Kluwer Academic Publishers, Math. and Its Appl. 520, 459-476, 2000

Chapter 21

ONE HUNDRED PROBLEMS IN COMMUTATIVE RING THEORY

Scott Chapman
Department of Mathematics
Trinity University
San Antonio, Texas 78212
schapman@trinity.edu

Sarah Glaz
Department of Mathematics
The University of Connecticut
Storrs, Connecticut 06269
glaz@uconnvm.uconn.edu

1. INTRODUCTION

This article consists of a collection of problems in Commutative Ring Theory sent to us, in response to our request, by the authors of articles in this volume. It also includes our contribution of a fair number of unsolved problems. Some of these one hundred problems already appear in other articles of this volume; some are related to the topics but do not appear in another article; yet others are problems unrelated to any of the articles, but that the authors consider of importance. For all problems, we gave a few useful references, which will lead readers to other relevant references. There is no attempt to be encyclopedic. We added definitions and clarifying comments to make the problems more self contained, but, as a rule, if undefined notions are used the reader can find the relevant definitions in the cited references. Finally, this is an article whose purpose is to generate many more articles. We hope the problems posed here will keep researchers busy for many years, and would appreciate very much being sent preprints of their solutions.

2. OPEN PROBLEMS

The underlying assumption is that, unless otherwise stated, all rings are commutative with 1, and the term “local ring” refers to a, not necessarily Noetherian, ring with only one maximal ideal.

1. Let D be a domain and let G(D) denote its group of divisibility. For which lattice ordered groups G is there a non-Bezout GCD domain D with G(D) order isomorphic to G? Useful references: [4, Problem # 28], [77], [5, Question 4.5].

2. Let R be a domain, let M be an R module and let $S_R(M)$ denote the symmetric algebra of M over R . When is $S_R(M)$ a GCD domain? ([5, Question 4.7]). Several answers to this question can be found in [78]. Another useful reference: [3].

3. Let R be a ring. When is the power series ring R[[x]] a GCD domain? ([5, Question 4.10], and [4, Problem #17]). Another useful reference: [3].

4. Let D be a domain. For a polynomial f in D[x], let c(f) denote the content of f. f is called a primitive polynomial if $c(f)_P = D$ for every prime ideal P of D. f is called a superprimitive polynomial if $(c(f)^{-1})^{-1} = D$. D is said to satisfy PSP if each primitive polynomial in D[x] is superprimitive. D is said to satisfy PSP2 if each primitive linear polynomial in D[x] is superprimitive. Does PSP2 imply PSP? Some useful references: [5, Question 6.1], [6].

5. Let R be a local ring and let f be a Gaussian polynomial in R[x], that is c(fg) = c(f)c(g) for every polynomial g in R[x]. Suppose that c(f) is a regular ideal. Is c(f) an invertible (principal) ideal? Kaplansky conjectured that the answer to this problem is affirmative for a domain R. For a Noetherian ring R the answer is affirmative [54], [62]. Other useful references: [7], [53], [63].

6. Which domains D satisfy $c(fg)^* = (c(f)c(g))^*$ for all nonzero power series f and g in D[[x]], where * denotes the star operation d or v or t? Versions of this question appear in [4, Problem # 17] , and in [5, Question 8.7]. Another useful reference: [7]

7. Which domains D satisfy $c(fg)^* = (c(f)c(g))^*$ for all nonzero (linear) f in D[x] and g in D[[x]], where * is the star operation d or t? Useful references: [5, Question 8.8], [7]

8. Let R be a domain, let $T_t(R)$ denote the group of t-invertible (fractional) t-ideals of R under t-multiplication, $Prin_t(R)$ be its subgroup of principal (fractional) ideals, and $Cl_t(R) = T_t(R)/Prin_t(R)$. Let R[x] be the polynomial ring in one variable over R. There is a natural morphism :$Cl_t(R)Cl_t(R[x])$ given by $[(I)] = [IR[x]]$. is an isomorphism if and only if R is integrally closed [42]. If R is not

Kluwer Academic Publishers, Math. and Its Appl. 520, 459-476, 2000
integrally closed does split? Determine $Cl_t(R[x])/Cl_t(R)$ when R is not integrally closed. Another useful reference: [11].

9. Let R be a domain. Compute $Cl_t(R)$ when R is a monoid domain, or more generally, a graded domain. Useful reference: [11].

