SMALL DEVIATIONS AND CHUNG'S LAW OF ITERATED LOGARITHM FOR A HYPOELLIPTIC BROWNIAN MOTION ON THE HEISENBERG GROUP

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ABSTRACT. A small ball problem and Chung's law of iterated logarithm for a hypoelliptic Brownian motion in Heisenberg group are proven. In addition, bounds on the limit in Chung's law are established.

CONTENTS

1. INTRODUCTION

Let E be a topological space and $\{X_t\}_{0 \leqslant t \leqslant 1}$ be an E-valued stochastic process with continuous paths such that $X_0 = x_0 \in E$ a.s. Denote by $W_{x_0}(E)$ the space of E-valued continuous functions on $[0, 1]$ starting at x_0, then we can view X_t as a $W_{x_0}(E)$-valued random variable. Given a norm $\| \cdot \|$ on $W_{x_0}(E)$, the small ball problem for X_t consists in finding the rate of explosion of

$$- \log \mathbb{P}(\|X\| < \varepsilon)$$

as $\varepsilon \to 0$. More precisely, a process X_t is said to satisfy a *small deviation principle* with rates α and β if there exist a constant $c > 0$ such that

(1.1) $$\lim_{\varepsilon \to 0} -\varepsilon^{\alpha} |\log \varepsilon|^{\beta} \log \mathbb{P}\left(\|X\| < \varepsilon\right) = c.$$

The values of α, β and c depend on the process X_t and on the chosen norm on $W_{x_0}(E)$. Small deviation principles have many applications including metric entropy estimates and Chung's law of the iterated logarithm. We refer to the survey paper [15] for more details. In our paper we are mostly interested in connections of a small deviation principle to Chung's law of the iterated logarithm.

We say that a process X_t satisfies *Chung's law of the iterated logarithm* with rate $a \in \mathbb{R}_+$ if there exists a constant C such that

(1.2) $$\liminf_{t \to \infty} \left(\frac{\log \log t}{t}\right)^a \max_{0 \leqslant s \leqslant t} |X_s| = C.$$

When X_t is a Brownian motion, it was proven in a famous paper by K.-L. Chung in 1948 that 1.2 holds with $a = \frac{1}{2}$ and $C = \frac{\pi}{\sqrt{8}}$. If $W_{x_0}(E)$ is a Banach space, and the law μ of X_t is a Gaussian measure on $W_{x_0}(E)$, then one can use a scaling property of the process X_t to prove Chung's law of the iterated logarithm from a small deviation principle.

Small deviation principle for a Brownian motion and related processes have been extensively studied, we mention only a few most relevant to our results. In [2] the authors considered the case of a one-dimensional Brownian motion and Hölder norms, in [14] a Brownian sheet in Hölder norms has been considered, [13] studied the integrated Brownian motion in the uniform norm, and [6] the m-fold integrated Brownian motion in both the uniform and L^2-norm. In [18] a small deviation principle and Chung's law of iterated logarithm is proven for some stochastic integrals and in particular for Lévy's stochastic area.

In the current paper we consider a hypoelliptic Brownian motion g_t on the Heisenberg group $\mathbb{H}$ starting at the identity e in $\mathbb{H}$. The group $\mathbb{H}$ is the simplest example of a sub-Riemannian manifold, and it comes with a natural left-invariant distance, the Carnot-Carathéodory distance d_{cc}. We then consider the uniform norm

$$\|g\|_{W_0(\mathbb{H})} := \max_{0 \leqslant t \leqslant 1} |g_t|$$

on the path space $W_0(\mathbb{H})$ of $\mathbb{H}$-valued continuous curves starting at the identity, where $|\cdot|$ is a norm on $\mathbb{H}$ equivalent to the Carnot-Carathéodory distance d_{cc}. We refer for details to Section 2.

Our main results include Theorem 3.2 where we prove Chung's law of the iterated logarithm with $a = \frac{1}{2}$ for a hypoelliptic Brownian motion g_t. As a consequence of Theorem 3.2, we prove Theorem 3.4 which represents a small deviation principle for the hypoelliptic diffusion g_t with respect to the norm $\|\cdot\|_{W_0(\mathbb{H})}$. More precisely, we prove that there exists a finite positive constant c such that (1.1) holds with $\alpha = 2$ and $\beta = 0$, and we provide a

lower and upper bound on c. Note that finding the constant c explicitly is difficult even in more studied cases, see for example [13, Remark 2.2].

Let us explain now how our setting differs from known results. First observe that the hypoelliptic Brownian motion g_t is an $\mathbb{R}^3$-valued stochastic process, but it is not a Gaussian process. Therefore we can not rely on the properties of Gaussian measures on Banach spaces, such as log-concavity and Anderson's inequality which are common tools in the subject. We refer to [1, 5] for more details about Gaussian measures on Banach spaces. These properties have been used to show the existence of a small deviations principle for some processes such as an integrated Brownian motion in [13], and a Brownian motion with values in a finite dimensional Banach space in [7].

Generally, if a small deviations principle is known, then it can be used together with scaling properties of the process to show Chung's law of the iterated logarithm. For example, in [18] a small deviation principle for Lévy's stochastic area A_t is first proven and then, using that $A_{\varepsilon t} \overset{(d)}{=} \varepsilon A_t$ for any t and $\varepsilon > 0$, Chung's law of the iterated logarithm for the process A_t follows. For related work we also refer to [8]. It is also possible to prove the converse. In [13] the authors first prove Chung's law of the iterated logarithm for the integrated one-dimensional Brownian motion $\int_0^t b_s ds$. Then, using the scaling property $\int_0^{\varepsilon t} b_s ds \overset{(d)}{=} \varepsilon^{\frac{3}{2}} \int_0^t b_s ds$, a small deviation principle for $\int_0^t b_s ds$ is shown.

Most relevant to our work is [13], where the existence of the limit (1.1) for $X_t := \int_0^t b_s ds$ follows from Anderson's inequality for Gaussian measures. Chung's law of the iterated logarithm is then used to prove that the limit is finite. This method can not be used directly in our setting since the hypoelliptic Brownian motion g_t is not a Gaussian process, and therefore we can not rely on Anderson's inequality. In our case we first prove in Proposition 4.3 that if the limit (1.1) exists then it is strictly positive and finite. We then prove Chung's law of the iterated logarithm for g_t and use it in place of Anderson's inequality to show the existence of the limit 1.1. As a by-product we have bounds on this limit in terms of the lowest Dirichlet eigenvalues as given in Theorem 5.6. The mathematical literature on the subject is vast, and we mention only the most relevant in terms of the techniques and results. In particular, a similar state space is considered in [16, 17] though the results are different.

The paper is organized as follows. In Section 2 we describe the Heisenberg group $\mathbb{H}$ and the corresponding sub-Laplacian and hypoelliptic Brownian motion. In Section 3 we state the main results of this paper, namely Chung's law of the iterated logarithm in Theorem 3.2 and a small deviation principle in Theorem 3.4. Section 4 contains estimates that are needed to prove Theorem 3.2 and Theorem 3.4. We conclude 5 with the proof of the main results.

2. Hypoelliptic Brownian motion on the Heisenberg group

2.1. **Heisenberg group as Lie group.** The Heisenberg group $\mathbb{H}$ as a set is $\mathbb{R}^3 \cong \mathbb{R}^2 \times \mathbb{R}$ with the group multiplication given by

$$(\mathbf{v}_1, z_1) \cdot (\mathbf{v}_2, z_2) := \left(x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2} \omega(\mathbf{v}_1, \mathbf{v}_2) \right),$$

where $\mathbf{v}_1 = (x_1, y_1)$, $\mathbf{v}_2 = (x_2, y_2) \in \mathbb{R}^2$,

$$\omega : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R},$$

$$\omega(\mathbf{v}_1, \mathbf{v}_2) := x_1 y_2 - x_2 y_1$$

is the standard symplectic form on $\mathbb{R}^2$. The identity in $\mathbb{H}$ is $e = (0, 0, 0)$ and the inverse is given by $(\mathbf{v}, z)^{-1} = (-\mathbf{v}, -z)$.

The Lie algebra of $\mathbb{H}$ can be identified with the space $\mathbb{R}^3 \cong \mathbb{R}^2 \times \mathbb{R}$ with the Lie bracket defined by

$$[(\mathbf{a}_1, c_1), (\mathbf{a}_2, c_2)] = (0, \omega(\mathbf{a}_1, \mathbf{a}_2)).$$

The set $\mathbb{R}^3 \cong \mathbb{R}^2 \times \mathbb{R}$ with this Lie algebra structure will be denoted by $\mathfrak{h}$.

