DIFFEOMORPHISMS OF THE CIRCLE AND BROWNIAN MOTIONS ON AN INFINITE-DIMENSIONAL SYMPLECTIC GROUP

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ABSTRACT. An embedding of the group $\mathrm{Diff}(S^1)$ of orientation preserving diffeomorphims of the unit circle S^1 into an infinite-dimensional symplectic group, $\mathrm{Sp}(\infty)$, is studied. The authors prove that this embedding is not surjective. A Brownian motion is constructed on $\mathrm{Sp}(\infty)$. This study is motivated by recent work of H. Airault, S. Fang and P. Malliavin.

1. Introduction

The group $\mathrm{Diff}(S^1)$ of orientation preserving diffeomorphims of the unit circle S^1 has been extensively studied for a long time. One of the goals of the research has been to construct and study the properties of a Brownian motion on this group. In [1] H. Airault and P. Malliavin considered an embedding of $\mathrm{Diff}(S^1)$ into an infinite-dimensional symplectic group.

This group, $\mathrm{Sp}(\infty)$, can be represented as a certain infinite-dimensional matrix group. For such matrix groups, the method of[6, 7] can be used to construct a Brownian motion living in the group. This construction relies on the fact that these groups can be embedded into a larger Hilbert space of Hilbert-Schmidt operators. We use the same method to construct a Brownian motion on $\mathrm{Sp}(\infty)$. One of the advantages of Hilbert-Schmidt groups is that one can associate an infinite-dimensional Lie algebra to such a group, and this Lie algebra is a Hilbert space. This is not the case with $\mathrm{Diff}(S^1)$, as an infinite-dimensional Lie algebra associated with $\mathrm{Diff}(S^1)$ is not a Hilbert space with respect to the inner product compatible with the symplectic structure on $\mathrm{Diff}(S^1)$.

In the current paper, we describe in detail the embedding of $\mathrm{Diff}(S^1)$ into $\mathrm{Sp}(\infty)$, and construct a Brownian motion on $\mathrm{Sp}(\infty)$. Our motivation comes from an attempt to use this embedding to better understand Brownian motion in $\mathrm{Diff}(S^1)$ as studied by H. Airault, S. Fang and P. Malliavin in a number of papers (e.g. [1, 2, 4, 5]). One of the main results of the paper is Theorem 4.6, where we describe the embedding of $\mathrm{Diff}(S^1)$ into $\mathrm{Sp}(\infty)$ and prove that the map is not surjective. Theorem 6.17 gives the construction of a Brownian motion on $\mathrm{Sp}(\infty)$. In order for this Brownian motion to live in the group we are forced to choose

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a non-Ad-invariant inner product on the Lie algebra of $\mathrm{Sp}(\infty)$. This fact has a potential implication for this Brownian motion not to be quasi-invariant for the appropriate choice of the Cameron-Martin subgroup of $\mathrm{Sp}(\infty)$. This is in contrast to results in [2]. The latter can be explained by the fact that the Brownian motion we construct in Section 6 lives in a subgroup of $\mathrm{Sp}(\infty)$ whose Lie algebra is much smaller than the full Lie algebra of $\mathrm{Sp}(\infty)$.

2. The spaces H and $\mathbb{H}_\omega$

Definition 2.1. Let H be the space of complex-valued C^∞ functions on the unit circle S^1 with the mean value 0. Define a bilinear form ω on H by

$$\omega(u,v)=\frac{1}{2\pi}\int_0^{2\pi} uv' d\theta, \qquad \text{for any } u,v\in H.$$

Remark 2.2. By using integration by parts, we see that the form ω is anti-symmetric, that is, $\omega(u,v)=-\omega(v,u)$ for any $u,v\in H$.

Next we define an inner product $(\cdot,\cdot)_\omega$ on H which is compatible with the form ω. First, we introduce a complex structure on H, that is, a linear map J on H such that $J^2=-id$. Then the inner product is defined by $(u,v)_\omega=\pm\omega(u,J\bar{v})$, where the sign depends on the choice of J. The complex structure J in this context is called the Hilbert transform.

Definition 2.3. Let $\mathbb{H}_0$ be the Hilbert space of complex-valued L^2 functions on S^1 with the mean value 0 equipped with the inner product

$$(u,v)=\frac{1}{2\pi}\int_0^{2\pi} u\bar{v} d\theta, \qquad \text{for any } u,v\in \mathbb{H}_0.$$

Notation 2.4. Denote $\hat{e}_n=e^{in\theta}, n\in\mathbb{Z}\backslash\{0\}$, and $\mathcal{B}_H=\{\hat{e}_n,\ n\in\mathbb{Z}\backslash\{0\}\}$. Let $\mathbb{H}^+$ and $\mathbb{H}^-$ be the closed subspaces of $\mathbb{H}_0$ spanned by $\{\hat{e}_n : n>0\}$ and $\{\hat{e}_n : n<0\}$, respectively. By π^+ and π^- we denote the projections of $\mathbb{H}_0$ onto subspaces $\mathbb{H}^+$ and $\mathbb{H}^-$, respectively. For $u\in\mathbb{H}_0$, we can write $u=u_++u_-$, where $u_+=\pi^+(u)$ and $u_-=\pi^-(u)$.

Definition 2.5. Define the **Hilbert transformation** J on $\mathcal{B}_H$ by

$$J:\hat{e}_n\mapsto i\,\mathrm{sgn}(n)\hat{e}_n$$

where $\mathrm{sgn}(n)$ is the sign of n, and then extended by linearity to $\mathbb{H}_0$.

Remark 2.6. In the above definition, J is defined on the space $\mathbb{H}_0$. We need to address the issue whether it is well–defined on the *subspace* H. That is, if $J(H)\subseteq H$. We will see that if we modify the space H a little bit, for example, if we let $C_0^1(S^1)$ be the space of complex-valued C^1 functions on the circle with mean value zero, then J is *not* well–defined on $C_0^1(S^1)$. This problem really lies in the heart of Fourier analysis. To see this, we need to characterize J by using the Fourier transform.

Notation 2.7. For $u\in\mathbb{H}_0$, let $\mathcal{F}:u\mapsto\hat{u}$ be the **Fourier transformation** with $\hat{u}(n)=(u,\hat{e}_n)$. Let $\hat{J}$ be a transformation on $l^2(\mathbb{Z}\backslash\{0\})$ defined by $\big(\hat{J}\hat{u}\big)(n)=i\,\mathrm{sgn}(n)\hat{u}(n)$ for any $\hat{u}\in l^2(\mathbb{Z}\backslash\{0\})$.

The Fourier transformation $\mathcal{F}: \mathbb{H}_0 \to l^2(\mathbb{Z}\backslash\{0\})$ is an isomorphism of Hilbert spaces, and $J = \mathcal{F}^{-1} \circ \hat{J} \circ \mathcal{F}$.

Proposition 2.8. *The Hilbert transformation J is well–defined on H, that is $J(H) \subseteq H$.*

Proof. The key of the proof is the fact that functions in H can be completely characterized by their Fourier coefficients. To be precise, let $u \in \mathbb{H}_0$ be continuous. Then u is in C^∞ if and only if $\lim_{n\to\infty} n^k \hat{u}(n) = 0$ for any $k \in \mathbb{N}$. From this fact, it follows immediately that J is well–defined on H, because J only changes the signs of the Fourier coefficients of a function $u \in H$.

For completeness of exposition, we give a proof of this characterization. Though this is probably a standard fact in Fourier analysis, we found a proof (in [8]) of only one direction.

We first assume that u is C^∞. Then $u(\theta) = u(0) + \int_0^\theta u'(t)dt$. So

$$\begin{aligned}\hat{u}(n) &= \frac{1}{2\pi}\Big(\int_0^{2\pi}\int_0^{2\pi} u'(t)\chi_{[0,\theta]}dt\Big)e^{-in\theta}d\theta = \frac{1}{2\pi}\int_0^{2\pi}\Big(\int_t^{2\pi} e^{-in\theta}d\theta\Big)u'(t)dt \\ &= -\frac{1}{2\pi in}\int_0^{2\pi} u'(t) - u'(t)e^{-int}dt = \frac{\widehat{u'}(n)}{in},\end{aligned}$$

where we have used Fubini's theorem and the continuity of u'. Now, u' is itself C^∞, so we can apply the procedure again. By induction, we get $\hat{u}(n) = \frac{\widehat{u^{(k)}}(n)}{(in)^k}$. But from the general theory of Fourier analysis, $\widehat{u^{(k)}}(n) \to 0$ as $n \to \infty$. Therefore $n^k\hat{u}(n) \to 0$ as $n \to \infty$.

Conversely, assume u is such that for any k, $n^k\hat{u}(n) \to 0$ as $n \to \infty$. Then the Fourier series of u converges uniformly. Also by assumption that u is continuous, the Fourier series converges to u for all $\theta \in S^1$ (see Corollary I.3.1 in [8]). So we can write $u(\theta) = \sum_{n\neq 0}\hat{u}(n)e^{in\theta}$. Fix a point $\theta \in S^1$, then

$$u'(\theta) = \frac{d}{dt}\Big|_{t=\theta}\sum_{n\neq 0}\hat{u}(n)e^{int} = \lim_{t\to\theta}\lim_{N\to\infty}\sum_{n=-N}^{N}\hat{u}(n)\frac{e^{int} - e^{in\theta}}{t-\theta}.$$

Note that the derivatives of $\cos nt$ and $\sin nt$ are all bounded by $|n|$. So by the mean value theorem, $|\cos nt - \cos n\theta| \le |n||t-\theta|$, and $|\sin nt - \sin n\theta| \le |n||t-\theta|$. So

$$\Big|\frac{e^{int} - e^{in\theta}}{t-\theta}\Big| \le 2|n|, \qquad \text{for any } t, \theta \in S^1.$$

Therefore, by the growth condition on the Fourier coefficients $\hat{u}$, we have

$$\lim_{N\to\infty}\sum_{n=-N}^{N}\hat{u}(n)\frac{e^{int} - e^{in\theta}}{t-\theta}$$

converges at the fixed $\theta \in S^1$ and the convergence is uniform in $t \in S^1$. Therefore we can interchange the two limits, and obtain

$$\Big(\sum_{n\neq 0}\hat{u}(n)e^{in\theta}\Big)' = \sum_{n\neq 0}\hat{u}(n)ine^{in\theta},$$

which means we can differentiate term by term. So the Fourier coefficients of u' are given by $\hat{u'}(n) = in\hat{u}(n)$. Clearly, $\hat{u'}$ satisfies the same condition as $\hat{u}$: $n^k\hat{u'}(n) \to 0$ as $n \to \infty$. By induction, u is j-times differentiable for any j. Therefore, u is in C^∞. □

Proposition 2.9. *Let $C_0^1(S^1)$ be the space of complex-valued C^1 functions on the circle with the mean value zero. Then the Hilbert transformation J is* not *well–defined on $C_0^1(S^1)$, i.e., $J(C_0^1(S^1)) \nsubseteq C_0^1(S^1)$.*

Proof. Let $C(S^1)$ be the space of continuous functions on the circle. In [8], it is shown that there exists a function in $C(S^1)$ such that the corresponding Fourier series does not converges *uniformly* [8, Theorem II.1.3], and therefore there exists an $f \in C(S^1)$ such that $Jf \notin C(S^1)$ [8, Theorem II.1.4]. Now take $u = f - f_0$ where f_0 is the mean value of f. Then u is a continuous function on the circle with the mean value zero, and Ju is *not* continuous.