10. Given a * operation one definesanalogously to the definition of $Cl_t(R)$, by replacing t by * (see Problem 8). For which * operations on a domain R, does the *-class group yield useful information about R. Useful reference: [11]

11. Let D be a domain and let I be an ideal of D. I is called strongly divisorial if it is a divisorial ideal satisfying $II^{-1} = I$. Is a domain D completely integrally closed if and only if no prime ideal is strongly divisorial? For a Mori domain the answer is affirmative [14, Corollary 14]. Another useful reference: [15].

12. Is a finite (or with finite character) intersection of strong Mori domains a strong Mori domain? Strong Mori domains are defined in [38]. This questions appears in [39]. Another useful reference: [15].

13. A Mori domain D is called a transcendental Mori domain if the polynomial ring D[x] is also Mori. Are the polynomial rings in two or more variables over a trancendental Mori domain, Mori? This question appears in [15, Section 6]. Another useful reference: [13].

14. Is the polynomial ring in infinitely many indeterminates over a Noetherian domain a Mori domain? This question appears in [15, Section 6].

15. Construct examples of strong Mori domains which are neither Noetherian nor Krull domains. Useful references: [15], [38], [39], [59], [60].

16. Is the complete integral closure of a strong Mori domain a Krull domain? Useful references: [15], [38], [39], [59], [60], [85].

17. Let E be a subset of a rank one valuation domain V, let K be the field of quotients of V and denote by Int (E, V) the ring of integer valued polynomials on E. Let denote the completion of E. The compactness of is a sufficient condition for Int (E, V) to be a Prufer domain which satisfies the almost strong Skolem property [22, Theorem 3.1, Proposition 3.4]. Is the compactness of necessary for Int (E, V) to be a Prufer domain? Is the compactness of necessary for Int (E, V) to satisfy the almost strong Skolem property? Useful references: [22, Remark 3.9], [21].

18. Let E be a subset of a domain D, and denote by Int (E, D) the ring of integer

valued polynomials on E. Is the almost strong Skolem property of Int (E, D) a local property, that is, if Int (E, D) satisfies the almost strong Skolem property does it follow that Int (E, D_m) satisfies the almost strong Skolem property for each maximal ideal m of D? The converse is true, and the answer is possitive in the Noetherian case, when E = D [22]. In particular, if D is a Dedekind domain, is it necessary for Int (E, D) to satisfy the almost strong Skolem property that, for each maximal ideal m, the completion of E in the m-adic topology is compact? Useful references: [22, Remark 3.9], [21].

19. Let E be a subset of a DVR, V with maximal ideal m, such that ,the completion of E in the m-adic topology is compact. In this case there is a one-to-one correspondence between the prime ideals of Int (E, V) lying over m and the elements of E [22, Theorem 2.8]. Does this result still hold if V is a one dimensional, local, Noetherian, possibly unibranched (or analytically irreducible, or with ch V/m > 0), domain? Useful reference: [22, Remarks 2.9 and 2.12].

20. Let E be a subset of a domain V and let Int (E, V) be the ring of integer valued polynomials on E. The Stone-Weirerstrass Theorem [22, Theorem 2.4], and the almost strong Skolem property [22, Proposition 3.4] extend from the case where V is a DVR to the case where V is a one dimensional, local, Noetherian domain which is analytically irreducible. Can these results be extended to a larger class of , not necessarily Noetherian, one dimensional, local domains V?

21. Let E be a subset of a domain D. We say that E does not have any polynomially isolated element if any polynomial with coefficients in the field of quotients of D which is integer-valued on the complement in E of a singleton {a}, is in fact integer-valued on E. We say that E is a coherent subset of D if any polynomial with coefficients in the field of quotients of D which is integer-valued on the complement in E of a finite subset of elements of E, is in fact integer-valued on E. Is a subset which does not have any polynomially isolated element a coherent subset? If D is completely integrally closed the answer is affirmative [22, Remark 3.9]. Another useful reference: [23].

22. A domain R is called a half-factorial domain (HFD), if every element of R can be written as a product of irreducible elements, and any two such factorizations of the same element have the same length. Is every HFD, R, a subring of a UFD, D, such that every irreducible element r in R is irreducible in D? Useful references: [30], [95], [96].

23. Let G be an abelian group. Does there exist a Dedekind domain D with class group G which is an HFD (see Problem 22)? Useful references: [30], [75], [96].

