r)$. The sub-Riemannian spheres have measure zero [44, Proposition 4.3], so we have $\mu_0(B_g(r)) = \mu_0(\bar{B}_g(r)) = \lim_{\epsilon\to 0} \mu_0(B_{g_\epsilon}(r))$, and by Theorem 1.2 each g_ϵ is volume doubling with constant at most D, so the result follows. □

Corollary 9.4. *For all $g_{(a_1,a_2,a_3)} \in \mathfrak{L}^*_{sub}(\mathrm{SU}(2))$, where we allow $a_3 = \infty$, the volume $V_{(a_1,a_2,a_3)}(r)$ is comparable to $\overline{V}_{(a_1,a_2,a_3)}(r)$ as defined in (6.1), uniformly in a_1, a_2, a_3, r.*

Note that for $a_3 = \infty$, the "Euclidean" regime, where volume scales as r^3, becomes empty, and for very small r, the volume scales as r^4 instead. This matches the Heisenberg behavior and corresponds to the fact that such a sub-Riemannian geometry has Hausdorff dimension 4.

Remark 9.5. The preceding corollaries may also be proved directly, instead of by approximating sub-Riemannian geometries by Riemannian geometries. Indeed, the proofs in Sections 3–6 go through without change if $a_3 = \infty$. (Note that Section 3, the Euclidean regime, becomes vacuous in that case.)

The results in Section 8 concerning the spectral gap λ_g, the heat kernel p_t^g, the eigenvalues $\lambda_{g,i}$ and the Weyl counting function $\mathfrak{W}_g$ all extend uniformly to sub-Riemannian geometries $g \in \mathfrak{L}^*_{\mathrm{sub}}(\mathrm{SU}(2))$, with Δ_g, $|\nabla_g f|^2$ redefined as above. It is only necessary to adjust the statements to replace all instances of μ_g by μ_0, since sub-Riemannian geometries do not admit a Riemannian volume, and scale appropriately. In particular, in this context the heat kernel p_t^g should be viewed as an integral kernel with respect to μ_0. The proofs need not be carried out by passing to the limit in the Riemannian statement; instead, the results follow because they are general consequences of uniform doubling and the uniform Poincaré inequality, for which the proof cited in Section 8.1 goes through without change in the sub-Riemannian setting.

Remark 9.6. One may also study the degenerate sub-Riemannian geometries, though this is more complicated because their topologies are not well behaved. For instance, with $g = g_{(a_1,\infty,\infty)}$, the ∞-metric space $(\mathrm{SU}(2), d_g)$ has uncountably many connected components, which are the left cosets of the one-dimensional subgroup $S = \{\exp(se_1) : s \in \mathbb{R}\}$, all isometric to S^1 and at pairwise distance infinity from one another. In particular, every ball of this metric has Haar measure zero, so statements about volume growth are not sensible. However, if we fix a sufficiently large R, then for all small ϵ the ball $B_{g_\epsilon}(R)$ is comparable to $B_{g_\epsilon}(S, R)$; by arguments similar to Proposition 5.3, one may see that $\mu_0(B_{g_\epsilon}(R)) \approx \epsilon^2 R^2$. On the other hand, (6.1) gives

$$\mu_0(B_{g_\epsilon}(r)) \approx \begin{cases} a_1^{-1}\epsilon^2 r^3, & 0 \leqslant r \leqslant a_1 \\ \epsilon^2 r^2, & a_1 \leqslant r \leqslant \epsilon^{-1} \\ 1, & r \geqslant \epsilon^{-1}. \end{cases}$$

As $\epsilon \to 0$, the ball $B_{g_\epsilon}(R)$ collapses to S, and we have

$$\frac{\mu_0(B_{g_\epsilon}(r))}{\mu_0(B_{g_\epsilon}(R))} \approx \begin{cases} \frac{r}{a_1}, & r \leqslant a_1 \\ 1, & r \geqslant a_1. \end{cases} \tag{9.4}$$

If we consider the circle S^1 as a Lie group equipped with its own normalized Haar measure μ_{S^1} and the metric $g = g_{a_1}$ which is the a_1-scaling of the unique left-invariant Riemannian metric on S^1, we can observe that (9.4) is comparable to the volume $\mu_{S^1}(B_g(r))$ of a ball in S^1. In particular, we recover the (trivial) fact that left-invariant Riemannian geometries on S^1 are uniformly volume doubling. This is perhaps not so interesting in our present context, but the idea of considering degenerate sub-Riemannian geometries may yield more useful insights when replacing SU(2) with other compact connected Lie groups K.

