A Note on Local Controllability on Matrix Groups

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Abstract

In this work we study linear control systems as recently addressed by Cardetti and Mittenhuber, 2005. In particular, we present local controllability results in the case when the Lie group G is a matrix Lie group. In the matrix Lie group case the subgroup $\pi(\widehat{H})$ where local controllability holds can be computed explicitly in terms of the control vector fields, using system theory on group manifolds. We give an example on the special linear group $SL(2,\mathbb{R})$ which shows that the identity of G is properly contained in $\pi(\widehat{H})$, i.e., $\pi(\widehat{H})$ can be nontrivial.

Key words:
Systems on matrix Lie groups, Non-invariant systems, Linear control systems, Local controllability
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1 Introduction

In this note we consider linear control systems of the type

$$\dot{x} = X(x) + \sum_{j=1}^{k} u_j Y^j(x), \tag{1.1}$$

evolving on a real finite dimensional Lie group G. Here the drift vector field X is an infinitesimal automorphism, that is, a vector field whose flow X_t is a

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one-parameter group of automorphisms of G. The input functions $u = (u_j)$ are piecewise constant functions, and the control vector fields Y^j are left-invariant.

The above definition, introduced in [2], generalizes the classical notion of linear systems on the Euclidean space $\mathbb{R}^n$. In [2], Ayala and Tirao consider the *ad-rank condition*, which is an extension of the Kalman rank condition, for controllability of the system Σ defined by (1.1). In fact, they prove that the ad-rank condition is sufficient for local controllability of Σ at the group identity e. Recently, in [4], the same result has been shown using a different approach, namely, Lie theory of semigroups (see [5]). The result is obtained by first constructing an augmented Lie group $\widehat{G}$, and then choosing a closed subgroup T of $\widehat{G}$ such that controllability properties of Σ on the homogeneous manifold $M = \widehat{G}/T$ correspond to controllability properties of Σ on G. Moreover, the authors prove that Σ is locally controllable at every point on $\pi(\widehat{H})$, where $\pi(\widehat{H})$ is a certain subgroup of G. In general, this subgroup cannot be easily described. The aim of this work is to present the local controllability results in the case when G is a matrix Lie group. In this case the subgroup $\pi(\widehat{H})$ can be computed explicitly in terms of the control vector fields. Assuming that G is a matrix Lie group is not too restrictive since these groups provide most of the interesting examples of Lie groups.

The question of whether local controllability on $\pi(\widehat{H})$ is really an extension of local controllability at the identity e was not addressed in [4]. In this note, we show an example on the special linear group $SL(2,\mathbb{R})$ in which $\{e\}$ is properly contained in $\pi(\widehat{H})$. In other words, $\pi(\widehat{H})$ is nontrivial. In this sense, the aforementioned result in [4] is indeed an extension of the main result in [2].

This work is organized as follows: in Section 2, we set up the notation and recall the results from [2] and [4]. Section 3 contains Theorem 3.1 which is the main result of this work, and in Section 4 the example on $SL(2,\mathbb{R})$ is presented.

2 Local Controllability

In this section we introduce some of the terminology, notation, and results from [2] and [4] that will be used in the subsequent sections.

The reachable set of Σ from a point x_0, denoted by $\mathcal{R}(x_0)$, is the set of all end-points of solutions of (1.1) that have initial condition x_0.

Definition 2.1. *A control system is said to be* locally controllable at a point x_0 *if the point belongs to the interior of its reachable set:* $x_0 \in \operatorname{Int} \mathcal{R}(x_0)$.

Throughout this work we identify the Lie algebra $\mathfrak{g}$ of G with the left-invariant vector fields. The main hypothesis for local controllability is defined in terms of the dimension of the subspace of $\mathfrak{g}$ defined as

$$\mathcal{H}^X = \operatorname{span}\{\operatorname{ad}^i(X)(Y^j) : i \geq 0,\ 1 \leq j \leq k\},$$

where $\operatorname{ad} X$ is the inner derivation $\operatorname{ad}(X)(Y) = [X, Y]$.

Ayala and Tirao proved in [2] that the linear control system Σ as defined in the introduction is locally controllable at the group identity e provided the system satisfies the following condition:
Definition 2.2 (ad-rank condition)**.** *A linear control system satisfies the* ad-rank condition *if $\mathcal{H}^X$ has full rank, that is,*

$$\dim(\mathcal{H}^X) = \dim(G).$$

Remark 2.3. *This property is a generalization of the well-known Kalman condition when $G = \mathbb{R}^n$. (See [1], or [4]).*
Theorem 2.4 (Theorem 3.5 in [2])**.** *Let G be a real finite-dimensional connected Lie group and let Σ be the linear control System (1.1). If Σ satisfies the ad-rank condition, then it is locally controllable at the identity e of G.*

Local controllability of these type of systems was also studied in [4] using Lie theory of semigroups. In this note, we adopt the notation and constructions from that paper. Specifically, the Lie group $\widehat{G}$ is defined as the semidirect product $G \rtimes_X \mathbb{R}$, with group multiplication given by

$$(g_1, t_1) \cdot (g_2, t_2) = (g_1 X_{t_1}(g_2), t_1 + t_2), \quad \text{for } g_i \in G, t_i \in \mathbb{R}, \text{ and } i = 1, 2. \tag{2.1}$$