24. Is there a Krull HFD (see Problem 22), R, which is not a Dedekind domain such that | Cl(R)| > 2 ? Useful references: [30], [96].

Kluwer Academic Publishers, Math. and Its Appl. 520, 459-476, 2000

25. Let G be an abelian group, and let S ⊆ G be a non empty subset. A Dedekind domain D with class group isomorphic to G such that the classes that contain maximal ideals are precisely the elements of S, is called the Dedekind domain associated with {G,S}. Given a finite abelian group G, which generating sets S ⊆ G yield associated Dedekind domains which are HFDs (see Problem 22)? Useful references: [12], [24], [28], [29], [30].

26. Describe all d < 0 for which the ring of algebraic integers in Q() is an HFD (see Problem 22)? Useful references: [30], [34].

27. A domain D is called atomic if every nonzero nonunit element of D can be written as a product of irreducible elements of D. Let R be an HFD (see Problem 22) and let S be an overring of R which posses a nonunit of boundary 0, then S is atomic [34]. Is the converse true? Other useful references: [30], [35].

28. Let R be an HFD (see Problem 22), and denote by, the integral closure of R. Ifis atomic, does it follow thatis an HFD? Useful references: [30], [35].

29. Are there an infinite number of (integrally closed) real quadratic HFDs (see Problem 22)? Useful references: [30], [34].

30. Does there exist a real quadratic HFD (see Problem 22) containing infinitely many orders that also have the half-factorial property? Useful references: [30], [34].

31.Let x and y be indeterminates over a ring R. If R[x] is an HFD, is R[x,y] an HFD? This question has an affirmative answer if R is Noetherian [26]. Other useful references: [30], [32], [35].

32. If R[[x]], the power series ring in one variable over a ring R, is an HFD (see Problem 22), does it follow that R is integrally closed? Useful references: [30], [35].

33. Let G be a finite abelian group. A zero sequence $\{g_1,...,g_n\}$ of G is a sequence of nonzero elements of G whose sum is equal to 0. If a zero sequence of G contains no proper zero subsequence it is called a minimal zero sequence of G. The Davenport constant of G, denoted by D(G), is the maximal length of a minimal zero sequence of G. Find a formula for the Davenport constant of an arbitrary finite abelian group G. Useful references: [24], [25].

34. Let G = be a finite abelian group with $n_i \mid n_{i+1}$ for 1 i t-1. Let D(G) denote the Davenport constant of G, and let M(G) = . It is known that D(G) = M(G) for p groups and for groups of rank 2, but not in general [24], [25]. Does this equality

hold if the rank of G is 3?

35. Let G be a finite abelian group and let g be a nonzero element of G. Find the maximal length of a minimal zero sequence of G containing g. Useful references: [25], [27].

36. Let G be a finite abelian group and let P be a minimal zero sequence of G. Find an algorithm for constructing a maximal length zero sequence of G containing P. Useful references: [25],[27].

37. Let D be a Dedekind domain and let (n) denote the nth generalized set of lengths for D [25, Definition 4.14]. Assume that in addition D has a prime ideal in every ideal class. Is (n) an interval for all positive integers n 2? Another useful reference: [27].

38. Let C_n be the cyclic group of order n. Does (3) = n + 1 for all positive integers n 2? For the definition of (n) see [27]. Another useful reference: [25].

39. For which finite abelian groups G does (3) = ? Here D(G) is the Davenport constant of G and (n) is defined in [27]. Another useful reference: [24].

40. Let (n) denote the cardinality of the set (n). Let D be a -finite domain, that is (n) is finite for each n. Does (n)/n exist? Useful references: [25], [26], [27], [28].

41. Let G be a finite abelian group. An r by s matrix with entries from G is a factorization matrix if the nonzero elements of each row and column form minimal zero sequences and each row and column contains nonzero entries. Let r = (3) = . Is there an r by 3 factorization matrix of G which has a column of nonzero entries? Useful references: [24], [25], [27].

42. If R is a going-down domain of Krull dimension 2, must the integral closure of R also be a going-down domain? Useful references: [36], [37].

43. If R is a domain such that the polynomial ring in one variable R[x] is catenarian, must R be universally catenarian? Useful references: [19], [37].