Using Notation 2.4 let us write $u = u_+ + u_-$. Then we can use the relation

$$iu + Ju = 2iu_+ \quad \text{and} \quad iu - Ju = 2iu_-.$$

to see that u_+ and u_- are *not* continuous. Integrating $u = u_+ + u_-$, we have

$$\int_0^t u(\theta)d\theta = \int_0^t u_+(\theta)d\theta + \int_0^t u_-(\theta)d\theta.$$

Denote the three functions in the above equation by v, v_1, v_2. By theorem I.1.6 in [8],

$$\hat{v}(n) = \frac{\hat{u}(n)}{in}, \quad \text{and} \quad \hat{v_1}(n) = \frac{\hat{u}_+(n)}{in}, \hat{v_2}(n) = \frac{1}{in}\hat{u}_-(n) \text{ for } n \neq 0.$$

Let $g = v - v_0$ where v_0 is the mean value of v. Then $g \in C_0^1(S^1)$. Write $g = g_+ + g_-$ 2.4. Then $g_+ = v_1 - (v_1)_0$ and $g_- = v_2 - (v_2)_0$ where $(v_1)_0$ and $(v_2)_0$ are the mean values of v_1 and v_2 respectively. Then $g_+, g_- \notin C_0^1(S^1)$ since $v_1' = u_+, v_2' = u_-$ are *not* continuous.

By the relation

$$ig + Jg = 2ig_+ \quad \text{and} \quad ig - Jg = 2ig_-,$$

we see that $Jg \notin C_0^1(S^1)$. □

Notation 2.10. Define an $\mathbb{R}$-bilinear form $(\cdot,\cdot)_\omega$ on H by

$$(u, v)_\omega = -\omega(u, J\bar{v}) \qquad \text{for any } u, v \in H.$$

Proposition 2.11. *$(\cdot,\cdot)_\omega$ is an inner product on H.*

Proof. We need to check that $(\cdot,\cdot)_\omega$ satisfies the following properties (1) $(\lambda u, v)_\omega = \lambda(u, v)_\omega$ for $\lambda \in \mathbb{C}$; (2) $(v, u)_\omega = \overline{(u, v)_\omega}$; (3) $(u, u)_\omega > 0$ unless $u = 0$.

(1) for $\lambda \in \mathbb{C}$,

$$(\lambda u, v)_\omega = -\omega(\lambda u, J\bar{v}) = -\lambda \cdot \omega(u, J\bar{v}) = \lambda \cdot (u, v)_\omega.$$

To prove (2) and (3), we need some simple facts: $H^+ = \pi^+(H) \subseteq H$ and $H^- = \pi^-(H) \subseteq H$, and $H = H^+ \oplus H^-$. If $u \in H^+, v \in H^-$, then $(u, v) = 0$. If $u \in H^+$, then $\bar{u} \in H^-, Ju = iu, Ju \in H^+$. If $u \in H^-$, then $\bar{u} \in H^+, Ju = -iu, Ju \in H^-$.

$J\bar{u} = \overline{Ju}$. $\widehat{u'}(n) = in\hat{u}(n)$. In particular, if $u \in H^+$, then $u' \in H^+$; if $u \in H^-$, then $u' \in H^-$.

(2) By definition,

$$(v, u)_\omega = -\omega(v, J\bar{u}) = \omega(J\bar{u}, v) = \frac{1}{2\pi}\int (J\bar{u})v' d\theta$$
$$\overline{(u, v)_\omega} = -\overline{\omega(u, J\bar{v})} = \overline{\omega(J\bar{v}, u)} = \frac{1}{2\pi}\int \overline{J\bar{v}}\bar{u}' d\theta = \frac{1}{2\pi}\int (Jv)\bar{u}' d\theta.$$

Write $u = u_+ + u_-$ and $v = v_+ + v_-$ as in Notation 2.4. Using the above fact, we can show that the above two quantities are equal to each other.

(3) Write $u = u_+ + u_-$, then

$$(u, u)_\omega = \frac{1}{2\pi}\int (-i\overline{u_+}u'_+ + i\overline{u_-}u'_-)d\theta = \sum_{n\neq 0} |n||\hat{u}(n)|^2.$$

Therefore, $(u, u)_\omega > 0$ unless $u = 0$. □

Definition 2.12. Let $\mathbb{H}_\omega$ be the completion of H under the norm $\|\cdot\|_\omega$ induced by the inner product $(\cdot,\cdot)_\omega$. Define

$$\mathcal{B}_\omega = \left\{\tilde{e}_n = \frac{1}{\sqrt{n}}e^{in\theta}, n > 0\right\} \cup \left\{\tilde{e}_n = \frac{1}{i\sqrt{|n|}}e^{in\theta}, n < 0\right\}.$$

Remark 2.13. $\mathbb{H}_\omega$ is a Hilbert space. Also the norm $\|\cdot\|_\omega$ induced by the inner product $(\cdot,\cdot)_\omega$ is *strictly* stronger than the norm $\|\cdot\|$ induced by the inner product $(\cdot,\cdot)$. So $\mathbb{H}_\omega$ can be identified as a *proper* subspace of $\mathbb{H}_0$. The inner product $(\cdot,\cdot)_\omega$ or the norm induced by it is sometimes called the $H^{1/2}$ metric or the $H^{1/2}$ norm on the space H.

One can verify that $\mathcal{B}_\omega$ is an orthonormal basis of $\mathbb{H}_\omega$. From the definition of the inner product $(\cdot,\cdot)_\omega$, we have the relation $\omega(u, v) = (u, \overline{Jv})_\omega$ for any $u, v \in H$. This can be used to *extend* the form ω to $\mathbb{H}_\omega$.

Finally, from the non–degeneracy of the inner product $(\cdot,\cdot)_\omega$, we see that the form $\omega(\cdot,\cdot)$ on $\mathbb{H}_\omega$ is also non–degenerate.

3. An infinite-dimensional symplectic group

Definition 3.1. Let $B(\mathbb{H}_\omega)$ be the space of **bounded operators** on $\mathbb{H}_\omega$ equipped with the operator norm. For an operator $A \in B(\mathbb{H}_\omega)$

(1) suppose $\bar{A}$ is an operator on $\mathbb{H}_\omega$ satisfying $\bar{A}u = \overline{A\bar{u}}$ for any $u \in \mathbb{H}_\omega$, then $\bar{A}$ is the **conjugate** of A;
(2) suppose $A^\dagger$ is an operator on $\mathbb{H}_\omega$ satisfying $(Au, v)_\omega = (u, A^\dagger v)_\omega$ for any $u, v \in \mathbb{H}_\omega$, then $A^\dagger$ is the **adjoint** of A;
(3) then $A^T = \bar{A}^\dagger$ is the **transpose** of A;
(4) suppose $A^\#$ is an operator on $\mathbb{H}_\omega$ satisfying $\omega(Au, v) = \omega(u, A^\# v)$ for any $u, v \in \mathbb{H}_\omega$, then $A^\#$ is the **symplectic adjoint** of A.
(5) A is said to **preserve the form** ω if $\omega(Au, Av) = \omega(u, v)$ for any $u, v \in \mathbb{H}_\omega$.

In the orthonormal basis $\mathcal{B}_\omega$, an operator $A \in B(\mathbb{H}_\omega)$ can be represented by an infinite-dimensional matrix, still denoted by A, with (m,n)th entry equal to $A_{m,n} = (A\tilde{e}_n, \tilde{e}_m)_\omega$.

Remark 3.2. If we represent an operator $A \in B(\mathbb{H}_\omega)$ by a matrix $\{A_{m,n}\}_{m,n\in\mathbb{Z}\setminus\{0\}}$, the indices m and n are allowed to be both positive and negative following Definition 2.12 of $\mathcal{B}_\omega$.

The next proposition collects some simple facts about operations on $B(\mathbb{H}_\omega)$ introduced in Definition 3.1.

Proposition 3.3. *Let $A, B \in B(\mathbb{H}_\omega)$. Then*

(1) $\overline{\tilde{e}_n} = i\tilde{e}_{-n}$, $J\tilde{e}_n = i\operatorname{sgn}(n)\tilde{e}_n$, $(\tilde{e}_n)' = in\tilde{e}_n$;
(2) $(\bar{A})_{m,n} = \overline{A_{-m,-n}}$;
(3) $(A^\dagger)_{m,n} = \overline{A_{n,m}}$;
(4) $\bar{A}^\dagger = \overline{A^\dagger}$, *and* $(A^T)_{m,n} = A_{-n,-m}$;
(5) *if* $A = \bar{A}$, *then* $(A^\#)_{m,n} = \operatorname{sgn}(mn)\overline{A_{n,m}}$;
(6) $\overline{AB} = \bar{A}\bar{B}$, $(AB)^\dagger = B^\dagger A^\dagger$, $(AB)^T = B^T A^T$, $(AB)^\# = B^\# A^\#$;
(7) *If A is invertible, then $\bar{A}, A^T, A^\dagger, A^\#$ are all invertible, and* $(\bar{A})^{-1} = \overline{A^{-1}}$, $(A^T)^{-1} = (A^{-1})^T$, $(A^\dagger)^{-1} = (A^{-1})^\dagger$, $(A^\#)^{-1} = (A^{-1})^\#$;
(8) $(\pi^+)_{m,n} = \frac{1}{2}(\delta_{mn} + \operatorname{sgn}(m)\delta_{mn})$, $(\pi^-)_{m,n} = \frac{1}{2}(\delta_{mn} - \operatorname{sgn}(m)\delta_{mn})$, $\overline{\pi^+} = \pi^-$, $\overline{\pi^-} = \pi^+$, $(\pi^+)^T = \pi^-$, $(\pi^-)^T = \pi^+$, $(\pi^+)^\dagger = \pi^+$, $(\pi^-)^\dagger = \pi^-$;
(9) $J_{m,n} = i\operatorname{sgn}(m)\delta_{mn}$, $\bar{J} = J$, $J = i(\pi^+ - \pi^-)$, $J^T = -J$, $J^\dagger = -J$, $J^2 = -id$;
(10) $(A^\#)_{m,n} = \operatorname{sgn}(mn)A_{-n,-m}$.

Proof. All of these properties can be checked by straight forward calculations. We only prove (10).

$$\begin{aligned}
(A^\#)_{m,n} &= (A^\#\tilde{e}_n, \tilde{e}_m)_\omega = -\omega(A^\#\tilde{e}_n, J\overline{\tilde{e}_m}) = \omega(J\overline{\tilde{e}_m}, A^\#\tilde{e}_n) \\
&= \omega(AJ\overline{\tilde{e}_m}, \tilde{e}_n) = -\omega(\tilde{e}_n, AJ\overline{\tilde{e}_m}) = -\omega(\tilde{e}_n, J(-J)AJ\overline{\tilde{e}_m}) \\
&= -\omega(\tilde{e}_n, J\overline{(-J\bar{A}J\tilde{e}_m)}),
\end{aligned}$$

where in the last equality we used property (6), $\overline{AB} = \bar{A}\bar{B}$, and property (9), $\bar{J} = J$, so that $\overline{-J\bar{A}J\tilde{e}_m} = -\bar{J}\bar{\bar{A}}\bar{J}\overline{\tilde{e}_m} = -JAJ\overline{\tilde{e}_m}$. Therefore,

$$\begin{aligned}
(A^\#)_{m,n} &= -\omega(\tilde{e}_n, J\overline{(-J\bar{A}J\tilde{e}_m)}) = (\tilde{e}_n, -J\bar{A}J\tilde{e}_m)_\omega = -(\tilde{e}_n, J\bar{A}J\tilde{e}_m)_\omega \\
&= -(J^\dagger\tilde{e}_n, \bar{A}J\tilde{e}_m)_\omega = -(-J\tilde{e}_n, \bar{A}J\tilde{e}_m)_\omega = (i\operatorname{sgn}(n)\tilde{e}_n, \bar{A}i\operatorname{sgn}(m)\tilde{e}_m)_\omega \\
&= \operatorname{sgn}(mn)(\tilde{e}_n, \bar{A}\tilde{e}_m)_\omega = \operatorname{sgn}(mn)\overline{(\bar{A}\tilde{e}_m, \tilde{e}_n)_\omega} = \operatorname{sgn}(mn)\overline{(\bar{A})_{n,m}} \\
&= \operatorname{sgn}(mn)A_{-n,-m}.
\end{aligned}$$

□

Notation 3.4. For $A \in B(\mathbb{H}_\omega)$, let $a = \pi^+A\pi^+$, $b = \pi^+A\pi^-$, $c = \pi^-A\pi^+$, and $d = \pi^-A\pi^-$, where $a : \mathbb{H}_\omega^+ \to \mathbb{H}_\omega^+$, $b : \mathbb{H}_\omega^- \to \mathbb{H}_\omega^+$, $c : \mathbb{H}_\omega^+ \to \mathbb{H}_\omega^-$, $d : \mathbb{H}_\omega^- \to \mathbb{H}_\omega^-$. Then $A = a + b + c + d$ can be represented as the following block matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If $A, B \in B(\mathbb{H}_\omega)$, then the block matrix representation for AB is exactly the multiplication of block matrices for A and B.