Acknowledgement. The third author would like to thank Dominique Bakry for inspiring discussions over many years. The authors are grateful for helpful and motivating conversations with Iddo Ben-Ari, Bruce K. Driver, Nicolas Juillet, Emilio A. Lauret, and K.-T. Sturm.

References

1. Andrei Agrachev, Davide Barilari, and Luca Rizzi, *Sub-Riemannian curvature in contact geometry*, J. Geom. Anal. **27** (2017), no. 1, 366–408. MR 3606555
2. Andrei Agrachev and Paul W. Y. Lee, *Generalized Ricci curvature bounds for three dimensional contact subriemannian manifolds*, Math. Ann. **360** (2014), no. 1-2, 209–253. MR 3263162

3. Pascal Auscher, Thierry Coulhon, Xuan Thinh Duong, and Steve Hofmann, *Riesz transform on manifolds and heat kernel regularity*, Ann. Sci. École Norm. Sup. (4) **37** (2004), no. 6, 911–957. MR 2119242
4. Zoltán M. Balogh, Jeremy T. Tyson, and Eugenio Vecchi, *Intrinsic curvature of curves and surfaces and a Gauss–Bonnet theorem in the Heisenberg group*, Mathematische Zeitschrift (2016), 1–38.
5. F. Baudoin, E. Grong, K. Kuwada, and A. Thalmaier, *Sub-Laplacian comparison theorems on totally geodesic Riemannian foliations*, arXiv:1706.08489, 2017.
6. Fabrice Baudoin and Michel Bonnefont, *The subelliptic heat kernel on* SU(2)*: representations, asymptotics and gradient bounds*, Math. Z. **263** (2009), no. 3, 647–672. MR 2545862 (2011d:58060)
7. Fabrice Baudoin and Nicola Garofalo, *Curvature-dimension inequalities and Ricci lower bounds for sub-Riemannian manifolds with transverse symmetries*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 1, 151–219. MR 3584561
8. A. Bendikov and L. Saloff-Coste, *Elliptic diffusions on infinite products*, J. Reine Angew. Math. **493** (1997), 171–220. MR 1491812
9. ———, *Central Gaussian semigroups of measures with continuous density*, J. Funct. Anal. **186** (2001), no. 1, 206–268. MR 1863298
10. M. Berger, *Sur les premières valeurs propres des variétés riemanniennes*, Compositio Math. **26** (1973), 129–149. MR 0316913
11. Ugo Boscain and Francesco Rossi, *Invariant Carnot-Caratheodory metrics on* S^3, SO(3), SL(2)*, and lens spaces*, SIAM J. Control Optim. **47** (2008), no. 4, 1851–1878. MR 2421332 (2009f:53043)
12. Peter Buser, *A note on the isoperimetric constant*, Ann. Sci. École Norm. Sup. (4) **15** (1982), no. 2, 213–230. MR 683635
13. Luca Capogna and Giovanna Citti, *Regularity for subelliptic PDE through uniform estimates in multi-scale geometries*, Bull. Math. Sci. **6** (2016), no. 2, 173–230. MR 3510691
14. Gilles Carron, *Inégalités isopérimétriques de Faber-Krahn et conséquences*, Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992), Sémin. Congr., vol. 1, Soc. Math. France, Paris, 1996, pp. 205–232. MR 1427759
15. Isaac Chavel, *Eigenvalues in Riemannian geometry*, Pure and Applied Mathematics, vol. 115, Academic Press, Inc., Orlando, FL, 1984, Including a chapter by Burton Randol, With an appendix by Jozef Dodziuk. MR 768584
16. Bennett Chow and Dan Knopf, *The Ricci flow: an introduction*, Mathematical Surveys and Monographs, vol. 110, American Mathematical Society, Providence, RI, 2004. MR 2061425
17. Thierry Coulhon and Adam Sikora, *Gaussian heat kernel upper bounds via the Phragmén-Lindelöf theorem*, Proc. Lond. Math. Soc. (3) **96** (2008), no. 2, 507–544. MR 2396848
18. P. Diaconis and L. Saloff-Coste, *Moderate growth and random walk on finite groups*, Geom. Funct. Anal. **4** (1994), no. 1, 1–36. MR 1254308
19. J.-H. Eschenburg, *Comparison theorems in Riemannian geometry*, Università delli studi di Trento, Dipartimento di matematica, 1994.
20. C. Fefferman and D. H. Phong, *Subelliptic eigenvalue problems*, Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981), Wadsworth Math. Ser., Wadsworth, Belmont, CA, 1983, pp. 590–606. MR 730094
21. A. A. Grigorʹyan, *The heat equation on noncompact Riemannian manifolds*, Mat. Sb. **182** (1991), no. 1, 55–87. MR 1098839
22. Alexander Grigorʹyan, *Heat kernel upper bounds on a complete non-compact manifold*, Rev. Mat. Iberoamericana **10** (1994), no. 2, 395–452. MR 1286481
23. Yves Guivarc'h, *Croissance polynomiale et périodes des fonctions harmoniques*, Bull. Soc. Math. France **101** (1973), 333–379. MR 0369608
24. Herbert Heyer, *Probability measures on locally compact groups*, Springer-Verlag, Berlin-New York, 1977, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 94. MR 0501241
25. Robert K. Hladky, *Connections and curvature in sub-Riemannian geometry*, Houston J. Math. **38** (2012), no. 4, 1107–1134. MR 3019025
26. David Jerison, *The Poincaré inequality for vector fields satisfying Hörmander's condition*, Duke Math. J. **53** (1986), no. 2, 503–523. MR 850547