44. Let R be a domain with field of quotient K. Denote by F(R) the set of all non zero R submodules of K. A mapping F(R) F(R), E E* is called a semistar operation on R if for all nonzero x ε K, and E, F ε F(R):
1. (xE)* = xE*,
2. E ⊆ E*, and E ⊆ F implies that E* ⊆ F*
3. E** = E*
Let $\{R_\alpha$ / αεA$\}$ be a family of overrings of a ring R. Assume that for each α ε A, $*_\alpha$ is a semistar operation on R_α, then E E^{*A}, where E^{*A} = $\cap_{\alpha\varepsilon A}(ER_\alpha)^{*\alpha}$,

Kluwer Academic Publishers, Math. and Its Appl. 520, 459-476, 2000
defines a semistar operation on R. Find conditions on $*^{A}$ to be of finite type. Useful references: [2], [40].

45. Let $\mathscr{F}$ be a localizing system of a domain R (for definition see [40]). For each E ε F(R), the map E $E_{\mathscr{F}} = \cup_{J \varepsilon \mathscr{F}}(E : J)$ defines a semistar operation on R denoted by $*^{\mathscr{F}}$. To a localizing system $\mathscr{F}$ one can associate a localizing system of finite type $\mathscr{F}_f$ ([40, Lemma 3.1]). To a semistar operation * one can associate a semistar operation of finite type $*_f$ ([40, Section 1]). Characterize all localizing systems $\mathscr{F}$ such that $*\mathscr{F}_f = (*\mathscr{F})_f$. Other useful references: [59], [41].

46. A semistar operation * on a domain R (see Problem 44) is called spectral, if there is a family Δ of prime ideal of R , such that for every E ε F(R), $E^* = \cap_{P \varepsilon \Delta} ER_P$. A semistar operation * on R is called quasi spectral if for every ideal I of R such that $I^* \cap R \neq R$, there exists a prime ideal P of R with $I \subseteq P$ and $P^* \cap R = P$. Spectral semistar operations are quasi spectral [40]. To a semistar operation * one can associate a spectral semistar operation $*_{sp}$ ([40, Section 4]). With the notation of Problem 45, if * is a quasi spectral semistar operation on a domain R, is $(*_f)_{sp} = (*_{sp})_f$?

47. Is there an example of a finitely spectral non-spectral localizing system? (See Problems 44-46). Useful reference: [40].

48. A pullback diagram of type ☐ is a diagram: R D

T k

where D and T are domains, M is a maximal ideal of T, kT/M, and :Tk is the canonical map. In a diagram of type ☐ we have
Krull dim (R) = max {ht_T (M) + Krull dim (D), Krull dim (T)} [44, Corollary 1.10]. Is there a similar formula for the t-dimension? If K is the field of quotients of R, this is related to the problem of determining whether, given a prime t-ideal P of R not containing M, Q is necessarily a t-ideal of T, where Q is the unique prime of T which satisfies QR = P.

49. A DVF domain is a domain such that each divisorial ideal is *v*-finite. Characterize pullback diagrams of the type ☐ (see Problem 48) where R is a DVF domain. This has been done in [43, Theorem 4.20] in the case where T is local. Another useful reference: [44].

50. Study the transfer of ideal-theoretic properties in a diagram which is of the type ☐ (see Problem 48) except that M is not necessarily assumed to be a maximal ideal of T. When T is a valuation domain, some results along these lines

have been obtained in [76]. Another useful reference: [44].

51. In a diagram of the type □ (see Problem 48), let I be an ideal of R, and assume that (I) can be generated by n elements in D, and that IT can be generated by r elements in T. Then I can be generated by n + r elements [44, Theorem 2.26]. Is this bound the best possible in general? This problem appears in [44, Problem 2.31].

52. Determine whether there exists a diagram of the type □ (see Problem 48) in which R is a Prufer domain containing a finitely generated ideal which requires more than two generators. This problem appears in [44, Problem 2.32].

53. Let R be a ring and let A(R) be the set of Artinian subrings of R. Determine conditions under which R can be expressed as a directed union of elements of A(R). Of particular interest is the case where R is Von Neumann regular. This question appears in [48, Remark 3.14]. Another useful reference: [49].

54. Let R be a ring, and denote by N(R) the nilradical of R. If R / N(R) can be expressed as a directed union of Artinian subrings, can R itself be so expressed? This question appears in [48, Remark 3.14]. Another useful reference: [49].