Proposition 3.5. *Suppose $A \in B(\mathbb{H}_\omega)$ with the matrix $\{A_{m,n}\}_{m,n\in\mathbb{Z}\setminus\{0\}}$. Then the following are equivalent*

(1) $A = \bar{A}$*;*
(2) *if $u = \bar{u}$, then $Au = \overline{Au}$;*
(3) $A_{m,n} = \overline{A_{-m,-n}}$ *(3.2);*
(4) *as a block matrix, A has the form* $\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$.

Proof. Equivalence of (1), (3) and (4) follows from Proposition3.3 and Notation3.4. First we show that (1) is equivalent to (2).

[(1)$\Longrightarrow$(2)]. If $u = \bar{u}$, then $Au = \bar{A}u = \overline{A\bar{u}} = \overline{Au}$.

[(2)$\Longrightarrow$(1)]. Let $u = \tilde{e}_n + \overline{\tilde{e}_n}$, and $v = \tilde{e}_{-n} + \overline{\tilde{e}_{-n}}$. Then u, v are real-valued functions on the circle. Using Proposition 3.3 we have $\overline{\tilde{e}_n} = i\tilde{e}_{-n}$, and therefore $Au = \overline{Au}$ and $Av = \overline{Av}$ imply

$$A\tilde{e}_n + iA\tilde{e}_{-n} = \overline{A\tilde{e}_n} - i\overline{A\tilde{e}_{-n}}$$
$$A\tilde{e}_n - iA\tilde{e}_{-n} = -\overline{A\tilde{e}_n} - i\overline{A\tilde{e}_{-n}}.$$

Solving the above two equations for $A\tilde{e}_n$, we have

$$A\tilde{e}_n = -i\overline{A\tilde{e}_{-n}} = \overline{A\overline{\tilde{e}_n}} = \bar{A}\tilde{e}_n$$

with this being true for any $n \neq 0$, and so $A = \bar{A}$. □

Proposition 3.6. *Let $A \in B(\mathbb{H}_\omega)$. The following are equivalent:*

(1) *A preserves the form ω;*
(2) *$\omega(Au, Av) = \omega(u, v)$ for any $u, v \in \mathbb{H}_\omega$;*
(3) *$\omega(A\tilde{e}_m, A\tilde{e}_n) = \omega(\tilde{e}_m, \tilde{e}_n)$ for any $m, n \neq 0$;*
(4) $A^T JA = J$*;*
(5) $\sum_{k\neq 0} \operatorname{sgn}(mk) A_{k,m} A_{-k,-n} = \delta_{m,n}$ *for any $m, n \neq 0$.*

If we further assume that $A = \bar{A}$, then the following two are equivalent to the above:

(I) $a^T\bar{a} - b^\dagger b = \pi^-$ *and* $a^T\bar{b} - b^\dagger a = 0$*;*
(II) $\sum_{k\neq 0} \operatorname{sgn}(mk) A_{k,m} \overline{A_{k,n}} = \delta_{m,n}$ *for any $m, n \neq 0$.*

Proof. Equivalence of (1),(2) and (3) follows directly from Definition 3.1. Let us check the equivalency of (2) and (4). First assume that (2) holds. By Remark 2.13 we have $\omega(u, v) = (u, J\bar{v})_\omega$, and therefore

$$\omega(Au, Av) = (Au, J\overline{Av})_\omega = (u, A^\dagger J\overline{Av})_\omega.$$

By assumption, $\omega(Au, Av) = \omega(u, v)$ for any $u, v \in \mathbb{H}_\omega$. So by the non-degeneracy of the inner product $(\cdot, \cdot)_\omega$, we have $A^\dagger J\overline{Av} = J\bar{v}$ for any $v \in \mathbb{H}_\omega$. By definition of $\bar{A}$, we have $\overline{Av} = \bar{A}\bar{v}$. So $A^\dagger J\bar{A}\bar{v} = J\bar{v}$ for any $v \in \mathbb{H}_\omega$, or $A^\dagger J\bar{A} = J$. Taking conjugation of both sides and using $\bar{J} = J$, we see that $A^T JA = J$.

Every step above is reversible, therefore we have implication in the other direction as well.

Now we check the equivalency of (3) and (5). First, by Remark 2.13 $\omega(u,v) = (u, J\bar{v})_\omega$ and Proposition 3.3

$$\omega(\tilde{e}_m, \tilde{e}_n) = (\tilde{e}_m, J\overline{\tilde{e}_n})_\omega = -\operatorname{sgn}(m)\delta_{m,-n}.$$

On the other hand, by the continuity of the form $\omega(\cdot,\cdot)$ in both variables, we have

$$\omega(A\tilde{e}_m, A\tilde{e}_n) = \omega\Big(\sum_k A_{k,m}\tilde{e}_k, \sum_k A_{l,n}\tilde{e}_l\Big)$$
$$= \sum_{k,l} A_{k,m}A_{l,n}(-\operatorname{sgn}(k))\delta_{k,-l} = -\sum_k \operatorname{sgn}(k)A_{k,m}A_{-k,n}.$$

Now assuming $\omega(A\tilde{e}_m, A\tilde{e}_n) = \omega(\tilde{e}_m, \tilde{e}_n)$, we have

$$-\sum_k \operatorname{sgn}(k)A_{k,m}A_{-k,n} = -\operatorname{sgn}(m)\delta_{m,-n}, \text{ for any } m,n \neq 0.$$

By multiplying by $\operatorname{sgn}(m)$ both sides, and replacing $-n$ with n, we get (5). Conversely, note that every step above is reversible, therefore we have implication in the other direction.

We have proved equivalence of (1)-(5). Now assume $A = \bar{A}$. To prove equivalence of (4) and (I), just notice that as block matrices, A, A^T and J have the form

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \quad \begin{pmatrix} a^\dagger & b^T \\ b^\dagger & a^T \end{pmatrix}, \quad \text{and} \quad i\begin{pmatrix} \pi^+ & 0 \\ 0 & -\pi^- \end{pmatrix}.$$

Equivalence of (5) and (II) follows from the relation $A_{-k,-n} = \overline{A_{k,n}}$. □

Proposition 3.7. *Let $A \in B(\mathbb{H}_\omega)$. If A preserves the form ω, then the following are equivalent:*

(1) *A is invertible.*
(2) *$AJA^T = J$.*
(3) *A^T preserves the form ω.*
(4) *$\sum_k \operatorname{sgn}(mk)A_{m,k}A_{-n,-k} = \delta_{m,n}$ for any $m,n \neq 0$.*

If we further assume that $A = \bar{A}$, then the following are equivalent to the above:

(I) *$\bar{a}a^T - \bar{b}b^T = \pi^-$ and $\bar{b}a^\dagger - \bar{a}b^\dagger = 0$.*
(II) *$\sum_k \operatorname{sgn}(mk)A_{m,k}\overline{A_{n,k}} = \delta_{m,n}$ for any $m,n \neq 0$.*

Proof. We will use several times the fact that if A preserves ω, then $A^TJA = J$.

[(1)⇒(2)] Multiplying on the left by $(A^T)^{-1}$ and multiplying on the right by A^{-1} both sides, we get $J = (A^T)^{-1}JA^{-1}$, and so $(A^{-1})^TJA^{-1} = J$. Taking inverse of both sides, and using $J^{-1} = -J$, we have $A^TJA = J$.

[(2)⇒(1)] As J is injective, so is A^TJA, and therefore A is injective. On the other hand, by assumption $AJA^T = J$. As J is surjective, so AJA^T is surjective too. This implies that A is surjective, and therefore A is invertible.

Equivalence of (2) and (3) follows from $(A^T)^T = A$ and Proposition 3.6. Equivalence of (3) and (4) follows directly from Proposition 3.6 and the fact that $(A^T)_{m,n} = A_{-n,-m}$.

Now assume that $A = \bar{A}$. Then equivalence of (3) and (I)can be checked by using multiplication of block matrices as in the proof of Proposition 3.6. Finally (4) is equivalent to (II) as if $A = \bar{A}$, then $A_{-m,-n} = \overline{A_{m,n}}$. □

Corollary 3.8. *Let $A \in B(\mathbb{H}_\omega)$ and $A = \bar{A}$. Then the following are equivalent:*

(1) *A preserves the form ω and is invertible;*
(2) *$A^\# A = A^\# A = id$;*

Proof. By Proposition 3.3

$$(A^\# A)_{m,n} = \sum_{k\neq 0} (A^\#)_{m,k} A_{k,n} = \sum_{k\neq 0} \operatorname{sgn}(mk) A_{k,n} \overline{A_{k,m}},$$
$$(AA^\#)_{m,n} = \sum_{k\neq 0} A_{m,k} (A^\#)_{k,n} = \sum_{k\neq 0} \operatorname{sgn}(nk) A_{m,k} \overline{A_{n,k}}.$$

Therefore, by (II) in Proposition 3.6 and (II) in Proposition 3.7 we have equivalence. □

Definition 3.9. Define a (semi)norm $\|\cdot\|_2$ on $B(\mathbb{H}_\omega)$ such that for $A \in B(\mathbb{H}_\omega)$, $\|A\|_2^2 = \operatorname{Tr}(b^\dagger b) = \|b\|_{HS}$, where $b = \pi^+ A \pi^-$. That is, the norm $\|A\|_2$ is just the Hilbert-Schmidt norm of the block b.

Definition 3.10. An **infinite-dimensional symplectic group** $\operatorname{Sp}(\infty)$ is the set of bounded operators A on H such that

(1) A is invertible;
(2) $A = \bar{A}$;
(3) A preserves the form ω;
(4) $\|A\|_2 < \infty$.

Remark 3.11. If A is a bounded operator on H, then A can be extended to a bounded operator on $\mathbb{H}_\omega$. Therefore, we can equivalently define $\operatorname{Sp}(\infty)$ to be the set of operators $A \in B(\mathbb{H}_\omega)$ such that

(1) A is invertible;
(2) $A = \bar{A}$;
(3) A preserves the form ω;
(4) $\|A\|_2 < \infty$.
(5) A is invariant on H, i.e., $A(H) \subseteq H$.

Remark 3.12. By Corollary 3.8, the definition of $\operatorname{Sp}(\infty)$ is also equivalent to

(1) $A = \bar{A}$;
(2) $A^\# A = AA^\# = id$;
(3) $\|A\|_2 < \infty$.

Proposition 3.13. $\operatorname{Sp}(\infty)$ *is a group.*

Proof. First we show that if $A \in \operatorname{Sp}(\infty)$, then $A^{-1} \in \operatorname{Sp}(\infty)$. By the assumption on A, it is easy to verify that A^{-1} satisfies (1), (2), (3) and (5) in Remark 3.11. We need to show that A^{-1} satisfies the condition (4), i.e. $\|A^{-1}\|_2 < \infty$. Suppose

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \quad \text{and} \quad A^{-1} = \begin{pmatrix} a' & b' \\ \overline{b'} & \overline{a'} \end{pmatrix},$$

where by our assumptions all blocks are bounded operators, and in addition b is a Hilbert-Schmidt operator. We want to prove b' is also a Hilbert-Schmidt operator. $AA^{-1} = I$ and $A^{-1}A = I$ imply that

$$ab' = -b\overline{a'}, \quad a'a + b'\bar{b} = I.$$

The last equation gives $a'ab' + b'\bar{b}b' = b'$, and so

$$b' = a'ab' + b'\bar{b}b' = -a'b\overline{a'} + b'\bar{b}b'$$

which is a Hilbert-Schmidt operator as b and $\bar{b}$ are Hilbert-Schmidt. Therefore $\|A^{-1}\|_2 < \infty$ and $A^{-1} \in \mathrm{Sp}(\infty)$.