Let us now recall some basic notation for Lie groups. Suppose G is a Lie group, then the left and right multiplication by an element $k \in G$ are denoted by

$$L_k : G \longrightarrow G, \qquad\qquad g \longmapsto k^{-1} g,$$
$$R_k : G \longrightarrow G, \qquad\qquad g \longmapsto gk.$$

Recall that the tangent space $T_e G$ can be identified with the Lie algebra $\mathfrak{g}$ of left-invariant vector fields on G, that is, vector fields X on G such that $dL_k \circ X = X \circ L_k$, where dL_k is the differential of L_k. More precisely, if A is a vector in $T_e G$, then we denote by $\tilde{A} \in \mathfrak{g}$ the (unique) left-invariant vector field such that $\tilde{A}(e) = A$. A left-invariant vector field is determined by its value at the identity, namely, $\tilde{A}(k) = dL_k \circ \tilde{A}(e)$.

For the Heisenberg group the differential of left and right multiplication can be described explicitly as follows.

Proposition 2.1. *Let $k = (k_1, k_2, k_3) = (\mathbf{k}, k_3)$ and $g = (g_1, g_2, g_3) = (\mathbf{g}, g_3)$ be two elements in $\mathbb{H}$. Then, for every $v = (v_1, v_2, v_3) = (\mathbf{v}, v_3)$ in $T_g \mathbb{H}$, the differentials (pushforward) of the left and right multiplication are given by*

$$dL_k = L_{k*} : T_g \mathbb{H} \longrightarrow T_{k^{-1}g} \mathbb{H},$$
$$dR_k = R_{k*} : T_g \mathbb{H} \longrightarrow T_{gk} \mathbb{H},$$
$$dL_k(v) = \left(v_1, v_2, v_3 + \frac{1}{2} \omega(\mathbf{v}, \mathbf{k}) \right),$$

(2.1)
$$dR_k(v) = \left(v_1, v_2, v_3 + \frac{1}{2} \omega(\mathbf{v}, \mathbf{k}) \right).$$

2.2. **Heisenberg group as a sub-Riemannian manifold.** The Heisenberg group $\mathbb{H}$ is the simplest non-trivial example of a sub-Riemannian manifold. We define X, Y and Z as the unique left-invariant vector fields satisfying $X_e = \partial_x$, $Y_e = \partial_y$ and $Z_e = \partial_z$ which are given by

$$X = \partial_x - \frac{1}{2}y\partial_z,$$
$$Y = \partial_y + \frac{1}{2}x\partial_z,$$
$$Z = \partial_z.$$

Note that the only non-zero Lie bracket for these left-invariant vector fields is $[X, Y] = Z$, so the vector fields $\{X, Y\}$ satisfy Hörmander's condition. We define the *horizontal distribution* as $\mathcal{H} := \operatorname{span}\{X, Y\}$ fiberwise, thus making $\mathcal{H}$ a sub-bundle in the tangent bundle $T\mathbb{H}$. To finish the description of the Heisenberg group as a sub-Riemannian manifold we need to equip the horizontal distribution $\mathcal{H}$ with an inner product. For any $p \in \mathbb{H}$ we define the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_p}$ on $\mathcal{H}_p$ so that $\{X(p), Y(p)\}$ is an orthonormal (horizontal) frame at any $p \in \mathbb{H}$. Vectors in $\mathcal{H}_p$ will be called *horizontal*, and the corresponding norm is denoted by $\|\cdot\|_{\mathcal{H}_p}$.

In addition, Hörmander's condition ensures that a natural sub-Laplacian on the Heisenberg group

$$(2.2) \qquad \Delta_{\mathcal{H}} = X^2 + Y^2$$

is a hypoelliptic operator by [10].

We recall now another notion in sub-Riemannian geometry, namely, of *horizontal curves*. Suppose $\gamma(t) = (x(t), y(t), z(t)) = (\mathbf{x}(t), z(t))$ is an absolutely continuous curve with values in $\mathbb{H}$, and the corresponding tangent vector $\gamma'(t)$ in $T\mathbb{H}_{\gamma(t)}$ is

$$\gamma'(t) = \left(x'(t), y'(t), z'(t)\right) = \left(\boldsymbol{x}'(t), z'(t)\right).$$

We denote by c_g the *Maurer–Cartan form* on $\mathbb{H}$, i.e. the $\mathfrak{h}$-valued 1-form on $\mathbb{H}$ defined by $c_g(v) = dL_g v$, $v \in T_g\mathbb{H}$. Note that the pushforward of a vector in $T_g\mathbb{H}$ along the left translation can be found explicitly. Namely for $\gamma(t) = (\mathbf{x}(t), z(t))$ the Maurer-Cartan form is

$$(2.3) \qquad c_\gamma(t) = c(t) = dL_{\gamma(t)}\left(\gamma'(t)\right)$$
$$= \left(\mathbf{x}'(t), z'(t) - \frac{1}{2}\omega(\mathbf{x}(t), \mathbf{x}'(t))\right),$$

where we used Proposition 2.1.

Definition 2.2. An absolutely continuous curve $t \longmapsto \gamma(t) \in \mathbb{H}, t \in [0, 1]$ is said to be *horizontal* if $\gamma'(t) \in \mathcal{H}_{\gamma(t)}$ for a.e. t, that is, the tangent vector to $\gamma(t)$ is horizontal for a.e. t. Equivalently we can say that γ is horizontal if $c_\gamma(t) \in \mathcal{H}_e$ for a.e. t.

Equation (2.3) can be used to characterize horizontal curves in terms of the components as follows. An absolutely continuous curve γ is horizontal if and only if

$$(2.4) \qquad z'(t) - \frac{1}{2}\omega(\mathbf{x}(t), \mathbf{x}'(t))) = 0 \text{ a.e. } t.$$

The *horizontal length* is defined as

$$L_{\mathcal{H}}(\gamma) := \int_0^1 |c_\gamma(s)|_{\mathcal{H}_e} ds,$$

where we set $L_{\mathcal{H}}(\gamma) = \infty$ if γ is not horizontal. The Heisenberg group as a sub-Riemannian manifold comes with a natural left-invariant distance.

Definition 2.3. For any $g_1, g_2 \in \mathbb{H}$ the *Carnot-Carathéodory distance* is defined as

$$d_{cc}(g_1, g_2) := \inf \{L(\gamma), \gamma : [0, 1] \longrightarrow \mathbb{H}, \gamma(0) = g_1, \gamma(1) = g_2\}.$$

Another consequence of Hörmander's condition for left-invariant vector fields X, Y and Z is that by the Chow–Rashevskii theorem there exists a horizontal curve connecting any two points in $\mathbb{H}$, and therefore the Carnot-Carathéodory distance is finite on $\mathbb{H}$.

In addition to the Carnot-Carathéodory distance on the Heisenberg group, we will use the following homogeneous distance

$$(2.5) \qquad \rho(g_1, g_2) := \left(\|\mathbf{x}_1 - \mathbf{x}_2\|_{\mathbb{R}^2}^4 + |z_1 - z_2 + \omega(\mathbf{x}_1, \mathbf{x}_2)|^2\right)^{\frac{1}{4}},$$

which is equivalent to the Carnot-Carathéodory distance, that is, there exist two positive constants c and C such that

$$(2.6) \qquad c\rho(g_1, g_2) \leqslant d_{cc}(g_1, g_2) \leqslant C\rho(g_1, g_2)$$

for all $g_1, g_2 \in \mathbb{H}$. We denote by $|\cdot|$ the norm on $\mathbb{H}$ induced by ρ, that is, $|g| = \rho(g, e)$ for all $g \in \mathbb{H}$. In particular, by the left-invariance of ρ we have that for any $g_1, g_2 \in \mathbb{H}$

$$(2.7) \quad |g_2^{-1}g_1| = \rho\left(g_2^{-1}g_1, e\right) = \rho(g_1, g_2) \leqslant \rho(g_1, e) + \rho(g_2, e) = |g_1| + |g_2|.$$

This is discussed in a more general setting in [4, Proposition 5.1.4].

Finally, we need to describe a hypoelliptic Brownian motion with values in $\mathbb{H}$. This is a stochastic process whose generator is the sub-Laplacian $\frac{1}{2}\Delta_{\mathcal{H}}$ defined by Equation (2.2).

Notation 2.4. Throughout the paper we use the following notation. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space. We denote the expectation under $\mathbb{P}$ by $\mathbb{E}$.

By a standard real-valued Brownian motion $\{B_t\}_{t \geqslant 0}$ we mean a continuous adapted $\mathbb{R}$-valued stochastic process on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ such that for all $0 \leqslant s \leqslant t$ the increment $B_t - B_s$ is independent of $\mathcal{F}_s$ and has a normal distribution with mean 0 and the variance $t - s$.

Definition 2.5. Let $W_t = (W_1(t), W_2(t), 0)$ be an $\mathfrak{h}$-valued stochastic process, where $\boldsymbol{W}_t := (W_1(t), W_2(t))$ is a standard two-dimensional Brownian motion. A hypoelliptic Brownian motion $g_t = (g_1(t), g_2(t), g_3(t))$ on $\mathbb{H}$ is the continuous $\mathbb{H}$-valued process defined by

$$(2.8) \qquad\qquad g_t := (\boldsymbol{W}_t, A_t),$$

where $A_t := \frac{1}{2} \int_0^t \omega\left(\boldsymbol{W}_s, d\boldsymbol{W}_s\right)$ is Lévy's stochastic area.