55. Let R be a ring and let A(R) denote the set of Artinian subrings of R. Determine when A(R) is a directed set. This question appears in: [48, Remark 3.14]. Another useful reference: [49].

56. Let F be an indexed family of fields and let C be a class of rings. Determine equivalent conditions in order for F to be the family of residue fields of a ring from C. A special case of the problem will be to determine what indexed families of fields can be realized as the family of residue fields of a commutative ring. [48, Section 6] discusses the case C = Z, [66] considers the case C = the class of PIDs.

57. Find an example of a ring or a domain which is a finite conductor ring [52, Definition 2.1], but not a quasi coherent ring [52, Definition 2.2]. In [43] it was speculated that the two properties might coincide in case the ring is a domain. Another useful reference: [51].

58. A ring R is called regular if every finitely generated ideal of R has finite projective dimension. In [51, Proposition 3.4] it is shown that a coherent regular ring is a G-GCD ring [51], [52, Definition 4.1]. Is a finite conductor regular ring a G-GCD ring?

59. Let R be a G-GCD ring. Is the polynomial ring in one variable R[x] a G-GCD ring? The answer is affirmative in case R is a domain [3]. The answer is also

Kluwer Academic Publishers, Math. and Its Appl. 520, 459-476, 2000
affirmative for another large class of G-GCD rings, namely the class of coherent regular rings [51], [52].

60. Let R be a ring. When is the power series ring R[[x]] a G-GCD ring? Useful references: [51], [52].

61. Let R be a ring, let M be an R module and let $S_R(M)$ denote the symmetric algebra of M over R. When is $S_R(M)$ a G-GCD ring? See Problem 2 for the case where R is a domain and the condition in question is the GCD property. The problem was partially solved for the condition of coherent regularity [56]. Other useful references: [51], [52].

62. Querre proved the following result in [82]: Let R be a domain and let I be a divisorial ideal of the polynomial ring in one variable R[x]. Then:
1. I R = J 0 implies that I = JR[x].
2. I R = 0 implies that I = fJR[x], for a divisorial ideal J $\subseteq$ R and an f ε R[x].
Can Querre's result be extended to rings with zero divisors? Other useful references: [51], [52].

63. Let R be an integrally closed coherent ring. Is the polynomial ring in one variable, R[x], a quasi coherent ring? The answer is yes in case R is a domain [51]. Another useful reference: [52].

64. A coherent ring R is called stably coherent if the polynomial rings in finitely many variables over R are coherent rings. Let R be a coherent ring with the polynomial ring in one variable R[x] coherent, is R a stably coherent ring? This is a question posed by Vasconcelos [89]. For an account of the known results regarding the coherence of polynomial rings see [55, Chapter 7].

65. Is the integral closure of a one dimensional coherent domain in its field of quotients a Prufer domain? This is a question posed by Vasconcelos which had been answered positively in many, but not all, cases. A useful reference which will lead to many other useful references: [55, Chapter 5 (Section 3) and Chapter 7 (Section 4)].

66. A ring R is called a total ring of quotients if every element of R is either a unit or a zero divisors. There are total ring of quotients which are not finite conductor rings [51, Example 3.5]. Find necessary and sufficient conditions for a total ring of quotients to be a finite conductor ring. Other useful references: [52], [55].

67. Let R be a ring, let G be an abelian group and denote by RG the group ring of G over R. Find necessary and sufficient conditions for RG to be a G-GCD ring, or a finite conductor ring. Necessary and sufficient conditions for RG to be

coherent or coherent regular were found in [57]. Other useful references: [51], [52], [55].

68. Let R be a ring, let G be a group of automorphisms of R, and denote by R^G the fixed ring of G, that is $R^G = \{ a \, \varepsilon \, R / \, g(a) = a \text{ for all } g \, \varepsilon \, G\}$. Conditions under which the finite conductor property descends from R to R^G were investigated in [51], [52]. Under what condition does the extension $R^G \subseteq R$ ascend the finite conductor property from R^G to R? The corresponding question for coherence was partially answered in [58, Theorem 4 and Example 5].

69. A local ring R with maximal ideal M is called a generalized local ring if M is finitely generated and $M^k = 0$. A local ring S with maximal ideal N is unramified with respect to a local ring R with maximal ideal M if R is a subring of S , N = MS and $N^k R = M^k$ for each positive integer k. Let F be a family of generalized local rings R_i with maximal ideals M_i , so that for any pair of rings R_i and R_j in F there exists an R_k in F such that R_k is unramified with respect to both R_i and R_j . If each R_i is Noetherian, is $S = R_i$ necessarily Noetherian? This is a question posed by I.S. Cohen in [31]. Another useful reference: [64].