Next we show that if $A, B \in \mathrm{Sp}(\infty)$, then $AB \in \mathrm{Sp}(\infty)$. By the assumption on A and B, it is easy to verify that AB satisfies (1), (2), (3) and (5) in Remark 3.11. We need to show that AB satisfies the condition (4), i.e. $\|AB\|_2 < \infty$. Suppose

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} c & d \\ \bar{d} & \bar{c} \end{pmatrix},$$

where all blocks are bounded, and $\|b\|_{HS}, \|d\|_{HS} < \infty$. Then

$$AB = \begin{pmatrix} ac + b\bar{d} & ad + b\bar{c} \\ \bar{b}c + \bar{a}\bar{d} & \bar{b}d + \bar{a}\bar{c} \end{pmatrix}.$$

Then

$$\|AB\|_2^2 = \|ad + b\bar{c}\|_{HS} \leqslant \|ad\|_2 + \|b\bar{c}\|_{HS} < \infty,$$

since both ad and $b\bar{c}$ are Hilbert-Schmidt operators. Therefore $\|AB\|_2 < \infty$ and $AB \in \mathrm{Sp}(\infty)$. □

4. Symplectic Representation of $\mathrm{Diff}(S^1)$

Definition 4.1. Let $\mathrm{Diff}(S^1)$ be the group of orientation preserving C^∞ diffeomorphisms of S^1. $\mathrm{Diff}(S^1)$ acts on H as follows

$$(\phi.u)(\theta) = u(\phi^{-1}(\theta)) - \frac{1}{2\pi}\int_0^{2\pi} u(\phi^{-1}(\theta))d\theta.$$

Note that if $u \in H$ is real-valued, then $\phi.u$ is real-valued as well.

Proposition 4.2. *The action of* $\mathrm{Diff}(S^1)$ *on* H *gives a group homomorphism*

$$\Phi : \mathrm{Diff}(S^1) \to \mathrm{Aut}\, H$$

defined by $\Phi(\phi)(u) = \phi.u$*, for* $\phi \in \mathrm{Diff}(S^1)$ *and* $u \in H$*, where* $\mathrm{Aut}\, H$ *is the group of automorphisms on* H*.*

Proof. Let $u \in H$, then $\phi.u$ is a C^∞ function with the mean value 0, and so $\phi.u \in H$. It is also clear that $\phi.(u + v) = \phi.u + \phi.v$ and $\phi.(\lambda u) = \lambda\phi.u$. So Φ is well–defined as a map from $\mathrm{Diff}(S^1)$ to $\mathrm{End}\, H$, the space of endomorphisms on H. Now let us check that Φ is a group homomorphism. Suppose $\phi, \psi \in \mathrm{Diff}(S^1)$ and $u \in H$, then

$$\begin{aligned}\Phi(\phi\psi)(u)(\theta) &= u\big((\phi\psi)^{-1}(\theta)\big) - \frac{1}{2\pi}\int_0^{2\pi} u\big((\phi\psi)^{-1}(\theta)\big)d\theta \\ &= u\big((\psi^{-1}\phi^{-1})(\theta)\big) - \frac{1}{2\pi}\int_0^{2\pi} u\big((\psi^{-1}\phi^{-1})(\theta)\big)d\theta.\end{aligned}$$

On the other hand,

$$\Phi(\phi)\Phi(\psi)(u)(\theta) = \Phi(\phi)\left[u(\psi^{-1}(\theta)) - \frac{1}{2\pi}\int_0^{2\pi} u(\psi^{-1}(\theta))d\theta\right]$$
$$= \Phi(\phi)\left[u(\psi^{-1}(\theta))\right] = u\big((\psi^{-1}\phi^{-1})(\theta)\big) - \frac{1}{2\pi}\int_0^{2\pi} u\big((\psi^{-1}\phi^{-1})(\theta)\big)d\theta.$$

So $\Phi(\phi\psi) = \Phi(\phi)\Phi(\psi)$. In particular, the image of Φ is in the $\operatorname{Aut} H$. □

Lemma 4.3. *Any $\phi \in \operatorname{Diff}(S^1)$ preserves the form ω, that is, $\omega(\phi.u, \phi.v) = \omega(u, v)$ for any $u, v \in H$.*

Proof. By Definition 4.1 $\phi.u = u(\psi) - u_0, \phi.v = v(\psi) - v_0$, where $\psi = \phi^{-1}$ and u_0, v_0 are the constants. Then

$$\begin{aligned}\omega(\phi.u, \phi.v) &= \omega(u(\psi) - u_0, v(\psi) - v_0)\\ &= \frac{1}{2\pi}\int_0^{2\pi} \big(u(\psi(\theta)) - u_0\big)\big(v(\psi(\theta)) - v_0\big)' d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi} u(\psi)v'(\psi)\psi'(\theta)d\theta - \frac{1}{2\pi}\int_0^{2\pi} u_0 v(\psi(\theta))d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi} u(\psi)v'(\psi)d\psi\\ &= \omega(u, v).\end{aligned}$$

□

We are going to prove that a diffeomorphism $\phi \in \operatorname{Diff}(S^1)$ acts on H as a bounded linear map, and that $\Phi(\phi)$ is in $\operatorname{Sp}(\infty)$. The next lemma is a generalization of a proposition in a paper of G. Segal[9].

Lemma 4.4. *Let $\psi \neq id \in \operatorname{Diff}(S^1)$ and $\phi = \psi^{-1}$. Let*

$$I_{n,m} = (\psi.e^{im\theta}, e^{in\theta}) = \frac{1}{2\pi}\int_0^{2\pi} e^{im\phi - in\theta}d\theta.$$

Then

(1) $\displaystyle\sum_{n>0,m<0} |n||I_{n,m}|^2 < \infty,$ *and* $\displaystyle\sum_{m>0,n<0} |n||I_{n,m}|^2 < \infty.$

(2) *For sufficiently large $|m|$ there is a constant C independent of m such that*

$$\sum_{n\neq 0} |n||I_{n,m}|^2 < C|m|. \tag{4.1}$$

Proof. Let

$$m_{\phi'} = \min\{\phi'(\theta)|\theta \in S^1\}; \quad \text{and} \quad M_{\phi'} = \max\{\phi'(\theta)|\theta \in S^1\}.$$

Since ϕ is a diffeomorphism, we have $0 < m_{\phi'} < M_{\phi'} < \infty$.

Take four points a, b, c, d on the unit circle such that a corresponds to $m_{\phi'}$ in the sense $\tan(a) = m_{\phi'}$, b corresponds to $M_{\phi'}$ in the sense $\tan(b) = M_{\phi'}$, c is opposite to a, i.e., $c = a + \pi$, d is opposite to b, i.e., $d = b + \pi$. The four points on the circle are arranged in the counter-clockwise order, and $0 < a < b < \frac{\pi}{2}$, $\pi < c < d < \frac{3}{2}\pi$.

Let $\tau \in S^1$ such that $\tau \neq \frac{\pi}{4}, \frac{5}{4}\pi$. Define a function ϕ_τ on S^1 by

$$\phi_\tau(\theta) = \frac{\cos\tau \cdot \phi(\theta) - \sin\tau \cdot \theta}{\cos\tau - \sin\tau}.$$

We will show that if $\tau \in (b, c)$ or $\tau \in (d, a)$, then ϕ_τ is an orientation preserving diffeomorphism of S^1, where (b, c) is the open arc from the point b to the point c, and (d, a) is the open arc from the point d to the point a.

Clearly ϕ_τ is a C^∞ function on S^1. Also, $\phi_\tau(0) = 0$ and $\phi_\tau(2\pi) = 2\pi$. Taking derivative with respect to θ, we have

$$\phi'_\tau(\theta) = \frac{\cos\tau \cdot \phi'(\theta) - \sin\tau}{\cos\tau - \sin\tau}.$$

By the choice of τ, we can prove that $\phi'_\tau(\theta) > 0$. Therefore, ϕ_τ is an orientation preserving diffeomorphism as claimed.

Let $m, n \in \mathbb{Z}\backslash\{0\}$. Let $\tau_{mn} = \operatorname{Arg}(m + in)$, i.e., the argument of the complex number $m+in$, considered to be in $[0, 2\pi]$. Then we have $m\phi - n\theta = (m-n)\phi_{\tau_{mn}}$.

If $\tau_{mn} \in (b, c)$, then $\phi_{\tau_{mn}}$ is a diffeomorphism. Let $\psi_{\tau_{mn}} = \phi^{-1}_{\tau_{mn}}$. Then

$$I_{n,m} = \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)\phi_{\tau_{mn}}} d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)\theta}\psi'_{\tau_{mn}}(\theta) d\theta,$$

where the last equality is by change of variable. On integration by parts k times, we have

$$I_{n,m} = \left(\frac{1}{i(m-n)}\right)^k \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)\theta}\psi^{(k+1)}_{\tau_{mn}}(\theta) d\theta.$$

Let $\alpha = [\alpha_0, \alpha_1]$ be a closed arc contained in the arc (b, c). Let S_α be the set of all pairs of nonzero integers (m, n) such that $\alpha_0 < \tau_{mn} < \alpha_1$, where $\tau_{mn} = \operatorname{Arg}(m + in)$. We are going to consider an upper bound of the sum $\sum_{(m,n)\in S_\alpha} |n||I_{n,m}|^2$.

For the pair (m, n), if $|m - n| = p$, the condition $\alpha_0 < \tau_{mn} < \alpha_1$ gives us both an upper bound and a lower bound for n:

$$\frac{m_{\phi'}}{m_{\phi'} - 1}p \leq n \leq \frac{M_{\phi'}}{M_{\phi'} - 1}p.$$

So $|n| \leq C_1 p$ where C_1 is a constant which does not depend on the pair (m, n). Also, the number of pairs $(m, n) \in S_\alpha$ such that $|m - n| = p$ is bounded by $C_2 p$ for some constant C_2. Let $C_3 = \max\left\{|\psi^{(k+1)}_\tau(\theta)| : \theta \in S^1, \tau \in [\alpha_0, \alpha_1]\right\}$. Then

$$|I_{n,m}| \leq C_3\left|\frac{1}{i(m-n)}\right|^k \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)\theta} d\theta = C_3 p^{-k}.$$

Therefore,

$$\begin{aligned}\sum_{(m,n)\in S} |n||I_{n,m}|^2 &= \sum_p \sum_{(m,n)\in S_\alpha; |m-n|=p} |n||I_{n,m}|^2 \\ &\leq \sum_p C_1 p \cdot C_3^2 p^{-2k} \cdot C_2 p = C_\alpha \sum_p p^{-(2k-2)},\end{aligned}$$

where the constant C_α depends on the arc α.

Similarly, for a closed arc $\beta = [\beta_0, \beta_1]$ contained in the arc (d, a), we have

$$\sum_{(m,n)\in S_\beta} |n||I_{n,m}|^2 \leq C_\beta \sum_p p^{-(2k-2)},$$

where the constant C_β depends on the arc β.

Now let $\alpha = [\frac{\pi}{2}, \pi]$, and $\beta = [\frac{3}{2}\pi, 2\pi]$. Then α is contained in (b, c) and β is contained in (d, a). We have

$$\sum_{n>0,m<0} |n||I_{n,m}|^2 = C_\alpha \cdot \sum_p p^{-(2k-2)} < \infty$$

and

$$\sum_{n<0,m>0} |n||I_{n,m}|^2 = C_\beta \cdot \sum_p p^{-(2k-2)} < \infty,$$

which proves (1) of the lemma.

To prove (2), we let $\alpha = [\alpha_0, \alpha_1]$ be a closed arc contained in the arc (b, c) such that $b < \alpha_0 < \frac{\pi}{2}$ and $\pi < \alpha_1 < c$, and $\beta = [\beta_0, \beta_1]$ be a closed arc contained in the arc (d, a) such that $d < \beta_0 < \frac{3}{2}\pi$ and $0 < \beta_1 < a$. Then we have

$$\sum_{(m,n)\in S_\alpha} |n||I_{n,m}|^2 + \sum_{(m,n)\in S_\beta} |n||I_{n,m}|^2 \leqslant C_{\alpha\beta}$$

for some constant $C_{\alpha\beta}$.