Note that we used Itô's integral in this definition rather than Stratonovich' integral. However, these two integrals are equal in our setting since the symplectic form ω is skew-symmetric, and therefore Lévy's stochastic area functional is the same for both integrals as was observed in [9, Remark 4.3].

One can also write a stochastic differential equation for $g_t = (x_t, y_t, z_t)$, $g_0 = (0, 0, 0) = e \in \mathbb{H}$ as a stochastic differential equation for a Lie group-valued Brownian motion

$$(2.9) \qquad\qquad L_{g_t*}(dg_t) = g_t^{-1} dg_t = dW_t,$$
$$g_0 = e.$$

Equation (2.8) gives an explicit solution to this stochastic differential equation for the Heisenberg group.

3. Main results

Notation 3.1. *Let X_t be a stochastic process with values in a metric space $(\mathfrak{X}, d)$ with $X_0 = x \in \mathfrak{X}$, then X_t^* denotes the process defined by*

$$X_t^* := \max_{0 \leqslant s \leqslant t} d\left(X_s, X_0\right).$$

For $\mathfrak{X} = \mathbb{H}$ we use the homogeneous distance ρ with $X_0 = e$, and on $\mathfrak{X} = \mathbb{R}^n$ we consider the standard Euclidean norm. Before formulating Chung's law of iterated logarithm for the hypoelliptic Brownian motion g_t we introduce the notation

$$\phi(t) := \sqrt{\frac{\log \log t}{t}}.$$

Theorem 3.2 (Chung's law of iterated logarithm)**.** *Let g_t be the hypoelliptic Brownian motion on the Heisenberg group $\mathbb{H}$ defined by (2.8). Then there exists a constant $c \in (0, \infty)$ such that*

$$(3.1) \qquad\qquad \liminf_{t \in \infty} \phi(t) g_t^* = c \quad a.s.$$

Remark 3.3. *Note that the hypoelliptic Brownian motion g_t has the same scaling property with respect to the norm induced by the homogeneous norm ρ as a standard Brownian motion in a Euclidean space. Indeed,*

$$|g_{\varepsilon t}| := \rho\left(g_{t\varepsilon}, e\right) = \sqrt[4]{|B_{t\varepsilon}|^4 + A_{t\varepsilon}^2}$$
$$\overset{(d)}{=} \sqrt[4]{|\sqrt{\varepsilon}B_t|^4 + (\varepsilon A_t)^2} = \sqrt{\varepsilon}\rho\left(g_t, e\right) = \sqrt{\varepsilon}|g_t|.$$

Therefore it is not surprising that the process g_t and the standard Brownian motion have the same rate $\phi(t)$ in Chung's law of iterated logarithm.

As a consequence of Theorem 3.2 we can prove a small deviation principle for g_t.

Theorem 3.4 (Small deviation principle)**.** *The limit*

(3.2) $$\lim_{\varepsilon \to 0} -\varepsilon^2 \log \mathbb{P}\left(g_1^* < \varepsilon\right) = c^2$$

exists with constant c being defined by (3.1).

Remark 3.5. *Using scaling properties of g_t we can formulate a small deviation principle over an interval $[0, T]$. Let $T > 0$ be fixed, then by Theorem 3.4 and Remark 3.3 we have that*

$$\lim_{\varepsilon \to 0} -\varepsilon^2 \log \mathbb{P}\left(g_T^* < \varepsilon\right) = c^2 T.$$

Indeed,

$$\lim_{\varepsilon \to 0} -\frac{\varepsilon^2}{T} \log \mathbb{P}\left(g_T^* < \varepsilon\right) = \lim_{\varepsilon \to 0} -\frac{\varepsilon^2}{T} \log \mathbb{P}\left(g_1^* < \frac{\varepsilon}{\sqrt{T}}\right) = c^2.$$

4. Preliminary estimates

We collect here several preliminary estimates that will be used throughout the paper.

Proposition 4.1. *Let Y_t be a positive real-valued process and assume there exist two finite positive constants $0 < a \leqslant b < \infty$ such that*

(4.1) $$\liminf_{t \to \infty} -\frac{1}{t} \log \mathbb{P}\left(Y_t < 1\right) \geqslant a$$

(4.2) $$\limsup_{t \to \infty} -\frac{1}{t} \log \mathbb{P}\left(Y_t < 1\right) \leqslant b.$$

Let c, x and y be real numbers such that $c > 1$, and $0 < x < a \leqslant b < y$, then there exists an $n_0 \in \mathbb{N}$ such that

(4.3) $$\sum_{n=n_0}^{\infty} \mathbb{P}\left(Y_{s_n} < 1\right) < \infty$$

where $s_n := \frac{1}{x} \log\log c^n$, and

(4.4) $$\sum_{n=n_0}^{\infty} \mathbb{P}\left(Y_{v_n} < 1\right) = \infty$$

where v_n is any positive sequence such that $v_n \to \infty$ as $n \to \infty$, and $v_n \leqslant \frac{1}{y} \log\log n^n$ for all $n \geqslant n_0$.

Proof. Let us first show (4.3). By (4.1) we have

$$\mathbb{P}\left(Y_t < 1\right) \leqslant e^{-at}$$

for all large enough t. Therefore there exists an $n_1 \in \mathbb{N}$ such that

$$\sum_{n=n_1}^{\infty} \mathbb{P}\left(Y_{s_n} < 1\right) \leqslant \sum_{n=n_1}^{\infty} e^{-as_n}$$

$$= \sum_{n=n_1}^{\infty} e^{-\frac{a}{x}\log\log c^n} = \sum_{n=n_1}^{\infty}\left(\frac{1}{n\log c}\right)^{\frac{a}{x}}$$

which is a convergent series since $x < a$.

Let us now show (4.4). By (4.1) we have that

$$\mathbb{P}\left(Y_t < 1\right) \geqslant e^{-bt}$$

for all large enough t, and hence there exists an $n_2 \in \mathbb{N}$ such that

$$\sum_{n=n_2}^{\infty} \mathbb{P}\left(Y_{v_n} < 1\right) \geqslant \sum_{n=n_2}^{\infty} e^{-bv_n}$$

$$\geqslant \sum_{n=n_2}^{\infty} e^{-\frac{b}{y}\log\log n^n} = \sum_{n=n_2}^{\infty}\left(\frac{1}{n\log n}\right)^{\frac{b}{y}}$$

which is divergent since $b < y$. The proof is then completed by taking $n_0 := \max\left(n_1, n_2\right)$. $\qquad\square$

We first prove a weaker version of Theorem 3.4, namely that if the limit in (3.2) exists, then it is finite and strictly positive. The estimates in Proposition 4.3 will be used in the proof of Chung's law of iterated logarithm. First we introduce the following notation.

Notation 4.2 (Dirichlet eigenvalues in $\mathbb{R}^n$). *We denote by $\lambda_1^{(n)}$ the lowest Dirichlet eigenvalue of $-\frac{1}{2}\Delta_{\mathbb{R}^n}$ on the unit ball in $\mathbb{R}^n$, where $0 < \lambda_1^{(n)} \leqslant \lambda_2^{(n)} \leqslant ...$ are Dirichlet eigenvalues for the Laplacian $-\frac{1}{2}\Delta_{\mathbb{R}^n}$ in the unit ball $D := \{x \in \mathbb{R}^n, |x| < 1\}$.*

Recall that the lowest Dirichlet eigenvalues appear in a small deviation principle for a Brownian motion in $\mathbb{R}^n$, see e.g. [11, Lemma 8.1]. Namely, suppose b_t is a standard Brownian motion in $\mathbb{R}^n$, then

$$(4.5) \qquad \lim_{\varepsilon \to 0} -\varepsilon^2 \log \mathbb{P}\left(b_1^* < \varepsilon\right) = \lambda_1^{(n)},$$

where $\lambda_1^{(n)}$ is as in Notation 4.2, and

$$b_1^* := \max_{0 \leqslant t \leqslant 1} |b_t|_{\mathbb{R}^n}.$$

Proposition 4.3. *Let g_t be the hypoelliptic Brownian motion on the Heisenberg group $\mathbb{H}$. Set*

$$c_- := \liminf_{\varepsilon \to 0} -\varepsilon^2 \log \mathbb{P}\left(g_1^* < \varepsilon\right),$$
$$c_+ := \limsup_{\varepsilon \to 0} -\varepsilon^2 \log \mathbb{P}\left(g_1^* < \varepsilon\right).$$

Then

$$(4.6) \qquad \lambda_1^{(2)} \leqslant c_- \leqslant c_+ \leqslant c\left(\lambda_1^{(1)}, \lambda_1^{(2)}\right),$$

where

$$c\left(\lambda_1^{(1)}, \lambda_1^{(2)}\right) := f(x^*) = \inf_{x \in (0,1)} f(x),$$

$$f(x) = \frac{\lambda_1^{(2)}}{\sqrt{1-x}} + \frac{\lambda_1^{(1)}\sqrt{1-x}}{4x},$$

$$x^* = \frac{\sqrt{(\lambda_1^{(1)})^2 + 32\lambda_1^{(1)}\lambda_1^{(2)}} - 3\lambda_1^{(1)}}{2\left(4\lambda_1^{(2)} - \lambda_1^{(1)}\right)},$$

and $\lambda_1^{(n)}$ are the lowest Dirichlet eigenvalues on the unit ball as defined in Notation 4.2.