70. Let R be a generalized local ring with maximal ideal M, and let J be an ideal of R which is closed in the M-adic topology. Assume that J is the closure of a properly smaller ideal. Does it follow that J is the closure of a non finitely generated ideal? Useful reference: [64].

71. Let R be a generalized local ring with maximal ideal M and with M-adic completion . If each primary ideal of R is contracted from , does it follow that R is Noetherian? Useful references: [31], [64].

72. Let R be a domain and let M be a maximal ideal of R. If M^2 is two-generated, does it follow that M is finitely generated? Useful references: [50], [64].

73. Let R be a seminormal (or an integrally closed, or a completely integrally closed) domain and let M be a maximal ideal of R with M^k finitely generated for some $k > 1$. Is M finitely generated? Useful references: [64], [86].

74. Let R be a domain. The trace of an R module M is the ideal of R generated by f(m), where m runs over all the elements of M and f runs over all the elements of $\mathrm{Hom}_R(M,R)$. R satisfies RTP if the trace of any ideal of R is either equal to R or is a radical ideal of R. R satisfies TPP if the trace of each primary ideal of R is either equal to R or is a prime ideal of R. Is RTP equivalent to TPP for any domain R? This questions appears in [68]. Another useful reference: [74].

75. Let R be a domain. R satisfies LTP if for each trace ideal I of R and each prime P minimal over I, $IR_P = PR_P$. With RTP and TPP as in Problem 74, is RTP

Kluwer Academic Publishers, Math. and Its Appl. 520, 459-476, 2000
or TPP equivalent to LTP for any domain R? Useful references: [68], [74].

76. Let K be a field, and let $\{V_i\}$ be a set of valuation domains all of which have residue fields equal to K. Give necessary and sufficient conditions on the set $\{V_i\}$,
so that V_i is a Prufer domain with residue field K. Useful references: [1], [47], [72], [73].

77. Swan [88] proved that for each positive integer n there is a Prufer domain with a finitely generated ideal requiring n generators. His proof and others' reproving this result use tools outside ring theory. Give a construction which yields, for each positive integer n, a Prufer domain containing a finitely generated ideal requiring n generators, such that the proof of the necessity of n generators can be carried out using elementary ring theoretical techniques without any reliance on geometry. Other useful references: [69], [70], [73].

78. A ring R is called t-closed if whenever $a^3 + arc - c^2 = 0$ for elements a, c and r in R, there exist an element b in R such that $b^2 - rb = a$ and $b^3 - rb^2 = c$. The notion of t-closedness (of rings and morphisms) originated in K-theory, but there is no characterization of this notion by means of K-theory functors. Find such a characterization. Useful references: [16], [79], [80].

79. Let R be a ring and let G be a group or a monoid. Find conditions on R and on G for the group (monoid) ring RG to be t-closed. This problem was solved for seminormality, a condition related to t-closedness, in [9], [10], [16], [87]. Another useful reference: [80].

80. Let R be a ring. Find general conditions for the power series ring in one variable R[[x]], to be t-closed. Partial answers were obtained in [17], [18], [79]. Another useful reference: [80].

81. Let RS be a ring extension. An element *b*B is subintegral over A if there exists a non negative integer *p* and elements $c_1,..., c_p$ in B such that R for all n large enough [84]. The morphism R S is subintegral if and only if every element of S is subintegral over R. Can one generalize the theory of subintegrality to infra-integrality defined in [80]. Another useful reference: [83].

82. Let R be a ring and denote by R+ , the additive group of R. R is an E-ring provided R is ring isomorphic to End (R+) under the map that sends rR to left multiplication by r. Is there a ring R such that REnd (R+), but R is not an E-ring? This question was posed by Schultz [20], and also appears (with a reward attached to its solution) in [90].

83. Let R be a ring. An R module M is called an E-module over R if $Hom_R(R,M) = Hom_Z(R,M)$. Which classes of groups are E-modules over an E-ring (see Problem 82)? This problem was posed by Pierce [81], and also appears in [90].