Let $m > 0$ be sufficiently large, and N_m be the largest integer less than or equal to $m\tan(\alpha_0)$,

$$\sum_{0<n\leqslant N_m} |I_{n,m}|^2 \leqslant \sum_{n\neq 0} |I_{n,m}|^2.$$

Note that $I_{n,m}$ is the nth Fourier coefficient of $\psi.e^{im\theta}$. Therefore,

$$\sum_{n\neq 0} |I_{n,m}|^2 = \|\psi.e^{im\theta}\|_{L^2}$$

which is bounded by a constant K. Therefore,

$$\sum_{0<n\leqslant N_m} |n||I_{n,m}|^2 \leqslant Km\tan(\alpha_0).$$

On the other hand,

$$\sum_{n<0} |n||I_{n,m}|^2 + \sum_{n>N_m} |n||I_{n,m}|^2 \leqslant \sum_{(m,n)\in S_\alpha} |n||I_{n,m}|^2 + \sum_{(m,n)\in S_\beta} |n||I_{n,m}|^2 = C_{\alpha\beta}.$$

Therefore,

$$\sum_{n\neq 0} |n||I_{n,m}|^2 \leqslant C_{\alpha\beta} + Km\tan(\alpha_0) \leqslant mC_+,$$

where C_+ can be chosen to be, for example, $K\tan(\alpha_0)+C_{\alpha\beta}$, which is independent of m.

Similarly, for $m < 0$ with sufficiently large $|m|$

$$\sum_{n\neq 0} |n||I_{n,m}|^2 \leqslant mC_-.$$

Let $C=\max\{C_+, C_-\}$. Then we have, for sufficiently large $|m|$,

$$\sum_{n\neq 0} |n||I_{n,m}|^2 \leqslant |m|C,$$

which proves (2) of the lemma. □

Lemma 4.5. *For any $\psi \in \mathrm{Diff}(S^1)$, $\Phi(\psi) \in B(H)$, the space of bounded linear maps on H. Moreover,*

$$\|\Phi(\psi)\| \leqslant C, \ \|\Phi(\psi)\|_2 \leqslant C,$$

where C is the constant in Equation 4.1.

Proof. First observe that the operator norm of $\Phi(\psi)$ is

$$\|\Phi(\psi)\| = \sup\{\|\psi.u\|_\omega \ \mid \ u \in H, \|u\|_\omega = 1\}.$$

For any $u \in H$, let $\hat{u}$ be its Fourier coefficients, that is $\hat{u}(n)=(u,\hat{e}_n)$, and let $\tilde{u}$ be defined by $\tilde{u}=(u,\tilde{e}_n)_\omega$ (2.10,2.12). It can be verified that the relation between $\hat{u}$ and $\tilde{u}$ is: if $n>0$, then $\tilde{u}(n)=\sqrt{n}\hat{u}(n)$; if $n<0$, then $\tilde{u}(n)=i\sqrt{|n|}\hat{u}(n)$. We have

$$\|u\|_\omega^2 = (u,u)_\omega = (\tilde{u},\tilde{u})_{l^2} = \sum_{n\neq 0}|\tilde{u}(n)|^2 = \sum_{n\neq 0}|n||\hat{u}(n)|^2.$$

Let $\phi=\psi^{-1}$. We have $u(\phi)=\sum_{m\neq 0}\hat{u}(m)e^{im\phi}$. Using the notation $I_{n,m}$ (4.4), we have

$$\begin{aligned}
\|\psi.u\|_\omega^2 &= \sum_{n\neq 0}|n||\widehat{\psi.u}(n)|^2 = \sum_{n\neq 0}|n|\Big|\frac{1}{2\pi}\int_0^{2\pi} u(\phi(\theta))e^{-in\theta}d\theta\Big|^2 \\
&= \sum_{n\neq 0}|n|\Big|\frac{1}{2\pi}\int_0^{2\pi}\sum_{m\neq 0}\hat{u}(m)e^{im\phi}e^{-in\theta}d\theta\Big|^2 \\
&= \sum_{n\neq 0}|n|\Big|\sum_{m\neq 0}\hat{u}(m)\frac{1}{2\pi}\int_0^{2\pi} e^{im\phi-in\theta}d\theta\Big|^2 \\
&= \sum_{n\neq 0}|n|\Big|\sum_{m\neq 0}\hat{u}(m)I_{n,m}\Big|^2 \\
&\leqslant \sum_{m,n\neq 0}|n||\hat{u}(m)|^2|I_{n,m}|^2 = \sum_{m\neq 0}|\hat{u}(m)|^2\sum_{n\neq 0}|n||I_{n,m}|^2 \\
&= \sum_{|m|\leqslant M_0}|\hat{u}(m)|^2\sum_{n\neq 0}|n||I_{n,m}|^2 + \sum_{|m|>M_0}|\hat{u}(m)|^2\sum_{n\neq 0}|n||I_{n,m}|^2,
\end{aligned}$$

where the constant M_0 in the last equality is a positive integer large enough so that we can apply part (2) of Lemma 4.4. It is easy to see that the first term in the last equality is finite. For the second term we use Lemma 4.4

$$\sum_{|m|>M_0}|\hat{u}(m)|^2\sum_{n\neq 0}|n||I_{n,m}|^2 \leqslant C\sum_{|m|>M_0}|\hat{u}(m)|^2|m| \leqslant C.$$

Thus for any $u \in H$ with $\|u\|_\omega = 1$, $\|\psi.u\|_\omega$ is uniformly bounded. Therefore, $\Phi(\psi)$ is a bounded operator on H.

Now we can use Lemma 4.4 again to estimate the norm $\|\Phi(\psi)\|_2$

$$\|\Phi(\psi)\|_2 = \sum_{n>0,m<0} |(\psi.\tilde{e}_m, \tilde{e}_n)_\omega|^2 = \sum_{n>0,m<0} |n||(\psi.\hat{e}_m, \hat{e}_n)|^2$$
$$= \sum_{n>0,m<0} |n||I_{n,m}|^2 < \infty.$$

□

Theorem 4.6. $\Phi : \mathrm{Diff}(S^1) \to \mathrm{Sp}(\infty)$ *is a group homomorphism. Moreover,* Φ *is injective, but not surjective.*

Proof. Combining Lemma 4.3 and Lemma 4.5 we see that for any diffeomorphism $\psi \in \mathrm{Diff}(S^1)$ the map $\Phi(\psi)$ is an invertible bounded operator on H, it preserves the form ω, and $\|\Phi(\psi)\|_2 < \infty$. In addition, by our remark after Definition 4.1 $\psi.u$ is real-valued, if u is real-valued. Therefore, Φ maps $\mathrm{Diff}(S^1)$ into $\mathrm{Sp}(\infty)$.

Next, we first prove that Φ is injective. Let $\psi_1, \psi_2 \in \mathrm{Diff}(S^1)$, and denote $\phi_1 = \psi_1^{-1}, \phi_2 = \psi_2^{-1}$. Suppose $\Phi(\psi_1) = \Phi(\psi_2)$, i.e. $\psi_1.u = \psi_2.u$, for any $u \in H$. In particular, $\psi_1.e^{i\theta} = \psi_2.e^{i\theta}$. Therefore

$$e^{i\phi_1} - C_1 = e^{i\phi_2} - C_2,$$

where $C_1 = \frac{1}{2\pi}\int_0^{2\pi} e^{i\phi_1} d\theta$, and $C_2 = \frac{1}{2\pi}\int_0^{2\pi} e^{i\phi_2} d\theta$. Note that $e^{i\phi_1}$ and $e^{i\phi_2}$ have the same image as maps from S^1 to $\mathbb{C}$. This implies $C_1 = C_2$, since otherwise $e^{i\phi_1} = e^{i\phi_2} + (C_1 - C_2)$ and $e^{i\phi_1}$ and $e^{i\phi_2}$ would have had different images. Therefore, we have $e^{i\phi_1} = e^{i\phi_2}$. But the function $e^{i\tau} : S^1 \to S^1$ is an injective function, so $\phi_1 = \phi_2$. Therefore $\psi_1 = \psi_2$, and so Φ is injective.

To prove that Φ is not surjective, we will construct an operator $A \in \mathrm{Sp}(\infty)$ which can not be written as $\Phi(\psi)$ for any $\psi \in \mathrm{Diff}(S^1)$. Let the linear map A be defined by the corresponding matrix $\{A_{m,n}\}_{m,n\in\mathbb{Z}}$ with the entries

$$A_{1,1} = A_{-1,-1} = \sqrt{2}$$
$$A_{1,-1} = i, A_{-1,1} = -i$$
$$A_{m,m} = 1, \quad \text{for } m \neq \pm 1$$

with all other entries being 0.

First we show that $A \in \mathrm{Sp}(\infty)$. For any $u \in H$, we can write $u = \sum_{n\neq 0} \tilde{u}(n)\tilde{e}_n$. Then A acting on u changes only $\tilde{e}_1$ and $\tilde{e}_{-1}$. Therefore, $Au \in H$, and clearly A is a well–defined bounded linear map on H to H. Moreover, $\|A\|_2 < \infty$. It is clear that $A_{m,n} = \overline{A_{-m,-n}}$, and therefore $A = \bar{A}$ by Proposition 3.3. Moreover, A preserves the form ω by part(II) of Proposition 3.6, as

$$\sum_{k\neq 0} \mathrm{sgn}(mk) A_{k,m}\overline{A_{k,n}} = \delta_{m,n}.$$

Finally, A is invertible, since $\{A_{k,m}\}_{m,n\in\mathbb{Z}}$ is, with the inverse $\{B_{k,m}\}_{m,n\in\mathbb{Z}}$ given by

$$B_{1,1} = B_{-1,-1} = \sqrt{2}$$
$$B_{1,-1} = -i, B_{-1,1} = i$$
$$B_{m,m} = 1, \quad \text{for } m \neq \pm 1$$

with all other entries being 0. Next we show that $A \neq \Phi(\psi)$ for any $\psi \in \mathrm{Diff}(S^1)$. First observe that if we look at any basis element $\tilde{e}_1 = e^{i\theta}$ as a function from S^1 to $\mathbb{C}$, then the image of this function lies on the unit circle. Clearly, when acted by a diffeomorphism $\phi \in \mathrm{Diff}(S^1)$, the image of the function $\phi.e^{i\theta}$ is still a circle with radius 1. But if we consider $A\tilde{e}_1$ as a function from S^1 to $\mathbb{C}$, we will show that the image of the function $A\tilde{e}_1 : S^1 \to \mathbb{C}$ is not a circle. Therefore, $A \neq \Phi(\psi)$ for any $\psi \in \mathrm{Diff}(S^1)$. Indeed, by definition of A we have

$$A\tilde{e}_1 = \sqrt{2}\tilde{e}_1 - i\tilde{e}_{-1}.$$

Let us write it as a function on S^1

$$A\tilde{e}_1(\theta) = \sqrt{2}e^{i\theta} - e^{-i\theta} = (\sqrt{2}-1)\cos\theta + i(\sqrt{2}+1)\sin\theta,$$

and then we see that the image lies on an ellipse, which is not the unit circle

$$\frac{x^2}{(\sqrt{2}-1)^2} + \frac{y^2}{(\sqrt{2}+1)^2} = 1.$$

□

5. The Lie algebra associated with $\mathrm{Diff}(S^1)$

Let $\mathrm{diff}(S^1)$ be the space of smooth vector fields on S^1. Elements in $\mathrm{diff}(S^1)$ can be identified with smooth functions on S^1. The space $\mathrm{diff}(S^1)$ is a Lie algebra with the following Lie bracket

$$[X, Y] = XY' - X'Y, \quad X, Y \in \mathrm{diff}(S^1),$$

where X' and Y' are derivatives with respect to θ.