The Lie algebra $\hat{\mathfrak{g}}$ is isomorphic as a vector space to $\mathfrak{g} \times \mathbb{R}$. Let $\widehat{X}$, and $\widehat{Y_j}$ be the right-invariant vector fields in $\widehat{G}$ defined at the identity as $\widehat{X} = (0, 1)$ and $\widehat{Y_j} = (Y_j, 0)$. The corresponding flows are the left translations by $\exp(t\widehat{X}) = (e, t)$ and $\exp(t\widehat{Y_j}) = (\exp tY^j, 0)$ respectively, where $\exp : \hat{\mathfrak{g}} \to \widehat{G}$ is the exponential map.

Let T be defined as the closed subgroup $T = \{e\} \rtimes \mathbb{R}$, which is also equal to the orbit of the identity under the flow of the vector field $\widehat{X}$, namely $T = \exp(\mathbb{R}\widehat{X})$. The map η defined by

$$\eta : \widehat{G} \times G \to G, \quad ((g_1, t_1), g) \mapsto g_1 X_{t_1}(g),$$

is a transitive action of the Lie group $\widehat{G}$ on G. Furthermore, η is a flow, and under the action η, T is the isotropy group of the identity $e \in G$, hence the mapping

$$\widehat{G}/T \to G, \quad (g_1, t_1)T \mapsto \eta_{(g_1,t_1)}(e) = g_1$$

is a $\widehat{G}$-flow isomorphism, i.e. an isomorphism of transformation groups. Denote by π the natural projection

$$\pi : \widehat{G} \rightarrow \widehat{G}/T$$

which induces a linear surjection

$$d\pi(\mathbf{1}) : T_{\mathbf{1}}\widehat{G} \rightarrow T_{x_0}\widehat{G}/T$$

where $x_0 = \pi(\mathbf{1})$ is the base point in $\widehat{G}/T$.

Let $\widehat{\mathfrak{h}}$ be the subalgebra generated by the lifted control vector fields. We use the subindex $\mathcal{L}.\mathcal{A}.$ to express that it is generated as a Lie algebra, thus in symbols $\widehat{\mathfrak{h}} = \mathrm{span}_{\mathcal{L}.\mathcal{A}.}\{\widehat{Y}_1, \ldots, \widehat{Y}_k\}$. Also let $\widehat{H}$ be the subgroup of $\widehat{G}$ defined by $\widehat{H} = \langle \exp \widehat{\mathfrak{h}} \rangle$. The main result of [4] is stated in the following theorem:
Theorem 2.5 (Theorem 4.9 in [4])**.** *Let G be a real finite-dimensional connected Lie group and let Σ be the linear control System (1.1). If Σ satisfies the ad-rank condition (2.2), then it is locally controllable at $p = \pi(h)$ for all $h \in \widehat{H}$.*

3 Local controllability on a matrix Lie group

In this section we present the local controllability result for the case when G is a matrix Lie group. It is important to remark here that in this case, the natural projection π, defined in the previous section, is the projection on the first coordinate of the semidirect product $\mathrm{pr} : \widehat{G} \rightarrow G$ (see [4]).
Theorem 3.1. *Let G be a connected matrix Lie group and let Σ be a linear control system defined by*

$$\dot{x}(t) = X(x(t)) + \sum_{j=1}^{k} u_j(t) Y^j(x(t)), \quad x(t) \in G,$$

where the drift vector field X is an infinitesimal automorphism, that is, a vector field whose flow X_t is a one-parameter group of automorphisms of G. The input functions $u = (u_j)$ are piecewise constant functions, and the control vector fields Y^j are left-invariant. If Σ satisfies the ad-rank condition (2.2), then Σ is locally controllable on the subgroup $\langle \exp \mathfrak{h} \rangle$.

Proof. The system Σ satisfies the ad-rank condition, thus by Theorem 2.5 we know the system is locally controllable on $\pi(\widehat{H})$. Recall that each $\widehat{Y_j}$ is defined as $\widehat{Y_j} = (Y^j, 0)$ with flow given by $\exp(t\widehat{Y_j}) = (\exp tY^j, 0)$. It is clear that if $k = 1$, that is, if there is only one control vector field,then

$$\pi(\exp t\widehat{Y_j}) = \exp tY^j \text{ for all } t \in \mathbb{R},$$

and the theorem clearly holds true.