84. Let R be a ring. Can the class of all E-modules over R be constructed from the torsion E-modules and the torsion-free ones (see Problem 82)? This problem was posed by Pierce [81], and also appears in [90].

85. Let R be a ring. The E-ring core of R, denoted by , is defined in [61], through a construction of nested rings inside R , , for every cardinality . The E-ring core of R is an E-ring [61]. Must the E-ring core of a strongly irreducible domain be strongly irreducible? For which rings R does ? These and other questions regarding the E-core of R are posed with more information in [61] and [90].

86. Investigate the structure of two-sided E-rings of finite rank. Two side E-rings are defined in [90], in which this problem appears and which provides additional useful information.

87. Extend results on E-rings, E-modules and related concepts to modules over more general rings. Useful reference: [90].

88. Let R be a ring for which there exists a positive integer n such that every finitely generated (finitely presented) module is a direct sum of modules generated by at most n elements. Such a ring is called an FG(n) ring. If R is a Noetherian FG(n) ring then R is FG(2) [91]. Is this true in general? How about FG(3)? This problem appears in [93, Question 1.8].

89. Let R be a ring. R is called an elementary divisor ring if every matrix (not necessarily square) is equivalent to a diagonal matrix. Equivalently, R is an elementary divisor ring if every finitely presented module is a direct sum of cyclic modules [71]. A classical open problem: Is every Bezout domain an elementary divisor ring? The answer for rings with zero divisors is no [45]. Another useful reference: [93 , Section 1].

90. Let R be a reduced Noetherian ring whose integral closure is finitely generated, and suppose direct-sum cancellation holds for finitely generated torsion-free R modules. Does direct-sum cancellation necessarily hold for all finitely generated modules? Characterize all rings R for which this is true. Useful reference: [93, Section 3].

91. Find new examples of non-Gorenstein local Cohen-Macaulay rings with finite representation type. There are no known examples of dimension greater than 3. For all undefined notions, as well as a broad perspective on the topic and more references see [93, Section 5].

Kluwer Academic Publishers, Math. and Its Appl. 520, 459-476, 2000

92. Characterize the posets that are order isomorphic to Spec R for some countable two dimensional (Noetherian) domain. Useful references: [93, Sections 2 and 4], [65].

93. Are Spec Q[x,y] and Spec [x,y] order isomorphic, where denotes the algebraic closure of the field of rational numbers Q? This problem appears as Question 4.8 in [93].

94. Consider the posets U_F = Spec F[x,y], where F ranges over countable fields of characteristic zero. Are all these posets order isomorphic? At the other extreme, if U_F and U_K are order isomorphic, must F and K be order isomorphic? Useful reference: [93, Sections 2 and 4].

95. Let R be a ring. Jspec R = {PSpec R / P is an intersection of maximal ideals of R}. In [92] the posets that arise as Jspec R for countable Noetherian rings R are characterized (see [93, Theorem 6.1]). Is this characterization still valid without both (on the ring R and on the poset U) countability assumption?

96. Let R be a ring. The maximal ideal space of R is the set of all maximal ideals of R with the induced Zariski topology. Is every compact metric space of dimension d homeomorphic to the maximal ideal space of some d-dimensional commutative ring? This problem appears as Question 6.4 in [93]. Another useful reference: [67].

97. Let DR be an extension of domains, where R is a generalized ring of fractions of D, and let I be a non zero finitely generated ideal of D. Is it true that $(IR)^{-1} = I^{-1}R$? Background material, the definition of a generalized ring of fractions and a more general version of this problem appear in [94, Question 2.7 and preceding paragraphs].

98. Let D be a domain. Denote by Max (D) the set of maximal ideals of D, and by t-Max (D) the set of maximal t-ideals of D. A set of prime ideals *F* of D is called a defining family of primes if . *F* is called of finite character if every nonzero nonunit of D belongs to finitely many of P*F*. *F* is called independent if no distinct primes P and Q of *F* contain a common nonzero prime ideal. A domain D with a defining family *F* which is independent and of finite character is said to be an *F*-IFC domain. Find an example of an *F*-IFC domain with *F*t-Max (D) and *F*Max (D). This problem appears in [94, Problem 3.6]. Another useful reference: [8].

99. Let D be a domain with a defining family of finite character (see Problem 98). Characterize all such domains in terms of unidirectional ideals. Useful

references: [8], [94, Section 3].