27. David Jerison and Antonio Sánchez-Calle, *Subelliptic, second order differential operators*, Complex analysis, III (College Park, Md., 1985–86), Lecture Notes in Math., vol. 1277, Springer, Berlin, 1987, pp. 46–77. MR 922334 (89b:35021)
28. Chris Judge and Russell Lyons, *Upper bounds for the spectral function on homogeneous spaces via volume growth*, arXiv:1704.01108, 2017.
29. Nicolas Juillet, *Geometric inequalities and generalized Ricci bounds in the Heisenberg group*, Int. Math. Res. Not. IMRN **13** (2009), 2347–2373. MR 2520783 (2011b:53088)
30. Bruce Kleiner, *A new proof of Gromov's theorem on groups of polynomial growth*, J. Amer. Math. Soc. **23** (2010), no. 3, 815–829. MR 2629989
31. Emilio A. Lauret, *The smallest Laplace eigenvalue of homogeneous 3-spheres*, arXiv:1801.04259, 2018.
32. Paul W. Y. Lee, *On measure contraction property without Ricci curvature lower bound*, Potential Anal. **44** (2016), no. 1, 27–41. MR 3455207
33. Paul W. Y. Lee, Chengbo Li, and Igor Zelenko, *Ricci curvature type lower bounds for sub-Riemannian structures on Sasakian manifolds*, Discrete Contin. Dyn. Syst. **36** (2016), no. 1, 303–321. MR 3369223
34. Peter Li, *Eigenvalue estimates on homogeneous manifolds*, Comment. Math. Helv. **55** (1980), no. 3, 347–363. MR 593051
35. P. Maheux and L. Saloff-Coste, *Analyse sur les boules d'un opérateur sous-elliptique*, Math. Ann. **303** (1995), no. 4, 713–740. MR 1359957
36. John Milnor, *Curvatures of left invariant metrics on Lie groups*, Advances in Math. **21** (1976), no. 3, 293–329. MR MR0425012 (54 #12970)
37. S. A. Molčanov, *Diffusion processes, and Riemannian geometry*, Uspehi Mat. Nauk **30** (1975), no. 1(181), 3–59. MR 0413289 (54 #1404)
38. Richard Montgomery, *A tour of subriemannian geometries, their geodesics and applications*, Mathematical Surveys and Monographs, vol. 91, American Mathematical Society, Providence, RI, 2002. MR 1867362 (2002m:53045)
39. Alexander Nagel, Elias M. Stein, and Stephen Wainger, *Balls and metrics defined by vector fields. I. Basic properties*, Acta Math. **155** (1985), no. 1-2, 103–147. MR 793239 (86k:46049)
40. Robert Neel, *The small-time asymptotics of the heat kernel at the cut locus*, Comm. Anal. Geom. **15** (2007), no. 4, 845–890. MR 2395259
41. Shin-ichi Ohta, *On the measure contraction property of metric measure spaces*, Comment. Math. Helv. **82** (2007), no. 4, 805–828. MR 2341840
42. Peter Petersen, *Riemannian geometry*, second ed., Graduate Texts in Mathematics, vol. 171, Springer, New York, 2006. MR 2243772
43. A. V. Podobryaev, *Diameter of the Berger Sphere*, Mat. Zametki **103** (2018), no. 5, 779–784, English preprint at arXiv:1710.02945. MR 3795126
44. L. Rifford and E. Trélat, *Morse-Sard type results in sub-Riemannian geometry*, Math. Ann. **332** (2005), no. 1, 145–159. MR 2139255
45. Ludovic Rifford, *Ricci curvatures in Carnot groups*, Math. Control Relat. Fields **3** (2013), no. 4, 467–487. MR 3110060
46. L. Saloff-Coste, *A note on Poincaré, Sobolev, and Harnack inequalities*, Internat. Math. Res. Notices **2** (1992), 27–38. MR 1150597 (93d:58158)
47. ———, *Convergence to equilibrium and logarithmic Sobolev constant on manifolds with Ricci curvature bounded below*, Colloq. Math. **67** (1994), no. 1, 109–121. MR 1292948
48. ———, *Parabolic Harnack inequality for divergence-form second-order differential operators*, Potential Anal. **4** (1995), no. 4, 429–467, Potential theory and degenerate partial differential operators (Parma). MR 1354894 (96m:35031)
49. ———, *Analysis on compact Lie groups of large dimension and on connected compact groups*, Colloq. Math. **118** (2010), no. 1, 183–199. MR 2600525
50. Laurent Saloff-Coste, *Uniformly elliptic operators on Riemannian manifolds*, J. Differential Geom. **36** (1992), no. 2, 417–450. MR MR1180389 (93m:58122)
51. ———, *Aspects of Sobolev-type inequalities*, London Mathematical Society Lecture Note Series, vol. 289, Cambridge University Press, Cambridge, 2002. MR 1872526 (2003c:46048)
52. Robert S. Strichartz, *The Campbell-Baker-Hausdorff-Dynkin formula and solutions of differential equations*, J. Funct. Anal. **72** (1987), no. 2, 320–345. MR 886816 (89b:22011)
53. K. T. Sturm, *Analysis on local Dirichlet spaces. III. The parabolic Harnack inequality*, J. Math. Pures Appl. (9) **75** (1996), no. 3, 273–297. MR 1387522