If $k > 1$, we use the following theorem:

Theorem 3.2 (Theorem 1 in [3]). *Let $\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_p$ be a collection of linear subspaces of $\mathbb{R}^{n\times n}$. Then $\langle \exp \mathcal{A}_1, \exp \mathcal{A}_2, \cdots, \exp \mathcal{A}_p\rangle = \langle \exp(\operatorname{span}_{\mathcal{L}.\mathcal{A}.}\{\mathcal{A}_1, \mathcal{A}_2, \cdots, \mathcal{A}_p\})\rangle$.*

Using Theorem 3.2, the subgroup $\widehat{H} = \langle \exp \widehat{\mathfrak{h}}\rangle = \langle \exp \widehat{Y_1}, \exp \widehat{Y_2}, \cdots, \exp \widehat{Y_k}\rangle$. Elements in $\langle \exp \widehat{\mathfrak{h}}\rangle$ consist of all the possible products of the different one-parameter subgroups. For simplicity we consider $h \in \langle \exp \widehat{\mathfrak{h}}\rangle$ expressed as

$$\begin{aligned}
h &= (\exp t_1\widehat{Y_1})(\exp t_2\widehat{Y_2}) \cdots (\exp t_l\widehat{Y_l}) \\
&= (\exp t_1Y^1, 0) \cdot (\exp t_2Y^2, 0) \cdots (\exp t_lY^l, 0) \\
&= (\exp t_1Y^1X_0(\exp t_2Y^2)X_0(\exp t_3Y^3) \cdots X_0(\exp t_lY^l), 0) \\
&= (\exp t_1Y^1 \exp t_2Y^2 \exp t_3Y^3 \cdots \exp t_lY^l, 0)
\end{aligned}$$

where $l = 1, \ldots, k$ and the multiplication in $\widehat{G}$ is as defined in (2.1).

From the above calculation we have that $\pi(h) = \exp t_1Y^1 \exp t_2Y^2 \exp t_3Y^3 \cdots \exp t_kY^k$. The same calculation holds when considering the more general form of h; therefore, the projection $\pi(\widehat{H})$ is contained in the subgroup $\langle \exp Y^1, \exp Y^2, \cdots, \exp Y^k\rangle$. To show $\pi(\widehat{H}) \subseteq \langle \exp \mathfrak{h}\rangle$ we use Theorem 3.2 again and obtain

$$\begin{aligned}
\pi(\widehat{H}) \subseteq \langle \exp Y^1, \exp Y^2, \cdots, \exp Y^k\rangle &= \langle \exp\{Y^1, Y^2, \cdots, Y^k\}_{\mathcal{L}.\mathcal{A}.}\rangle \\
&= \langle \exp \mathfrak{h}\rangle.
\end{aligned}$$

Clearly, working the above steps backward we see that $\langle \exp \mathfrak{h}\rangle \subseteq \pi(\widehat{H})$ also holds true and the proof is finished. $\square$

4 Example on $SL(2, \mathbb{R})$

In this section we present an example on a matrix Lie group which shows that Theorem 2.5 is really an extension of Theorem 2.4. In other words, $\pi(\widehat{H})$ is a nontrivial subgroup of G.

Let G be the Lie group $SL(2, \mathbb{R})$ of all 2×2 real matrices with determinant one, and let $\mathfrak{sl}_2(\mathbb{R})$ denote its Lie algebra; that is, the vector space of all 2×2 real matrices with trace zero. Consider a fixed matrix A in $\mathfrak{sl}_2(\mathbb{R})$, and define the vector field X by

$$X(g) = Ag - gA, \text{ for all } g \in G.$$

Then X is the infinitesimal generator of the 1-parameter group X_t given by

$$X_t(g) = e^{tA}\ g\ e^{-tA}.$$

Clearly, X_t is a diffeomorphism and it preserves the group operation. Thus, X is an infinitesimal automorphism of G. Systems with the drift vector field defined in this form were studied by Markus in [6].

Now, let $\{E, F, H\}$ be the standard basis of $\mathfrak{sl}(2, \mathbb{R})$ given by

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ \ F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \text{and}\ \ H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then the following Lie bracket identities are satisfied: $[H, E] = 2E,\ [H, F] = -2F,$ and $[E, F] = H$. Consider the following linear control system

$$\dot{x}(t) = X(x(t)) + u(t)E(x(t)), \quad x(t) \in SL(2, \mathbb{R}), \tag{4.1}$$

where X is the vector field associated to the matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Statement 4.1. $\pi(\widehat{H}) = \left\{ \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} :\ t \in \mathbb{R} \right\}$

Proof. Using the Lie bracket identities, we have

$$\begin{aligned} \dim \mathcal{H}^X &= \dim\langle \{ad^i(X)(Y^j) |\ i \geq 0,\ 1 \leq j \leq k\} \rangle \\ &= \dim\langle \{E, [X, E] = H, [X, H] = 2F\} \rangle \\ &= \dim(G). \end{aligned}$$

Thus System (4.1) satisfies the ad-rank condition. Hence by Theorem 3.1, System 4.1 is locally controllable on the subgroup $\pi(\widehat{H})$. Recall that the subgroup $\widehat{H}$ is defined as $\widehat{H} = \langle \exp \mathbb{R}\widehat{Y} \rangle$, and in this example $Y = E$; therefore, after some computations, we have

$$\exp t\widehat{Y} = (\exp tE, 0) = \left(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, 0 \right).$$

Consequently, the system is locally controllable on the upper triangular group

$$\pi(\widehat{H}) = \left\{ \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} : \; t \in \mathbb{R} \right\}.$$

Clearly, the identity matrix is properly contained in this subgroup, that is, $\pi(\widehat{H})$ is nontrivial. □

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