Proof. The lower bound in (4.6) follows from the small deviation principle 4.5 for $\mathbb{R}^n$-valued Brownian motion and the fact that $\mathbb{P}\left(g_1^* < \varepsilon\right) \leqslant \mathbb{P}\left(B_1^* < \varepsilon\right)$.

Let us prove now the upper bound. For any $x \in (0,1)$ we have

$$\mathbb{P}\left(g_1^* < \varepsilon\right) = \mathbb{P}\left(\max_{0 \leqslant s \leqslant 1} \left(|B_s|_{\mathbb{R}^2}^4 + |A_s|_{\mathbb{R}}^2\right) < \varepsilon^4\right)$$
$$\geqslant \mathbb{P}\left(B_1^* < (1-x)^{\frac{1}{4}}\varepsilon, \ A_1^* < \sqrt{x}\varepsilon^2\right).$$

It is well-known that $A_t = b_{\tau(t)}$ where b_t is a one-dimensional Brownian motion independent of B_t, and $\tau(t) = \frac{1}{4}\int_0^t |B_s|_{\mathbb{R}^2}^2 ds$, see for example [11, Chapter 7, Section 6, Example 6.1]. Therefore we have

$$\mathbb{P}\left(B_1^* < \varepsilon (1-x)^{\frac{1}{4}}, \sup_{0 \leqslant t \leqslant 1} |b_{\tau(t)}|_{\mathbb{R}} < \varepsilon^2 \sqrt{x}\right)$$

$$= \mathbb{P}\left(B_1^* < \varepsilon (1-x)^{\frac{1}{4}}, \sup_{0 \leqslant t \leqslant \tau(1)} |b_t|_{\mathbb{R}} < \varepsilon^2 \sqrt{x}\right)$$

$$\geqslant \mathbb{P}\left(B_1^* < \varepsilon (1-x)^{\frac{1}{4}}, \sup_{0 \leqslant t \leqslant \frac{\varepsilon^2}{4}(1-x)^{\frac{1}{2}}} |b_t|_{\mathbb{R}} < \varepsilon^2 \sqrt{x}\right)$$

$$= \mathbb{P}\left(B_1^* < \varepsilon (1-x)^{\frac{1}{4}}, b_1^* < \frac{2\varepsilon\sqrt{x}}{(1-x)^{\frac{1}{4}}}\right)$$

$$= \mathbb{P}\left(B_1^* < \varepsilon (1-x)^{\frac{1}{4}}\right) \mathbb{P}\left(b_1^* < \frac{2\varepsilon\sqrt{x}}{(1-x)^{\frac{1}{4}}}\right).$$

Thus

$$\log \mathbb{P}\left(g_1^* < \varepsilon\right) \geqslant \log \mathbb{P}\left(B_1^* < \varepsilon (1-x)^{\frac{1}{4}}\right) + \log \mathbb{P}\left(b_1^* < \frac{2\varepsilon\sqrt{x}}{(1-x)^{\frac{1}{4}}}\right),$$

and hence

$$- \varepsilon^2 \log \mathbb{P}\left(g_1^* < \varepsilon\right) \leqslant -\varepsilon^2 (1-x)^{\frac{1}{2}} \log \mathbb{P}\left(B_1^* < \varepsilon (1-x)^{\frac{1}{4}}\right) \frac{1}{(1-x)^{\frac{1}{2}}}$$

$$- \varepsilon^2 \frac{4x}{(1-x)^{\frac{1}{2}}} \log \mathbb{P}\left(b_1^* < \frac{2\sqrt{x}}{(1-x)^{\frac{1}{4}}}\varepsilon\right) \frac{(1-x)^{\frac{1}{2}}}{4x}.$$

From the small deviation principle (4.5) for a $\mathbb{R}^n$-valued Brownian motion applied to B_t and b_t it follows that

$$\limsup_{\varepsilon \to 0} -\varepsilon^2 \log \mathbb{P}\left(g_1^* < \varepsilon\right) \leqslant \frac{\lambda_1^{(2)}}{\sqrt{1-x}} + \frac{\lambda_1^{(1)}\sqrt{1-x}}{4x}$$

for all x in $(0,1)$. Note that

$$f(x) := \frac{\lambda_1^{(2)}}{\sqrt{1-x}} + \frac{\lambda_1^{(1)}\sqrt{1-x}}{4x} > 0 \text{ for all } x \in (0,1)$$

always has a local minimum over $(0,1)$ even if we do not rely on the known values of the eigenvalues $\lambda_1^{(2)}$ and $\lambda_1^{(1)}$. It is easy to see that this minimum is achieved at

$$x^* = \frac{\sqrt{(\lambda_1^{(1)})^2 + 32\lambda_1^{(1)}\lambda_1^{(2)}} - 3\lambda_1^{(1)}}{2\left(4\lambda_1^{(2)} - \lambda_1^{(1)}\right)} \in (0,1)$$

which gives Equation (4.6). □

5. Proof of the main results

5.1. **Chung's law of iterated logarithm for** g_t. The goal of this section is to prove Theorem 3.2. Later in Proposition 5.5 we prove that $c := \liminf_{t \to \infty} \phi(t) g_t^*$ is constant a.s., where $\phi(t) := \sqrt{\frac{\log \log t}{t}}$. For now c is a random variable for which we first show lower and upper bounds in Proposition 5.1 and Proposition 5.3.

Proposition 5.1 (Lower bound). *For the lowest eigenvalue* $\lambda_1^{(2)}$ *as introduced in Notation 4.2 we have*

$$c = \liminf_{t \in \infty} \phi(t) g_t^* \geqslant \sqrt{\lambda_1^{(2)}} \quad a.s.$$

Proof. While this proof is motivated by [18], we provide a detailed argument for completeness. Let $r > 0$ be such that $0 < r < \sqrt{\lambda_1^{(2)}}$. Then we can find a constant $M > 1$ such that $rM < \sqrt{\lambda_1^{(2)}}$. We will show that

$$\mathbb{P}(c < r) = 0 \text{ for all } 0 < r < \sqrt{\lambda_1^{(2)}}.$$

We have

$$\mathbb{P}(c < r) = \mathbb{P}\left(\liminf_{t \to \infty} \phi(t) g_t^* < r\right) \leqslant \mathbb{P}\left(\bigcap_{k \geqslant 1} \bigcup_{n \geqslant k} \left\{ \inf_{M^n \leqslant t \leqslant M^{n+2}} \phi(t) g_t^* < r \right\}\right)$$

$$\leqslant \mathbb{P}\left(\bigcap_{k \geqslant 1} \bigcup_{n \geqslant k} \left\{ \frac{1}{M} \phi(M^n) g_{M^n}^* < r \right\}\right) = \mathbb{P}\left(\bigcap_{k \geqslant 1} \bigcup_{n \geqslant k} \left\{ g^*\left(\frac{M^{n-2}}{r^2} \phi(M^n)^2\right) < 1 \right\}\right),$$

where $g^*\left(\frac{M^{n-2}}{r^2}\phi(M^n)^2\right) := g^*_{\frac{M^{n-2}}{r^2}\phi(M^n)^2}$. Here we used that

$$\inf_{a \leqslant t \leqslant b} \phi(t) g_t^* \geqslant \frac{\sqrt{a}}{\sqrt{b}} \phi(a) g_a^*$$

for any $0 < a < b < \infty$. It is enough to show that

$$\sum_{n=1}^{\infty} \mathbb{P}\left(g^*\left(\frac{M^{n-2}}{r^2}\phi(M^n)^2\right) < 1\right) < \infty,$$

and then the result follows from the Borel-Cantelli Lemma. By (4.6) and the scaling property of g_t, it follows that

$$\lambda_1^{(2)} \leqslant \liminf_{\varepsilon \to 0} -\varepsilon^2 \mathbb{P}(g_1^* < \varepsilon) = \liminf_{\varepsilon \to 0} -\varepsilon^2 \mathbb{P}\left(\max_{0 \leqslant s \leqslant 1} |g_{\frac{1}{\varepsilon^2}s}| < 1\right)$$

$$= \liminf_{t \to \infty} -\frac{1}{t} \mathbb{P}\left(\max_{0 \leqslant s \leqslant t} |g_s| < 1\right) = \liminf_{t \to \infty} -\frac{1}{t} \mathbb{P}(g_t^* < 1).$$

Moreover,

$$\frac{M^{n-2}}{r^2}\phi\left(M^n\right)^2 = \frac{1}{M^2 r^2}\log\log M^n,$$

and hence we can apply Proposition 4.1 with $Y_t = g_t^*$, $a = \lambda_1^{(2)}$, $s_n = \frac{1}{x}\log\log M^n$, and $x = M^2 r^2$, since $M^2 r^2 < \lambda_1^{(2)}$. $\square$

Our next step is to show that c is finite almost surely. To do so, we need the following Lemma.