Let $X \in \mathrm{diff}(S^1)$, and ρ_t be the corresponding flow of diffeomorphisms. We define an action of $\mathrm{diff}(S^1)$ on H as follows: for $X \in \mathrm{diff}(S^1)$ and $u \in H$, $X.u$ is a function on S^1 defined by

$$(X.u)(\theta) = \left.\frac{d}{dt}\right|_{t=0} [(\rho_t.u)(\theta)],$$

where ρ_t acts on u via the representation $\Phi : \mathrm{Diff}(S^1) \to \mathrm{Sp}(\infty)$.

The next proposition shows that the action is well–defined, and also gives an explicit formula of $X.u$.

Proposition 5.1. *Let $X \in \mathrm{diff}(S^1)$. Then*

$$(X.u)(\theta) = u'(\theta)(-X(\theta)) - \frac{1}{2\pi}\int_0^{2\pi} u'(\theta)(-X(\theta))d\theta,$$

that is, $X.u$ is the function $-u'X$ with the 0th Fourier coefficient replaced by 0.

Proof. Let ρ_t be the flow that corresponds to X, and λ_t be the flow that corresponds to $-X$. Then λ_t is the inverse of ρ_t for all t.

$$(X.u)(\theta) = \left.\frac{d}{dt}\right|_{t=0} [(\rho_t.u)(\theta)] = \left.\frac{d}{dt}\right|_{t=0} \left[u(\lambda_t(\theta)) - \frac{1}{2\pi}\int_0^{2\pi} u(\lambda_t(\theta))d\theta\right].$$

Using the chain rule, we have

$$\left.\frac{d}{dt}\right|_{t=0} u(\lambda_t(\theta)) = u'(\theta)(-\widetilde{X}(\theta)),$$

and

$$\left.\frac{d}{dt}\right|_{t=0} \frac{1}{2\pi}\int_0^{2\pi} u(\lambda_t(\theta))d\theta = \frac{1}{2\pi}\int_0^{2\pi} u'(\theta)(-X(\theta))d\theta.$$

□

Notation 5.2. We consider diff(S^1) as a subspace of the space of real-valued L^2 functions on S^1. The space of real-valued L^2 functions on S^1 has an orthonormal basis

$$\mathcal{B} = \{X_l = \cos(m\theta), Y_k = \sin(k\theta), l = 0,1,...,k = 1,2,...\}$$

which is contained in diff(S^1).

Let us consider how these basis elements act on H.

Proposition 5.3. *For any $l = 0,1,...,k = 1,2,...$ the basis elements X_l, Y_k act on H as linear maps. In the basis $\mathcal{B}_\omega$ of H, they are represented by infinite dimensional matrices with (m,n)th entries equal to*

$$(X_l)_{m,n} = (X_l.\tilde{e}_n, \tilde{e}_m)_\omega = s(m,n)\frac{1}{2}\sqrt{|mn|}(\delta_{m-n,l} + \delta_{n-m,l})$$
$$(Y_k)_{m,n} = (Y_k.\tilde{e}_n, \tilde{e}_m)_\omega = s(m,n)(-i)\frac{1}{2}\sqrt{|mn|}(\delta_{m-n,k} - \delta_{n-m,k})$$

where $m,n \neq 0$,

$$s(m,n) = \begin{cases} -i & m,n>0 \\ 1 & m>0, n<0 \\ 1 & m<0, n>0 \\ i & m,n<0. \end{cases}$$

Proof. By Proposition 5.1 and a simple verification depending on the signs of m,n we see that

$$X_l.e^{in\theta} = -ine^{in\theta}\cos(l\theta) = -\frac{1}{2}in\left[e^{i(n+l)\theta} + e^{i(n-l)\theta}\right]$$
$$Y_k.e^{in\theta} = -ine^{in\theta}\sin(k\theta) = -\frac{1}{2}n\left[e^{i(n+k)\theta} - e^{i(n-k)\theta}\right].$$

Indeed, recall that a basis element $\tilde{e}_n \in \mathcal{B}_\omega$ has the form

$$\tilde{e}_n = \begin{cases} \frac{1}{\sqrt{n}}e^{in\theta} & n>0 \\ \frac{1}{i\sqrt{|n|}}e^{in\theta} & n<0. \end{cases}$$

Suppose $m,n>0$

$$X_l.\tilde{e}_n = \frac{1}{\sqrt{n}}X_l.e^{in\theta} = -\frac{1}{2}i\sqrt{n}\left[e^{i(n+l)\theta} + e^{i(n-l)\theta}\right],$$

and

$$(e^{i(n+l)\theta}, \tilde{e}_m)_\omega = \sqrt{m}\delta_{m-n,k}; \qquad (e^{i(n-l)\theta}, \tilde{e}_m)_\omega = \sqrt{m}\delta_{n-m,l}.$$

Therefore,

$$(X_l)_{m,n} = (X_l.\tilde{e}_n, \tilde{e}_m)_\omega = (-i)\frac{1}{2}\sqrt{|mn|}(\delta_{m-n,l} + \delta_{n-m,l}).$$

All other cases can be verified similarly. □

Remark 5.4. Recall that $\mathbb{H}_\omega$ is the completion of H under the metric $(\cdot,\cdot)_\omega$. The above calculation shows that the trigonometric basis X_l, Y_k of $\mathrm{diff}(S^1)$ act on $\mathbb{H}_\omega$ as *unbounded* operators. They are densely defined on the subspace $H \subseteq \mathbb{H}_\omega$.

6. Brownian motion on $\mathrm{Sp}(\infty)$

Notation 6.1. As in [1], let $\mathfrak{sp}(\infty)$ be the set of infinite-dimensional matrices A which can be written as block matrices of the form

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$$

such that $a + a^\dagger = 0$, $b = b^T$, and b is a Hilbert-Schmidt operator.

Remark 6.2. The set $\mathfrak{sp}(\infty)$ has a structure of Lie algebra with the operator commutator as a Lie bracket, and we associate this Lie algebra with the group $\mathrm{Sp}(\infty)$.

Proposition 6.3. *Let* $\{A_{m,n}\}_{m,n\in\mathbb{Z}\setminus\{0\}}$ *be the matrix corresponding to an operator* A. *Then any* $A \in \mathfrak{sp}(\infty)$ *satisfies (1)* $A_{m,n} = \overline{A_{-m,-n}}$*; (2)* $A_{m,n} + \overline{A_{n,m}} = 0$*, for* $m, n > 0$*; (3)* $A_{m,n} = A_{-n,-m}$*, for* $m > 0, n < 0$.

Moreover, $A \in \mathfrak{sp}(\infty)$ *if and only if (1)* $A = \bar{A}$*; (2)* $\pi^+ A \pi^-$ *is Hilbert-Schmidt; (3)* $A + A^\# = 0$.

Proof. The first part follows directly from definition of $\mathfrak{sp}(\infty)$. Then we can use this fact and the formula for the matrix entries of $A^\#$ in Proposition 3.3 to prove the second part. □

Definition 6.4. Let HS be the space of Hilbert-Schmidt matrices viewed as a real vector space, and $\mathfrak{sp}_{\mathrm{HS}} = \mathfrak{sp}(\infty) \cap HS$.

The space HS as a real Hilbert space has an orthonormal basis

$$\mathcal{B}_{HS} = \{e^{Re}_{mn} : m, n \neq 0\} \cup \{e^{Im}_{mn} : m, n \neq 0\},$$

where e^{Re}_{mn} is a matrix with (m,n)-th entry 1 all other entries 0, and e^{Im}_{mn} is a matrix with (m,n)th entry i all other entries 0.

The space $\mathfrak{sp}_{\mathrm{HS}}$ is a closed subspace of HS, and therefore a real Hilbert space. According to the symmetry of the matrices in $\mathfrak{sp}_{\mathrm{HS}}$, we define a projection $\pi : HS \to \mathfrak{sp}_{\mathrm{HS}}$, such that

$$\begin{aligned}
\pi(e^{Re}_{mn}) &= \frac{1}{2}\big(e^{Re}_{mn} - e^{Re}_{nm} + e^{Re}_{-m,-n} - e^{Re}_{-n,-m}\big), && \text{if } \mathrm{sgn}(mn) > 0 \\
\pi(e^{Im}_{mn}) &= \frac{1}{2}\big(e^{Im}_{mn} + e^{Im}_{nm} - e^{Im}_{-m,-n} - e^{Im}_{-n,-m}\big), && \text{if } \mathrm{sgn}(mn) > 0 \\
\pi(e^{Re}_{mn}) &= \frac{1}{2}\big(e^{Re}_{mn} + e^{Re}_{-n,-m} + e^{Re}_{-m,-n} + e^{Re}_{n,m}\big), && \text{if } \mathrm{sgn}(mn) < 0 \\
\pi(e^{Im}_{mn}) &= \frac{1}{2}\big(e^{Im}_{mn} + e^{Im}_{-n,-m} - e^{Im}_{-m,-n} - e^{Im}_{nm}\big), && \text{if } \mathrm{sgn}(mn) < 0
\end{aligned}$$

Notation 6.5. We choose $\mathcal{B}_{\mathfrak{sp}_{\mathrm{HS}}} = \pi(\mathcal{B}_{HS})$ to be the orthonormal basis of $\mathfrak{sp}_{\mathrm{HS}}$.

Clearly, if $A \in \mathfrak{sp}_{\mathrm{HS}}$, then $|A|_{\mathfrak{sp}_{\mathrm{HS}}} = |A|_{HS}$.

Definition 6.6. Let W_t be a Brownian motion on $\mathfrak{sp}_{\mathrm{HS}}$ which has the mean zero and covariance Q, where Q is assumed to be a positive symmetric trace class operator on H. We further assume that Q is diagonal in the basis $\mathcal{B}_{\mathfrak{sp}_{\mathrm{HS}}}$.

Remark 6.7. Q can also be viewed as a positive function on the set $\mathcal{B}_{\mathfrak{sp}_{\mathrm{HS}}}$, and the Brownian motion W_t can be written as

$$W_t = \sum_{\xi \in \mathcal{B}_{\mathfrak{sp}_{\mathrm{HS}}}} \sqrt{Q(\xi)} B_t^{\xi} \xi, \tag{6.1}$$

where $\{B_t^{\xi}\}_{\xi \in \mathcal{B}_{\mathfrak{sp}_{\mathrm{HS}}}}$ are standard real-valued mutually independent Brownian motions.

Our goal now is to construct a Brownian motion on the group Sp(∞) using the Brownian motion W_t on $\mathfrak{sp}_{\mathrm{HS}}$. This is done by solving the Stratonovich stochastic differential equation

$$\delta X_t = X_t \delta W_t. \tag{6.2}$$

This equation can be written as the following Itô stochastic differential equation

$$dX_t = X_t dW_t + \frac{1}{2} X_t D dt, \tag{6.3}$$

where $D = \mathrm{Diag}(D_m)$ is a diagonal matrix with entries

$$D_m = -\frac{1}{4}\,\mathrm{sgn}(m) \sum_k \mathrm{sgn}(k)\left[Q_{mk}^{Re} + Q_{mk}^{Im}\right] \tag{6.4}$$

with $Q_{mk}^{Re} = Q(\pi(e_{mk}^{Re}))$ and $Q_{mk}^{Im} = Q(\pi(e_{mk}^{Im}))$.

Notation 6.8. Denote by $\mathfrak{sp}_{\mathrm{HS}}^{\mathrm{Q}} = Q^{1/2}(\mathfrak{sp}_{\mathrm{HS}})$ which is a subspace of $\mathfrak{sp}_{\mathrm{HS}}$. Define an inner product on $\mathfrak{sp}_{\mathrm{HS}}^{\mathrm{Q}}$ by $\langle u, v\rangle_{\mathfrak{sp}_{\mathrm{HS}}^{\mathrm{Q}}} = \langle Q^{-1/2}u, Q^{-1/2}v\rangle_{\mathfrak{sp}_{\mathrm{HS}}}$. Then $\mathcal{B}_{\mathfrak{sp}_{\mathrm{HS}}^{\mathrm{Q}}} = \{\hat{\xi} = Q^{1/2}\xi : \xi \in \mathcal{B}_{\mathfrak{sp}_{\mathrm{HS}}}\}$ is an orthonormal basis of the Hilbert space $\mathfrak{sp}_{\mathrm{HS}}^{\mathrm{Q}}$.