54. Karl-Theodor Sturm, *Analysis on local Dirichlet spaces. II. Upper Gaussian estimates for the fundamental solutions of parabolic equations*, Osaka J. Math. **32** (1995), no. 2, 275–312. MR 1355744
55. Hajime Urakawa, *On the least positive eigenvalue of the Laplacian for compact group manifolds*, J. Math. Soc. Japan **31** (1979), no. 1, 209–226. MR 519046
56. N. Th. Varopoulos, *Small time Gaussian estimates of heat diffusion kernels. I. The semigroup technique*, Bull. Sci. Math. **113** (1989), no. 3, 253–277. MR 1016211
57. Nicolas Th. Varopoulos, *Fonctions harmoniques sur les groupes de Lie*, C. R. Acad. Sci. Paris Sér. I Math. **304** (1987), no. 17, 519–521. MR 892879
58. Norbert J. Wielenberg, *Three-dimensional hyperbolic spaces*, Rocky Mountain J. Math. **11** (1981), no. 4, 659–671. MR 639450
59. Jay A. Wood, *Maximal abelian subgroups of spinor groups and error-correcting codes*, Algebraic topology (Evanston, IL, 1988), Contemp. Math., vol. 96, Amer. Math. Soc., Providence, RI, 1989, pp. 333–350. MR 1022693

† School of Mathematical Sciences, University of Northern Colorado, Greeley, CO 80639, U.S.A.

Email address: neldredge@unco.edu

†† Department of Mathematics, University of Connecticut, Storrs, CT 06269, U.S.A.

Email address: maria.gordina@uconn.edu

‡ Department of Mathematics, Cornell University, Ithaca, NY 14853, U.S.A

Email address: lsc@math.cornell.edu