Lemma 5.2. *Set $t_n = n^n$. Then for every $\varepsilon > 0$*

$$(5.1) \qquad \mathbb{P}\left(\bigcap_{k \geqslant 1}\bigcup_{n \geqslant k}\left\{\phi\left(t_n\right)g_{t_{n-1}}^* > \varepsilon\right\}\right) = 0$$

Proof. It is enough to show that for any $\varepsilon > 0$

$$(5.2) \qquad \mathbb{P}\left(\bigcap_{k \geqslant 1}\bigcup_{n \geqslant k}\left\{\phi\left(t_n\right)B_{t_{n-1}}^* > \varepsilon\right\}\right) = 0 \text{ and}$$

$$(5.3) \qquad \mathbb{P}\left(\bigcap_{k \geqslant 1}\bigcup_{n \geqslant k}\left\{\phi\left(t_n\right)^2 A_{t_{n-1}}^* > \varepsilon\right\}\right) = 0.$$

Indeed,

$$\left\{\phi\left(t_n\right)g_{t_{n-1}}^* > \varepsilon\right\} = \left\{\phi\left(t_n\right)^4 \max_{0 \leqslant s \leqslant t_{n-1}}\left(|B_s|_{\mathbb{R}^2}^4 + |A_s|^2\right) > \varepsilon^4\right\}$$

$$\subset \left\{B_{t_{n-1}}^* > \frac{\varepsilon}{\sqrt[4]{2}\phi\left(t_n\right)}\right\} \cup \left\{A_{t_{n-1}}^* > \frac{\varepsilon^2}{\sqrt{2}\phi\left(t_n\right)^2}\right\},$$

and hence

$$\mathbb{P}\left(\bigcap_{k \geqslant 1}\bigcup_{n \geqslant k}\left\{\phi\left(t_n\right)g_{t_{n-1}}^* > \varepsilon\right\}\right)$$

$$\leqslant \mathbb{P}\left(\bigcap_{k \geqslant 1}\bigcup_{n \geqslant k}\left\{\phi\left(t_n\right)B_{t_{n-1}}^* > \frac{\varepsilon}{\sqrt[4]{2}}\right\}\right)$$

$$+ \mathbb{P}\left(\bigcap_{k \geqslant 1}\bigcup_{n \geqslant k}\left\{\phi\left(t_n\right)^2 A_{t_{n-1}}^* > \frac{\varepsilon^2}{\sqrt{2}}\right\}\right).$$

Let us first prove (5.2). For every ε fixed we have that

$$\left\{ \omega \; : \; \limsup_{n \to \infty} \phi\left(t_n\right) \max_{0 \leqslant s \leqslant t_{n-1}} |B_s(\omega)|_{\mathbb{R}^2} = 0 \right\}$$
$$\subset \bigcup_{k \geqslant 1} \bigcap_{n \geqslant k} \left\{ \omega \; : \; \phi\left(t_n\right) \max_{0 \leqslant s \leqslant t_{n-1}} |B_s(\omega)|_{\mathbb{R}^2} < \varepsilon \right\}.$$

Moreover, by [12, Lemma 1] we have that for $t_n = n^n$

$$\limsup_{n \to \infty} \phi\left(t_n\right) \max_{0 \leqslant s \leqslant t_{n-1}} |B_s|_{\mathbb{R}^2} = 0 \quad \text{a.s..}$$

Combining everything together we have that for $t_n = n^n$

$$1 = \mathbb{P}\left(\limsup_{n \to \infty} \phi\left(t_n\right) \max_{0 \leqslant s \leqslant t_{n-1}} |B_s|_{\mathbb{R}^2} = 0 \right)$$
$$\leqslant \mathbb{P}\left(\bigcup_{k \geqslant 1} \bigcap_{n \geqslant k} \left\{ \phi\left(t_n\right) \max_{0 \leqslant s \leqslant t_{n-1}} |B_s(\omega)|_{\mathbb{R}^2} < \varepsilon \right\} \right),$$

and (5.2) is proven since

$$\mathbb{P}\left(\bigcap_{k \geqslant 1} \bigcup_{n \geqslant k} \left\{ \phi\left(t_n\right) B_{t_{n-1}}^* > \varepsilon \right\} \right)$$
$$= 1 - \mathbb{P}\left(\bigcup_{k \geqslant 1} \bigcap_{n \geqslant k} \left\{ \phi\left(t_n\right) \max_{0 \leqslant s \leqslant t_{n-1}} |B_s(\omega)|_{\mathbb{R}^2} < \varepsilon \right\} \right).$$

Let us now prove (5.3). It follows from [3, Example on pp. 449-451] that there exists a finite constant $d > 0$ such that with probability one we have $A_{t_{n-1}}^* \leqslant d\, t_{n-1}^2 \phi\left(t_{n-1}\right)^2$ eventually, that is,

$$\mathbb{P}\left(\bigcup_{k \geqslant 1} \bigcap_{n \geqslant k} \left\{ A_{t_{n-1}}^* \leqslant d\, t_{n-1}^2 \phi\left(t_{n-1}\right)^2 \right\} \right) = 1.$$

For any $\varepsilon > 0$ there exists an N_ε such that for any $n \geqslant N_\varepsilon$ we have that $d\, t_{n-1}^2 \phi\left(t_{n-1}\right)^2 \phi\left(t_n\right)^2 \leqslant \varepsilon$. Set

$$E_k := \bigcap_{n \geqslant k} \left\{ \phi\left(t_n\right)^2 A_{t_{n-1}}^* \leqslant d\, t_{n-1}^2 \phi\left(t_{n-1}\right)^2 \phi\left(t_n\right)^2 \right\}.$$

Then the family E_k is an increasing sequence of sets, and it the follows that

$$\bigcup_{k \geqslant 1} \bigcap_{n \geqslant k} \left\{ A^*_{t_{n-1}} \leqslant d\, t^2_{n-1} \phi\, (t_{n-1})^2 \right\} = \bigcup_{k \geqslant 1} E_k \subset \bigcup_{k \geqslant N_\varepsilon} E_k$$

$$= \bigcup_{k \geqslant N_\varepsilon} \bigcap_{n \geqslant k} \left\{ \phi\, (t_n)^2\, A^*_{t_{n-1}} \leqslant d\, t^2_{n-1} \phi\, (t_{n-1})^2\, \phi\, (t_n)^2 \right\}$$

$$\subset \bigcup_{k \geqslant N_\varepsilon} \bigcap_{n \geqslant k} \left\{ \phi\, (t_n)^2\, A^*_{t_{n-1}} \leqslant \varepsilon \right\} \subset \bigcup_{k \geqslant 1} \bigcap_{n \geqslant k} \left\{ \phi\, (t_n)^2\, A^*_{t_{n-1}} \leqslant \varepsilon \right\}.$$

Therefore for any $\varepsilon > 0$ we have that

$$\mathbb{P} \left(\bigcup_{k \geqslant 1} \bigcap_{n \geqslant k} \left\{ \phi\, (t_n)^2\, A^*_{t_{n-1}} \leqslant \varepsilon \right\} \right) = 1,$$

and so (5.3) is proven. $\qquad \square$

Proposition 5.3 (Upper bound). *Let* $c\left(\lambda^{(1)}_1, \lambda^{(2)}_1 \right)$ *be as in Proposition 4.3, then*

$$c = \liminf_{t \in \infty} \phi\, (t)\, g^*_t \leqslant \sqrt{ c\left(\lambda^{(1)}_1, \lambda^{(2)}_1 \right) } \quad a.s.$$

Proof. Set $t_n = n^n$. We will show that $\mathbb{P}\, (c > r) = 0$ for any $r > \sqrt{ c\left(\lambda^{(1)}_1, \lambda^{(2)}_1 \right) }$. Since

$$\mathbb{P} \left(\liminf_t \phi(t) g^*_t > r \right) \leqslant \mathbb{P} \left(\bigcup_{k \geqslant 1} \bigcap_{n \geqslant k} \left\{ \phi(t_n) g^*_{t_n} > r \right\} \right)$$

$$= 1 - \mathbb{P} \left(\bigcap_{k \geqslant 1} \bigcup_{n \geqslant k} \left\{ \phi(t_n) g^*_{t_n} \leqslant r \right\} \right),$$

it is sufficient to show that for any $r > \sqrt{ c\left(\lambda^{(1)}_1, \lambda^{(2)}_1 \right) }$

$$(5.4) \qquad \mathbb{P} \left(\bigcap_{k \geqslant 1} \bigcup_{n \geqslant k} \left\{ \phi\, (t_n)\, g^*_{t_n} \leqslant r \right\} \right) = 1.$$