Notation 6.9. Let L_2^0 be the space of Hilbert-Schmidt operators from $\mathfrak{sp}_{\mathrm{HS}}^{\mathrm{Q}}$ to $\mathfrak{sp}_{\mathrm{HS}}$ with the norm

$$|\Phi|_{L_2^0}^2 = \sum_{\hat{\xi} \in \mathcal{B}_{\mathfrak{sp}_{\mathrm{HS}}^{\mathrm{Q}}}} |\Phi\hat{\xi}|_{\mathfrak{sp}_{\mathrm{HS}}}^2 = \sum_{\xi, \zeta \in \mathcal{B}_{\mathfrak{sp}_{\mathrm{HS}}}} Q(\xi) |\langle \Phi\xi, \zeta\rangle_{\mathfrak{sp}_{\mathrm{HS}}}|^2 = \mathrm{Tr}[\Phi Q \Phi^*],$$

where $Q(\xi)$ means Q evaluated at ξ as a positive function on $\mathcal{B}_{\mathfrak{sp}_{\mathrm{HS}}}$.

Lemma 6.10. *If $\Psi \in L(\mathfrak{sp}_{HS}, \mathfrak{sp}_{HS})$, a bounded linear operator from $\mathfrak{sp}_{HS}$ to $\mathfrak{sp}_{HS}$, then Ψ restricted on $\mathfrak{sp}_{HS}^{Q}$ is a Hilbert-Schmidt operator from $\mathfrak{sp}_{HS}^{Q}$ to $\mathfrak{sp}_{HS}$, and $|\Psi|_{L_2^0} \leqslant \mathrm{Tr}(Q)\|\Psi\|^2$, where $\|\Psi\|$ is the operator norm of Ψ.*

Proof.

$$
\begin{aligned}
|\Psi|_{L_2^0}^2 &= \sum_{\hat{\xi}\in\mathcal{B}_{\mathfrak{sp}_{\mathrm{HS}}^{\mathrm{Q}}}} |\Psi\hat{\xi}|_{\mathfrak{sp}_{\mathrm{HS}}}^2 \leqslant \|\Psi\|^2 \sum_{\hat{\xi}\in\mathcal{B}_{\mathfrak{sp}_{\mathrm{HS}}^{\mathrm{Q}}}} |\hat{\xi}|_{\mathfrak{sp}_{\mathrm{HS}}}^2 \\
&= \|\Psi\|^2 \sum_{\xi\in\mathcal{B}_{\mathfrak{sp}_{\mathrm{HS}}}} \langle Q^{1/2}\xi, Q^{1/2}\xi\rangle_{\mathfrak{sp}_{\mathrm{HS}}} = \|\Psi\|^2 \sum_{\xi\in\mathcal{B}_{\mathfrak{sp}_{\mathrm{HS}}}} \langle Q\xi,\xi\rangle_{\mathfrak{sp}_{\mathrm{HS}}} = \|\Psi\|^2\,\mathrm{Tr}(Q)
\end{aligned}
$$

□

Notation 6.11. Define $B:\mathfrak{sp}_{\mathrm{HS}}\to L_2^0$ by $B(Y)A=(I+Y)A$ for $A\in\mathfrak{sp}_{\mathrm{HS}}^{\mathrm{Q}}$, and $F:\mathfrak{sp}_{\mathrm{HS}}\to\mathfrak{sp}_{\mathrm{HS}}$ by $F(Y)=\frac{1}{2}(I+Y)D$.

Note that B is well–defined by Lemma 6.10. Also $D\in\mathfrak{sp}_{\mathrm{HS}}$, and so $F(Y)\in\mathfrak{sp}_{\mathrm{HS}}$ and F is well–defined as well.

Theorem 6.12. *The stochastic differential equation*

$$
\begin{aligned}
dY_t &= B(Y_t)dW_t + F(Y_t)dt \\
Y_0 &= 0
\end{aligned}
\tag{6.5}
$$

has a unique solution, up to equivalence, among the processes satisfying

$$
P\left(\int_0^T |Y_s|_{\mathfrak{sp}_{HS}}^2 ds<\infty\right)=1.
$$

Proof. To prove this theorem we will use Theorem 7.4 from the book by G. DaPrato and J. Zabczyk [3] as it has been done in [6, 7]. It is enough to check

1. B is a measurable mapping.
2. $|B(Y_1)-B(Y_2)|_{L_2^0}\leqslant C_1|Y_1-Y_2|_{\mathfrak{sp}_{\mathrm{HS}}}$ for $Y_1,Y_2\in\mathfrak{sp}_{\mathrm{HS}}$;
3. $|B(Y)|_{L_2^0}^2\leqslant K_1(1+|Y|_{\mathfrak{sp}_{\mathrm{HS}}}^2)$ for any $Y\in\mathfrak{sp}_{\mathrm{HS}}$;
4. F is a measurable mapping.
5. $|F(Y_1)-F(Y_2)|_{\mathfrak{sp}_{\mathrm{HS}}}\leqslant C_2|Y_1-Y_2|_{\mathfrak{sp}_{\mathrm{HS}}}$ for $Y_1,Y_2\in\mathfrak{sp}_{\mathrm{HS}}$;
6. $|F(Y)|_{\mathfrak{sp}_{\mathrm{HS}}}^2\leqslant K_2(1+|Y|_{\mathfrak{sp}_{\mathrm{HS}}}^2)$ for any $Y\in\mathfrak{sp}_{\mathrm{HS}}$.

Proof of 1. By the proof of 2, B is a continuous mapping, therefore it is measurable.
Proof of 2.

$$
\begin{aligned}
|B(Y_1)-B(Y_2)|_{L_2^0}^2 &= \sum_{\hat{\xi}\in\mathcal{B}_{\mathfrak{sp}_{\mathrm{HS}}^{\mathrm{Q}}}} |(Y_1-Y_2)\hat{\xi}|_{\mathfrak{sp}_{\mathrm{HS}}}^2 = \sum_{\xi\in\mathcal{B}_{\mathfrak{sp}_{\mathrm{HS}}}} Q(\xi)|(Y_1-Y_2)\xi|_{\mathfrak{sp}_{\mathrm{HS}}}^2 \\
&\leqslant \sum_{\xi\in\mathcal{B}_{\mathfrak{sp}_{\mathrm{HS}}}} Q(\xi)\|\xi\|^2|Y_1-Y_2|_{\mathfrak{sp}_{\mathrm{HS}}}^2 \leqslant \max_{\xi\in\mathcal{B}_{\mathfrak{sp}_{\mathrm{HS}}}}\|\xi\|^2\left(\sum_{\xi\in\mathcal{B}_{\mathfrak{sp}_{\mathrm{HS}}}} Q(\xi)\right)|Y_1-Y_2|_{\mathfrak{sp}_{\mathrm{HS}}}^2 \\
&= \mathrm{Tr}\,Q\left(\max_{\xi\in\mathcal{B}_{\mathfrak{sp}_{\mathrm{HS}}}}\|\xi\|^2\right)|Y_1-Y_2|_{\mathfrak{sp}_{\mathrm{HS}}}^2 = C_1^2|Y_1-Y_2|_{\mathfrak{sp}_{\mathrm{HS}}}^2,
\end{aligned}
$$

where $\|\xi\|$ is the operator norm of ξ, which is uniformly bounded for all $\xi\in\mathcal{B}_{\mathfrak{sp}_{\mathrm{HS}}}$.

Proof of 3.

$$|B(Y_1)|^2_{L_2^0} = \sum_{\hat{\xi}\in\mathcal{B}_{\mathfrak{sp}^Q_{\mathrm{HS}}}} |(I+Y)\hat{\xi}|^2_{\mathfrak{sp}_{\mathrm{HS}}} = \sum_{\xi\in\mathcal{B}_{\mathfrak{sp}_{\mathrm{HS}}}} Q(\xi)|(I+Y)\xi|^2_{\mathfrak{sp}_{\mathrm{HS}}}$$
$$\leqslant |(I+Y)\xi|^2_{\mathfrak{sp}_{\mathrm{HS}}} \sum_{\xi\in\mathcal{B}_{\mathfrak{sp}_{\mathrm{HS}}}} Q(\xi)\|\xi\|^2 \leq (1+|Y|^2_{\mathfrak{sp}_{\mathrm{HS}}})\cdot K_1.$$

Proof of 4. By the proof of 5, F is a continuous mapping, therefore it is measurable.
Proof of 5.

$$|F(Y_1)-F(Y_2)|_{\mathfrak{sp}_{\mathrm{HS}}} = |\frac{1}{2}(Y_1-Y_2)D|_{\mathfrak{sp}_{\mathrm{HS}}} \leq \|\frac{1}{2}D\||Y_1-Y_2|_{\mathfrak{sp}_{\mathrm{HS}}}$$

Proof of 6.

$$|F(Y)|^2_{\mathfrak{sp}_{\mathrm{HS}}} = |\frac{1}{2}(I+Y)D|^2_{\mathfrak{sp}_{\mathrm{HS}}} \leq \|\frac{1}{2}D\|^2|I+Y|^2_{\mathfrak{sp}_{\mathrm{HS}}} \leqslant K_2(1+|Y|^2_{\mathfrak{sp}_{\mathrm{HS}}}).$$

□

Notation 6.13. Let $B^{\#}:\mathfrak{sp}_{\mathrm{HS}}\to L_2^0$ be the operator $B^{\#}(Y)A = A^{\#}(I+Y)$, and $F^{\#}:\mathfrak{sp}_{\mathrm{HS}}\to\mathfrak{sp}_{\mathrm{HS}}$ be the operator $F^{\#}(Y)=\frac{1}{2}D^{\#}(Y+I)$.

Proposition 6.14. *If Y_t is the solution to the stochastic differential equation*

$$dX_t = B(X_t)dW_t + F(X_t)dt$$
$$X_0 = 0,$$

where B and F are defined in Notation 6.11, then $Y_t^{\#}$ is the solution to the stochastic differential equation

$$dX_t = B^{\#}(X_t)dW_t + F^{\#}(X_t)dt \tag{6.6}$$
$$X_0 = 0,$$

where $B^{\#}$ and $F^{\#}$ are defined in Notation 6.13.

Proof. This follows directly from the property $(AB)^{\#} = B^{\#}A^{\#}$ for any A and B, which can be verified by using part (5) of Proposition 3.3. □

Lemma 6.15. *Let U and H be real Hilbert spaces. Let $\Phi: U\to H$ be a bounded linear map. Let $G: H\to H$ be a bounded linear map. Then*

$$\mathrm{Tr}_H(G\Phi\Phi^*) = \mathrm{Tr}_U(\Phi^*G\Phi)$$

Proof.

$$\mathrm{Tr}_H(G\Phi\Phi^*) = \sum_{i,j\in H;k\in U} G_{ij}\Phi_{jk}(\Phi^*)_{ki} = \sum_{i,j\in H;k\in U} G_{ij}\Phi_{jk}\Phi_{ik}$$
$$\mathrm{Tr}_U(\Phi^*G\Phi) = \sum_{i,j\in H;k\in U} (\Phi^*)_{ki}G_{ij}\Phi_{jk} = \sum_{i,j\in H;k\in U} G_{ij}\Phi_{jk}\Phi_{ik}.$$

Therefore $\mathrm{Tr}_H(G\Phi\Phi^*) = \mathrm{Tr}_U(\Phi^*G\Phi)$. □

Lemma 6.16.

$$\sum_{\xi\in\mathcal{B}_{\mathfrak{sp}_{HS}}} \left(Q^{1/2}\xi\right)\left(Q^{1/2}\xi\right)^{\#} = -D$$

Proof. If $\xi \in \mathcal{B}_{\mathfrak{sp}_{\mathrm{HS}}}$, then $\xi \in \mathfrak{sp}(\infty)$, so $\xi^{\#} = -\xi$. We will use the fact that

$$(e_{ij}^{Re} e_{kl}^{Re})_{pq} = \delta_{ip}\delta_{jk}\delta_{lq}$$

where e_{ij}^{Re} is the matrix with the (i,j)th entry being 1 and all other entries being zero. Using this fact, we see

(1) for $\xi = \frac{1}{2}\left(e_{mn}^{Re} - e_{nm}^{Re} + e_{-m,-n}^{Re} - e_{-n,-m}^{Re}\right)$ with $\mathrm{sgn}(mn) > 0$,

$$\left(Q^{1/2}\xi\right)\left(Q^{1/2}\xi\right)^{\#} = -\frac{1}{4}Q_{mn}^{Re}\left[-e_{mm}^{Re} - e_{nn}^{Re} - e_{-m,-m}^{Re} - e_{-n,-n}^{Re}\right]$$

(2) for $\xi = \frac{1}{2}\left(e_{mn}^{Im} + e_{nm}^{Im} - e_{-m,-n}^{Im} - e_{-n,-m}^{Im}\right)$ with $\mathrm{sgn}(mn) > 0$,

$$\left(Q^{1/2}\xi\right)\left(Q^{1/2}\xi\right)^{\#} = -\frac{1}{4}Q_{mn}^{Im}\left[-e_{mm}^{Re} - e_{nn}^{Re} - e_{-m,-m}^{Re} - e_{-n,-n}^{Re}\right]$$

(3) for $\xi = \frac{1}{2}\left(e_{mn}^{Re} + e_{-n,-m}^{Re} + e_{-m,-n}^{Re} + e_{n,m}^{Re}\right)$ with $\mathrm{sgn}(mn) < 0$,

$$\left(Q^{1/2}\xi\right)\left(Q^{1/2}\xi\right)^{\#} = -\frac{1}{4}Q_{mn}^{Re}\left[e_{mm}^{Re} + e_{nn}^{Re} + e_{-m,-m}^{Re} + e_{-n,-n}^{Re}\right]$$

(4) for $\xi = \frac{1}{2}\left(e_{mn}^{Im} + e_{-n,-m}^{Im} - e_{-m,-n}^{Im} - e_{nm}^{Im}\right)$ with $\mathrm{sgn}(mn) < 0$,

$$\left(Q^{1/2}\xi\right)\left(Q^{1/2}\xi\right)^{\#} = -\frac{1}{4}Q_{mn}^{Im}\left[e_{mm}^{Re} + e_{nn}^{Re} + e_{-m,-m}^{Re} + e_{-n,-n}^{Re}\right].$$

Each of the above is a diagonal matrix. The lemma can be proved by looking at the diagonal entries of the sum. □

Theorem 6.17. *Let Y_t be the solution to Equation 6.5. Then $Y_t + I \in \mathrm{Sp}(\infty)$ for any $t > 0$ with probability* 1.

Proof. The proof is adapted from papers by M. Gordina [6, 7]. Let Y_t be the solution to Equation (6.5) and $Y_t^{\#}$ be the solution to Equation (6.6). Consider the process $\mathbf{Y}_t = (Y_t, Y_t^{\#})$ in the product space $\mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}$. It satisfies the following stochastic differential equation

$$d\mathbf{Y}_t = (B(Y_t), B^{\#}(Y_t^{\#}))dW + (F(Y_t), F^{\#}(Y_t^{\#}))dt.$$

Let G be a function on the Hilbert space $\mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}$ defined by $G(Y_1, Y_2) = \Lambda((Y_1 + I)(Y_2 + I))$, where Λ is a nonzero linear real bounded functional from $\mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}$ to $\mathbb{R}$. We will apply Itô's formula to $G(\mathbf{Y}_t) = G(Y_t, Y_t^{\#})$. Then $(Y_t + I)(Y_t^{\#} + I) = I$ if and only if $\Lambda((Y_t + I)(Y_t^{\#} + I) - I) = 0$ for any Λ.

In order to use Itô's formula we must verify that G and the derivatives G_t, $G_{\mathbf{Y}}$, $G_{\mathbf{YY}}$ are uniformly continuous on bounded subsets of $[0,T] \times \mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}$, where $G_{\mathbf{Y}}$ is defined as follows

$$G_{\mathbf{Y}}(\mathbf{Y})(\mathbf{S}) = \lim_{\epsilon \to 0} \frac{G(\mathbf{Y} + \epsilon\mathbf{S}) - G(\mathbf{Y})}{\epsilon} \quad \text{for any } \mathbf{Y}, \mathbf{S} \in \mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}$$

and $G_{\mathbf{YY}}$ is defined as follows

$$G_{\mathbf{YY}}(\mathbf{Y})(\mathbf{S} \otimes \mathbf{T}) = \lim_{\epsilon \to 0} \frac{G_{\mathbf{Y}}(\mathbf{Y} + \epsilon\mathbf{T})(\mathbf{S}) - G_{\mathbf{Y}}(\mathbf{Y})(\mathbf{S})}{\epsilon}$$

for any $\mathbf{Y}, \mathbf{S}, \mathbf{T} \in \mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}$. Let us calculate G_t, $G_{\mathbf{Y}}$, $G_{\mathbf{YY}}$. Clearly, $G_t = 0$. It is easy to verify that for any $\mathbf{S} = (S_1, S_2) \in \mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}$

$$G_{\mathbf{Y}}(\mathbf{Y})(\mathbf{S}) = \Lambda(S_1(Y_2 + I) + (Y_1 + I)S_2)$$

and for any $\mathbf{S} = (S_1, S_2) \in \mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}$ and $\mathbf{T} = (T_1, T_2) \in \mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}$

$$G_{\mathbf{YY}}(\mathbf{Y})(\mathbf{S} \otimes \mathbf{T}) = \Lambda(S_1 T_2 + T_1 S_2).$$

So the condition is satisfied.

We will use the following notation

$$G_{\mathbf{Y}}(\mathbf{Y})(\mathbf{S}) = \langle \bar{G}_{\mathbf{Y}}(\mathbf{Y}), \mathbf{S} \rangle_{\mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}}$$
$$G_{\mathbf{YY}}(\mathbf{Y})(\mathbf{S} \otimes \mathbf{T}) = \langle \bar{G}_{\mathbf{YY}}(\mathbf{Y})\mathbf{S}, \mathbf{T} \rangle_{\mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}},$$

where $\bar{G}_{\mathbf{Y}}(\mathbf{Y})$ is an element of $\mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}$ corresponding to the functional $G_{\mathbf{Y}}(\mathbf{Y})$ in $(\mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}})^*$ and $\bar{G}_{\mathbf{YY}}(\mathbf{Y})$ is an operator on $\mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}$ corresponding to the functional $G_{\mathbf{YY}}(\mathbf{Y}) \in ((\mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}) \otimes (\mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}))^*$.

Now we can apply Itô's formula to $G(\mathbf{Y}_t)$

$$\begin{aligned}
G(\mathbf{Y}_t) - G(\mathbf{Y}_0) = &\int_0^t \langle \bar{G}_{\mathbf{Y}}(\mathbf{Y}_s), \big(B(Y_s)dW_s, B^{\#}(Y_s^{\#})dW_s\big) \rangle_{\mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}} \\
&+ \int_0^t \langle \bar{G}_{\mathbf{Y}}(\mathbf{Y}_s), \big(F(Y_s), F^{\#}(Y_s^{\#})\big) \rangle_{\mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}} ds \\
&+ \int_0^t \frac{1}{2} \operatorname{Tr}_{\mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}} \Big[\bar{G}_{\mathbf{YY}}(\mathbf{Y}_s) \Big(B(Y_s)Q^{1/2}, B^{\#}(Y_s^{\#})Q^{1/2} \Big) \\
&\qquad \Big(B(Y_s)Q^{1/2}, B^{\#}(Y_s^{\#})Q^{1/2} \Big)^* \Big] ds.
\end{aligned}$$

Let us calculate the three integrands separately. The first integrand is

$$\begin{aligned}
\langle \bar{G}_{\mathbf{Y}}(\mathbf{Y}_s), &\big(B(Y_s)dW_s, B^{\#}(Y_s^{\#})dW_s\big) \rangle_{\mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}} \\
&= \Big(B(Y_s)dW_s\Big)(Y_s^{\#} + I) + (Y_s + I)\Big(B^{\#}(Y_s^{\#})dW_s\Big) \\
&= (Y_s + I)dW_s(Y_s^{\#} + I) + (Y_s + I)dW_s^{\#}(Y_s^{\#} + I) = 0.
\end{aligned}$$

The second integrand is

$$\begin{aligned}
\langle \bar{G}_{\mathbf{Y}}(\mathbf{Y}_s), &\big(F(Y_s), F^{\#}(Y_s^{\#})\big) \rangle_{\mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}} \\
&= F(Y_s)(Y_s^{\#} + I) + (Y_s + I)F^{\#}(Y_s^{\#}) \\
&= \frac{1}{2}(Y_s + I)D(Y_s^{\#} + I) + \frac{1}{2}(Y_s + I)D^{\#}(Y_s^{\#} + I) \\
&= \frac{1}{2}(Y_s + I)(D + D^{\#})(Y_s^{\#} + I) \\
&= (Y_s + I)D(Y_s^{\#} + I),
\end{aligned}$$

where we have used the fact that $D = D^{\#}$, since D is a diagonal matrix with all real entries.

The third integrand is

$$
\begin{aligned}
&\frac{1}{2}\operatorname{Tr}_{\mathfrak{sp}_{\mathrm{HS}}\times\mathfrak{sp}_{\mathrm{HS}}} \\
&\left[\bar{G}_{\mathbf{YY}}(\mathbf{Y}_s)\left(B(Y_s)Q^{1/2}, B^{\#}(Y_s^{\#})Q^{1/2}\right)\left(B(Y_s)Q^{1/2}, B^{\#}(Y_s^{\#})Q^{1/2}\right)^*\right] \\
&= \frac{1}{2}\operatorname{Tr}_{\mathfrak{sp}_{\mathrm{HS}}}\left[\left(B(Y_s)Q^{1/2}, B^{\#}(Y_s^{\#})Q^{1/2}\right)^*\bar{G}_{\mathbf{YY}}(\mathbf{Y}_s)\left(B(Y_s)Q^{1/2}, B^{\#}(Y_s^{\#})Q^{1/2}\right)\right] \\
&= \frac{1}{2}\sum_{\xi\in\mathcal{B}_{\mathfrak{sp}_{\mathrm{HS}}}} G_{\mathbf{YY}}(\mathbf{Y}_s)\Bigg(\left(B(Y_s)Q^{1/2}\xi, B^{\#}(Y_s^{\#})Q^{1/2}\xi\right) \\
&\qquad\qquad\qquad\qquad\otimes\left(B(Y_s)Q^{1/2}\xi, B^{\#}(Y_s^{\#})Q^{1/2}\xi\right)\Bigg) \\
&= \sum_{\xi\in\mathcal{B}_{\mathfrak{sp}_{\mathrm{HS}}}}\left(B(Y_s)Q^{1/2}\xi\right)\left(B^{\#}(Y_s^{\#})Q^{1/2}\xi\right) \\
&= \sum_{\xi\in\mathcal{B}_{\mathfrak{sp}_{\mathrm{HS}}}}(Y_s+I)\Big(\left(Q^{1/2}\xi\right)\left(Q^{1/2}\xi\right)^{\#}\Big)(Y_s^{\#}+I) \\
&= -(Y_s+I)D(Y_s^{\#}+I),
\end{aligned}
$$

where the second equality follows from Lemma 6.15, and the last equality follows from Lemma 6.16.

The above calculations show that the stochastic differential of G is zero. So $G(\mathbf{Y}_t) = G(\mathbf{Y}_0) = \Lambda(I)$ for any $t > 0$ and any nonzero linear real bounded functional Λ on $\mathfrak{sp}_{\mathrm{HS}} \times \mathfrak{sp}_{\mathrm{HS}}$. This means $(Y_t + I)(Y_t^{\#} + I) = I$ almost surely for any $t > 0$. Similarly we can show $(Y_t^{\#} + I)(Y_t + I) = I$ almost surely for any $t > 0$. Therefore $Y_t + I \in \mathrm{Sp}(\infty)$ almost surely for any $t > 0$. □

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