Fix $r > \sqrt{ c\left(\lambda^{(1)}_1, \lambda^{(2)}_1 \right) }$ and choose r_1 such that $\sqrt{ c\left(\lambda^{(1)}_1, \lambda^{(2)}_1 \right) } < r_1 < r$. Let us define the events

$$A_n := \left\{ \phi\, (t_n) \max_{t_{n-1} \leqslant s \leqslant t_n} |g^{-1}_{t_{n-1}} g_s| < r_1 \right\},$$

$$B_n := \left\{ \phi\, (t_n)\, g^*_{t_{n-1}} < \frac{r - r_1}{2} \right\}.$$

Then by (2.7)on the event $A_n \cap B_n$ we have

$$
\begin{aligned}
\phi\left(t_n\right) g_{t_n}^* &\leqslant \phi\left(t_n\right) g_{t_{n-1}}^* + \phi\left(t_n\right) \max_{t_{n-1} \leqslant s \leqslant t_n} |g_{t_{n-1}} g_{t_{n-1}}^{-1} g_s| \\
&\leqslant \phi\left(t_n\right) g_{t_{n-1}}^* + \phi\left(t_n\right) |g_{t_{n-1}}| + \phi\left(t_n\right) \max_{t_{n-1} \leqslant s \leqslant t_n} |g_{t_{n-1}}^{-1} g_s| \\
&\leqslant \frac{r - r_1}{2} + \frac{r - r_1}{2} + r_1 = r,
\end{aligned}
$$

and hence

$$
\mathbb{P}\left(A_n \cap B_n\right) \leqslant \mathbb{P}\left(\phi\left(t_n\right) g_{t_n}^* \leqslant r\right).
$$

By Lemma 5.2 we have $\mathbb{P}\left(\bigcap_{k \geqslant 1} \bigcup_{n \geqslant k} B_n^c\right) = 0$, and therefore

$$
\begin{aligned}
& \mathbb{P}\left(\bigcap_{k \geqslant 1} \bigcup_{n \geqslant k}\left\{\phi\left(t_n\right) \max_{0 \leqslant s \leqslant t_n - t_{n-1}} |g_{t_{n-1}}^{-1} g_{s+t_{n-1}}| < r_1\right\}\right) \\
&= \mathbb{P}\left(\bigcap_{k \geqslant 1} \bigcup_{n \geqslant k} A_n\right) = \mathbb{P}\left(\bigcap_{k \geqslant 1} \bigcup_{n \geqslant k}\left(A_n \cap B_n\right)\right) \\
&\leqslant \mathbb{P}\left(\bigcap_{k \geqslant 1} \bigcup_{n \geqslant k}\left\{\phi\left(t_n\right) g^*\left(t_n\right) \leqslant r\right\}\right).
\end{aligned}
$$

Equation (5.4) holds if we can show that

$$
\mathbb{P}\left(\bigcap_{k \geqslant 1} \bigcup_{n \geqslant k}\left\{\phi\left(t_n\right) \max_{0 \leqslant s \leqslant t_n - t_{n-1}} |g_{t_{n-1}}^{-1} g_{s+t_{n-1}}| < r_1\right\}\right) = 1.
$$

Note that $g_{t_{n-1}}^{-1} g_{s+t_{n-1}} \overset{(d)}{=} g_s$ and it is independent of $\mathcal{F}_{t_{n-1}}$. Indeed,

(5.5)

$$
\begin{aligned}
& g_{t_{n-1}}^{-1} g_{s+t_{n-1}} \\
&= \left(B_{s+t_{n-1}} - B_{t_{n-1}}, \frac{1}{2} \int_{t_{n-1}}^{s+t_{n-1}} \omega\left(B_u, dB_u\right) + \frac{1}{2} \omega\left(B_{s+t_{n-1}}, B_{t_{n-1}}\right)\right) \\
&\overset{(d)}{=} \left(B_s, \frac{1}{2} \int_0^s \omega\left(B_u, dB_u\right)\right) = g_s.
\end{aligned}
$$

The assumptions of the Borel-Cantelli Lemma are satisfied, therefore we only need to show that the series with the term

$$
\mathbb{P}\left(\phi\left(t_n\right) \max_{0 \leqslant s \leqslant t_n - t_{n-1}} |g_{t_{n-1}}^{-1} g_{s+t_{n-1}}| < r_1\right)
$$

diverges. We have

$$\mathbb{P}\left(\phi\left(t_n\right)\max_{0\leqslant s\leqslant t_n-t_{n-1}}|g^{-1}_{t_{n-1}}g_{s+t_{n-1}}| < r_1\right)$$

$$= \mathbb{P}\left(\max_{0\leqslant s\leqslant t_n-t_{n-1}}\frac{\phi\left(t_n\right)}{r_1}|g_s| < 1\right)$$

$$= \mathbb{P}\left(g^*\left(\frac{t_n-t_{n-1}}{r_1^2}\phi\left(t_n\right)^2\right) < 1\right),$$

where $g^*\left(\frac{t_n-t_{n-1}}{r_1^2}\phi\left(t_n\right)^2\right) := g^*_{\frac{t_n-t_{n-1}}{r_1^2}\phi(t_n)^2}$.

By (4.6) and the scaling property of g_t it follows that

$$c\left(\lambda_1^{(1)},\lambda_1^{(2)}\right) \geqslant \limsup_{\varepsilon\to0}-\varepsilon^2\mathbb{P}\left(g_1^* < \varepsilon\right) = \limsup_{\varepsilon\to0}-\varepsilon^2\mathbb{P}\left(\max_{0\leqslant s\leqslant1}|g_{\frac{1}{\varepsilon^2}s}| < 1\right)$$

$$= \limsup_{t\to\infty}-\frac{1}{t}\mathbb{P}\left(\max_{0\leqslant s\leqslant t}|g_s| < 1\right) = \limsup_{t\to\infty}-\frac{1}{t}\mathbb{P}\left(g_t^* < 1\right).$$

Moreover

$$\frac{t_n-t_{n-1}}{r_1^2}\phi\left(t_n\right)^2 = \frac{1}{r_1^2}\frac{t_n-t_{n-1}}{t_n}\log\log t_n$$

$$\leqslant \frac{1}{r_1^2}\log\log t_n = \frac{1}{r_1^2}\log\log n^n$$

and hence we can apply Proposition 4.1 with $Y_t = g_t^*$, $b = c\left(\lambda_1^{(1)},\lambda_1^{(2)}\right)$, $v_n = \frac{1}{y}\log\log n^n$, and $y = r_1^2$, since $r_1^2 > c\left(\lambda_1^{(1)},\lambda_1^{(2)}\right)$. □

Remark 5.4. *In the proof of Proposition 5.3 we used left increments $g^{-1}_{t_{n-1}}g_{s+t_{n-1}}$. It is easy to check that the argument does not work if one considers the right*

increments $g_{s+t_{n-1}} g_{t_{n-1}}^{-1}$ *instead. Indeed,*

$$g_{s+t_{n-1}} g_{t_{n-1}}^{-1}$$

$$= \left(B_{s+t_{n-1}} - B_{t_{n-1}}, \frac{1}{2} \int_{t_{n-1}}^{s+t_{n-1}} \omega\left(B_u, dB_u\right) - \frac{1}{2}\omega\left(B_{s+t_{n-1}}, B_{t_{n-1}}\right) \right)$$

$$= \left(B_{s+t_{n-1}} - B_{t_{n-1}}, \frac{1}{2} \int_0^s \omega\left(B_{u+t_{n-1}} - B_{t_{n-1}}, dB_{u+t_{n-1}}\right) \right.$$

$$+ \frac{1}{2} \int_0^s \omega\left(B_{t_{n-1}}, dB_{u+t_{n-1}}\right) - \frac{1}{2}\omega\left(B_{s+t_{n-1}}, B_{t_{n-1}}\right) \Big)$$

$$= \left(B_{s+t_{n-1}} - B_{t_{n-1}}, \frac{1}{2} \int_0^s \omega\left(B_{u+t_{n-1}} - B_{t_{n-1}}, dB_{u+t_{n-1}}\right) \right.$$

$$+ \frac{1}{2}\omega\left(B_{t_{n-1}}, B_{s+t_{n-1}} - B_{t_{n-1}}\right) - \frac{1}{2}\omega\left(B_{s+t_{n-1}}, B_{t_{n-1}}\right) \Big)$$

$$= \left(B_{s+t_{n-1}} - B_{t_{n-1}}, \frac{1}{2} \int_0^s \omega\left(B_{u+t_{n-1}} - B_{t_{n-1}}, dB_{u+t_{n-1}}\right) \right.$$

$$+ \omega\left(B_{t_{n-1}}, B_{s+t_{n-1}} - B_{t_{n-1}}\right) \Big)$$

$$\stackrel{(d)}{=} \left(B_s, \frac{1}{2} \int_0^s \omega\left(B_u, dB_u\right) + \omega\left(B_{t_{n-1}}, B_s\right) \right)$$

$$\neq \left(B_s, \frac{1}{2} \int_0^s \omega\left(B_u, dB_u\right) \right) = g_s.$$

This is a consequence of our choice of the (left) Brownian motion g_t defined by the left translation in (2.9).

The next Proposition completes the proof of Theorem 3.2.

Proposition 5.5. *Let c be given by* (3.1), *then c is constant a.s.*

Proof. Let $\mathcal{T}_u := \sigma\{B_r, r \geq u\}$, and $\mathcal{T} := \cap_{u>0}\mathcal{T}_u$ be the tail σ-algebra generated by the Brownian motion, which is trivial by Kolmogorov's 0-1 law. We will show that

$$(5.6) \qquad c^4 = \liminf_{t\to\infty} \phi(t)^4 \max_{u\leq s\leq t} \left[|B_s|^4 + \left(\frac{1}{2} \int_u^s \omega\left(B_v, dB_v\right) \right)^2 \right].$$

This means that the random variable c is $\mathcal{T}_u$-measurable for every u and hence $\mathcal{T}$-measurable. Since $\mathcal{T}$ is trivial, then c is constant a.s. because $\mathcal{T}$ is trivial. Let us now prove (5.6). Suppose u is fixed, and note that

$$(5.7) \qquad\qquad c = \liminf_{t\to\infty} \phi(t) \max_{u\leq s\leq t} |g_s|.$$

Indeed,

$$\max_{u\leq s\leq t} |g_s| \leq \max_{0\leq s\leq t} |g_s| \leq \max_{u\leq s\leq t} |g_s| + \max_{0\leq s\leq u} |g_s|,$$

and (5.7) follows from the fact that $\lim_{t\to\infty}\phi(t) = 0$. Using the triangular inequality one can show that

$$\max_{u\leqslant s\leqslant t}\left[|B_s|^4 + \left(\frac{1}{2}\int_u^s \omega\left(B_v, dB_v\right)\right)^2\right] - A_u^2 - 2|A_u|\max_{0\leqslant s\leqslant t}|A_s|$$

$$(5.8)\qquad \leqslant \max_{u\leqslant s\leqslant t}|g_s|^4$$

$$\leqslant \max_{u\leqslant s\leqslant t}\left[|B_s|^4 + \left(\frac{1}{2}\int_u^s \omega\left(B_v, dB_v\right)\right)^2\right] + 2A_u^2 + |A_u|\max_{0\leqslant s\leqslant t}|A_s|.$$

Indeed,

$$\max_{u\leqslant s\leqslant t}|g_s|^4 = \max_{u\leqslant s\leqslant t}\left[|B_s|^4 + A_s^2\right]$$

$$= \max_{u\leqslant s\leqslant t}\left[|B_s|^4 + \left(\frac{1}{2}\int_u^s \omega\left(B_v, dB_v\right) + A_u\right)^2\right]$$

$$= \max_{u\leqslant s\leqslant t}\left[|B_s|^4 + \left(\frac{1}{2}\int_u^s \omega\left(B_v, dB_v\right)\right)^2 + A_u^2 + A_u\int_u^s \omega\left(B_v, dB_v\right)\right]$$

$$\leqslant \max_{u\leqslant s\leqslant t}\left[|B_s|^4 + \left(\frac{1}{2}\int_u^s \omega\left(B_v, dB_v\right)\right)^2\right] + A_u^2 + A_u\max_{u\leqslant s\leqslant t}\left|\int_u^s \omega\left(B_v, dB_v\right)\right|$$

$$= \max_{u\leqslant s\leqslant t}\left[|B_s|^4 + \left(\frac{1}{2}\int_u^s \omega\left(B_v, dB_v\right)\right)^2\right] + A_u^2 + A_u\max_{u\leqslant s\leqslant t}|A_s - A_u|$$

$$\leqslant \max_{u\leqslant s\leqslant t}\left[|B_s|^4 + \left(\frac{1}{2}\int_u^s \omega\left(B_v, dB_v\right)\right)^2\right] + 2A_u^2 + |A_u|\max_{0\leqslant s\leqslant t}|A_s|,$$

and the upper bound in (5.8) is proven. Le us now show the lower bound. We have that

$$\max_{u\leqslant s\leqslant t}\left[|B_s|^4 + \left(\frac{1}{2}\int_u^s \omega\left(B_v, dB_v\right)\right)^2\right] = \max_{u\leqslant s\leqslant t}\left[|B_s|^4 + (A_s - A_u)^2\right]$$

$$= \max_{u\leqslant s\leqslant t}\left[|B_s|^4 + A_s^2 + A_u^2 - 2A_uA_s\right] \leqslant \max_{u\leqslant s\leqslant t}|g_s|^4 + A_u^2 + 2\max_{u\leqslant s\leqslant t}\left(-A_uA_s\right)$$

$$\leqslant \max_{u\leqslant s\leqslant t}|g_s|^4 + A_u^2 + 2|A_u|\max_{0\leqslant s\leqslant t}|A_s|,$$

and the lower bound is also proven.

Before finishing the proof we recall that by [18, Theorem 1]

$$\liminf_{t\to\infty}\phi(t)^2\max_{0\leqslant s\leqslant t}|A_s| = \frac{\pi}{4}\ \text{ a.s.}$$

Then (5.6) follows by (5.8) and the fact that $\lim_{t\to\infty}\phi(t) = 0$. $\qquad\square$

We have actually proven a more quantitative version of Theorem 3.2 as follows.

Theorem 5.6 (Chung's law of iterated logarithm with bounds). *The constant $c = \liminf_{t \to \infty} \phi(t) g_t^*$ in Theorem 3.2 satisfies*

$$\sqrt{\lambda_1^{(2)}} \leqslant c \leqslant \sqrt{c\left(\lambda_1^{(1)}, \lambda_1^{(2)}\right)},$$

where $\lambda_1^{(2)}$ is defined in Notation 4.2, and $c\left(\lambda_1^{(1)}, \lambda_1^{(2)}\right)$ is defined in Proposition 4.3.

5.2. **Small deviations for g_t.** We are now ready to prove Theorem 3.4, that is, the small deviation principle for the hypoelliptic Brownian motion g_t.

Proof of Theorem 3.4. We recall the notation

$$c_- := \liminf_{\varepsilon \to 0} -\varepsilon^2 \log \mathbb{P}\left(g_1^* < \varepsilon\right),$$

$$c_+ := \limsup_{\varepsilon \to 0} -\varepsilon^2 \log \mathbb{P}\left(g_1^* < \varepsilon\right),$$

and

$$c := \liminf_{t \to \infty} \phi(t) g_t^*.$$

We first show that

$$(5.9) \qquad\qquad\qquad c_+ \leqslant c^2.$$

Let $k \in (0, c_+)$ be a fixed number, that is, $k \leqslant \limsup_{\varepsilon \to 0} -\varepsilon^2 \log \mathbb{P}\left(g_1^* < \varepsilon\right)$. This means that there exists an $\varepsilon(k)$ such that

$$(5.10) \qquad\qquad\qquad \mathbb{P}\left(g_1^* < \varepsilon\right) \leqslant \exp\left(-\frac{k}{\varepsilon^2}\right),$$

for any $\varepsilon \leqslant \varepsilon(k)$.

Now fix $R > 1$ and a number γ such that $0 < R\gamma < k$. Define $t_n = R^n > 1$, and $\varepsilon_n = \varepsilon_n(k, \gamma, R) := \frac{\sqrt{\gamma}}{\phi(t_{n+1})\sqrt{t_n}}$. Note that ε_n goes to zero as n goes to infinity, and hence there exists an $N = N(k, \gamma, R)$ such that $\varepsilon_n \leqslant \varepsilon(k)$ for any $n \geqslant N(k, \gamma, R)$. Then we have that

$$\mathbb{P}\left(g_{t_n}^* < \frac{\sqrt{\gamma}}{\phi(t_{n+1})}\right) = \mathbb{P}\left(g_1^* < \frac{\sqrt{\gamma}}{\phi(t_{n+1})\sqrt{t_n}}\right) = \mathbb{P}\left(g_1^* < \varepsilon_n\right)$$

$$\leqslant \exp\left(-\frac{k}{\varepsilon_n^2}\right) = \exp\left(-\frac{k}{\gamma}\phi(t_{n+1})^2 t_n\right) = \exp\left(-\frac{k}{\gamma}\frac{t_n}{t_{n+1}}\log\log t_{n+1}\right)$$

$$= \exp\left(-\frac{k}{R\gamma}\log\log R^{n+1}\right) = \left(\frac{1}{(n+1)\log R}\right)^{\frac{k}{R\gamma}},$$

for all $n \geqslant N(k, \gamma, R)$, which is a term of a convergent series since $R\gamma < k$. Therefore for any $0 < k < c_+$, $R > 1$, $0 < R\gamma < k$, and $t_n = R^n$ we have

that

$$\sum_{n=1}^{\infty} \mathbb{P}\left(g_{t_n}^* < \frac{\sqrt{\gamma}}{\phi\left(t_{n+1}\right)}\right) < \infty.$$

Hence by the Borel-Cantelli Lemma it follows that

$$\mathbb{P}\left(\bigcup_{k \geqslant 1}\bigcap_{n \geqslant k}\left\{g_{t_n}^* < \frac{\sqrt{\gamma}}{\phi\left(t_{n+1}\right)}\right\}\right) = 0, \text{ that is,}$$

$$\mathbb{P}\left(\bigcap_{k \geqslant 1}\bigcup_{n \geqslant k}\left\{g_{t_n}^* \geqslant \frac{\sqrt{\gamma}}{\phi\left(t_{n+1}\right)}\right\}\right) = 1.$$

Therefore almost surely for all large n, $g_{t_n}^* \geqslant \frac{\sqrt{\gamma}}{\phi(t_{n+1})}$. The function $\phi(t)$ is decreasing for $t > 1$, therefore $t \in [t_n, t_{n+1}]$ we have

$$g_t^* \geqslant g_{t_n}^* \geqslant \frac{\sqrt{\gamma}}{\phi\left(t_{n+1}\right)} \geqslant \frac{\sqrt{\gamma}}{\phi\left(t\right)}$$

which yields

$$c := \liminf_{t \to \infty} \phi(t) g_t^* \geqslant \sqrt{\gamma} \quad \text{a.s.}$$

for any $\gamma < \frac{k}{R} < \frac{c_+}{R}$, and hence (5.9) is proven by letting first R go to 1, and then k to c_+.

Let us now show that

(5.11) $$c^2 \leqslant c_-.$$

Suppose $k > c_-$, then $k \geqslant \liminf_{\varepsilon \to 0} -\varepsilon^2 \log \mathbb{P}\left(g_1^* < \varepsilon\right)$. Therefore there exists an $\varepsilon'(k)$ such that

(5.12) $$\mathbb{P}\left(g_1^* < \varepsilon\right) \geqslant \exp\left(-\frac{k}{\varepsilon^2}\right),$$

for any $\varepsilon \leqslant \varepsilon'(k)$. Now set $t_n = n^n$ and define

$$\varepsilon_n = \varepsilon_n(k) := \frac{k}{\sqrt{c_-}} \frac{1}{\sqrt{t_n - t_{n-1}}\phi\left(t_n\right)},$$

$$E_n^k := \left\{\max_{t_{n-1} \leqslant s \leqslant t_n} |g_{t_{n-1}}^{-1} g_s| < \frac{k}{\sqrt{c_-}} \frac{1}{\phi\left(t_n\right)}\right\}.$$

Note that ε_n goes to zero as n goes to infinity, and hence there exists an $N(k)$ such that $\varepsilon_n \leqslant \varepsilon'(k)$ for any $n \geqslant N(k)$. We claim that, for any $k > c_-$, $\sum_{n=1}^{\infty} \mathbb{P}\left(E_n^k\right) = \infty$. Indeed, since by (5.5) the left increment $g_{t_{n-1}}^{-1} g_{s+t_{n-1}} \overset{(d)}{=}$

g_t, we can use (5.12) to see that for $n \geqslant N(k)$

$$\mathbb{P}\left(E_n^k\right) = \mathbb{P}\left(\max_{t_{n-1} \leqslant s \leqslant t_n} |g_{t_{n-1}}^{-1} g_s| < \frac{k}{\sqrt{c_-}} \frac{1}{\phi(t_n)}\right)$$

$$= \mathbb{P}\left(\max_{0 \leqslant s \leqslant t_n - t_{n-1}} |g_{t_{n-1}}^{-1} g_{s+t_{n-1}}| < \frac{k}{\sqrt{c_-}} \frac{1}{\phi(t_n)}\right) = \mathbb{P}\left(\max_{0 \leqslant s \leqslant t_n - t_{n-1}} |g_s| < \frac{k}{\sqrt{c_-}} \frac{1}{\phi(t_n)}\right)$$

$$= \mathbb{P}\left(g_1^* < \varepsilon_n\right) \geqslant \exp\left(-\frac{k}{\varepsilon_n^2}\right) = \exp\left(-\frac{c_-}{k} \frac{t_n - t_{n-1}}{t_n} \log \log t_n\right)$$

$$\geqslant \exp\left(-\frac{c_-}{k} \log \log t_n\right) = \left(\frac{1}{n \log n}\right)^{\frac{c_-}{k}}.$$

This yields $\sum_{n=1}^{\infty} \mathbb{P}\left(E_n^k\right) = \infty$ since $k > c_-$. Note that the events E_n are independent because the increments are independent and

$$\max_{t_{n-1} \leqslant s \leqslant t_n} |g_{t_{n-1}}^{-1} g_s| = \max_{0 \leqslant s \leqslant t_n - t_{n-1}} |g_{t_{n-1}}^{-1} g_{s+t_{n-1}}|$$

and $g_{t_{n-1}}^{-1} g_{s+t_{n-1}}$ is independent of $\mathcal{F}_{t_{n-1}}$ as shown in the proof of Proposition 5.3. Hence by the Borel-Cantelli Lemma we have that

$$\mathbb{P}\left(\limsup_{n \to \infty} E_n^k\right) = \mathbb{P}\left(\bigcap_{j \geqslant 1} \bigcup_{n \geqslant j} \left\{\phi(t_n) \max_{t_{n-1} \leqslant s \leqslant t_n} |g_{t_{n-1}}^{-1} g_s| < \frac{k}{\sqrt{c_-}}\right\}\right) = 1,$$

which yields

$$\liminf_{n \to \infty} \phi(t_n) \max_{t_{n-1} \leqslant s \leqslant t_n} |g_{t_{n-1}}^{-1} g_s| < \frac{k}{\sqrt{c_-}} \quad \text{a.s. for all } k > c_-,$$

and hence

(5.13) $$\liminf_{n \to \infty} \phi(t_n) \max_{t_{n-1} \leqslant s \leqslant t_n} |g_{t_{n-1}}^{-1} g_s| \leqslant \sqrt{c_-} \quad \text{a.s.}$$

We will show later in the proof that

(5.14) $$\lim_{n \to \infty} \phi(t_n) g_{t_{n-1}}^* = 0 \quad \text{a.s.}$$

Assume (5.14) for now, we can use 2.7 to see that

$$\phi(t_n) g_{t_n}^* \leqslant \phi(t_n) g_{t_{n-1}}^* + \phi(t_n) \max_{t_{n-1} \leqslant s \leqslant t_n} |g_s|$$

$$\leqslant \phi(t_n) g_{t_{n-1}}^* + \phi(t_n) \max_{t_{n-1} \leqslant s \leqslant t_n} |g_{t_{n-1}}^{-1} g_s| + \phi(t_n) |g_{t_{n-1}}|$$

$$\leqslant 2\phi(t_n) g_{t_{n-1}}^* + \phi(t_n) \max_{t_{n-1} \leqslant s \leqslant t_n} |g_{t_{n-1}}^{-1} g_s|.$$

Therefore by (5.13) and (5.14) we have that

$$c := \liminf_{t \to \infty} \phi(t) g_t^* \leqslant \liminf_{n \to \infty} \phi(t_n) g_{t_n}^*$$

$$\leqslant \liminf_{n \to \infty} \left(2\phi(t_n) g_{t_{n-1}}^* + \phi(t_n) \max_{t_{n-1} \leqslant s \leqslant t_n} |g_s g_{t_{n-1}}^{-1}|\right) \leqslant \sqrt{c_-} \quad \text{a.s.,}$$

which proves (5.11). Let us now show (5.14). By Lemma 5.2 we have that for any $\varepsilon > 0$

$$1 = \mathbb{P}\left(\bigcup_{k \geqslant 1} \bigcap_{n \geqslant k} \left\{ \phi\left(t_n\right) g_{t_{n-1}}^* < \varepsilon \right\} \right) \leqslant \mathbb{P}\left(\limsup_{n \to \infty} \phi\left(t_n\right) g_{t_{n-1}}^* < \varepsilon \right).$$

So for every ε we have that

$$\limsup_{n \to \infty} \phi\left(t_n\right) g_{t_{n-1}}^* < \varepsilon \quad \text{a.s.}$$

and hence $\limsup_{n \to \infty} \phi\left(t_n\right) g_{t_{n-1}}^* = 0$ a.s., which implies (5.14). $\qquad\square$

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