100. A domain D is called a fgv domain if every finitely generated ideal of D is a *v*-ideal. Is it true that if D is a domain with w = t , then for every maximal t-ideal M of D, D_M is a fgv domain? The converse is true, that is if D is a domain and for every maximal t-ideal M of D, D_M is a fgv domain then $A_t = A_w$ for every fractional ideal of D (M. Zafrullah, private communication). Useful references: [94, Section 4], [38].

References

[1] S. Abhyankar, P. Eakin and W. Heinzer, On The Uniqueness Of The Coefficient Ring In A Polynomial Ring, *J. Algebra* **23** (1972), 310-342.
[2] D.D. Anderson, Star Operations Induced By Overrings, *Comm. Alg*. **16** (1988), 2535-2553.
[3] D.D. Anderson and D.F. Anderson, Generalized GCD Domains, Comment. *Math. Univ. St. Pauli* **XXVIII** (1979), 215-221.
[4] D.D. Anderson, Some Problems In Commutative Ring Theory, Lecture Notes in Pure and Applied Math # 171, Marcel Dekker (1995), 363- 372.
[5] D.D. Anderson, GCD Domains, Gauss' Lemma, And Contents Of Polynomials, this volume.
[6] D.D. Anderson and R.O. Quintero, Some Generalizations Of GCD Domains, Lecture Notes in Pure and Applied Math # 189, Marcel Dekker (1997), 189-195.
[7] D.D. Anderson and B.G. Kang, Content Formulas For Polynomials And Power Series And Complete Integral Closure, *J. Algebra* **181** (1996), 82-94.
[8] D.D. Anderson and M. Zafrullah, Independent Locally-Finite Intersections Of Localizations, *Houston J. Math*. **25** (1999), 433-452.
[9] D.F. Anderson, The Picard Group Of A Monoid Domain, *J. Algebra* **115** (1988), 342-351.
[10] D.F. Anderson. The Picard Group Of A Monoid Domain II, *Arch. Math*. **55** (1990), 143-145.
[11] D.F. Anderson, The Class Group And Local Class Group Of An Integral Domain, this volume.
[12] D.F. Anderson, S.T. Chapman and W.W. Smith, On Krull Half-Factorial Domains With Infinite Cyclic Divisor Class Group, *Houston J. Math*. **20** (1994), 561-570.
[13] B. Ballet and N. Dessagnes, Anneaux De Polynomes Sur Un Anneau De Mori, *C.R. Math. Rep. Acad. Sci. Canada* **8** (1986), 393-398.
[14] V. Barucci, Strongly Divisorial Ideals And Complete Integral Closure Of An Integral Domain, *J. Algebra* **99** (1986), 132-142.
[15] V. Barucci, Mori Domains, this volume.
[16] H. Bass and M.P. Murthy, Grothendieck Groups And Picard Groups Of Abelian Group Rings, *Ann. Math*. **86** (1976), 16-73.
[17] A. Benhissi, Dependance Integrale Dans Les Anneaux De Series Formelles, *Comm. Algebra* **27** (1999), 1369-1383.
[18] A. Benhissi, t-closed Rings Of Formal Power Series, *Arch. Math*. **72** (1999), 1-5.
[19] A. Bouvier, D.E. Dobbs and M. Fontana, Universally Catenarian Integral Domains, *Advances in Math*. **72** (1988), 211-231.
[20] R.A. Bowshell and P. Schultz, Unital Rings Whose Additive Endomorphisms Commute" *Math. Ann*. **228** (1977), 197-214.
[21] P.J. Cahen and J.L. Chabert, "Integer Valued Polynomials", AMS Survey and Monographs 48 (1997).
[22] P.J. Cahen and J.L. Chabert, What's New About Integer Valued Polynomials On A Subset?, this volume.
[23] J.L. Chabert, S.T. Chapman and W.W. Smith, The Skolem Property In Rings Of Integer

Valued Polynomials, *Proc. AMS* **126** (1998), 3151-3159.
[24] S.T. Chapman, On The Davenport Constant, The Cross Number And Their Application In Factorization Theory, Lecture Notes in Pure and Appl. Math # 189, Marcel Dekker (1995), 167-190.
[25] S.T. Chapman, M. Freeze and W. Smith, On Generalized Length Of Factorizations In Dedekind And Krull Domains, this volume.
[26] S.T. Chapman and W.W. Smith, An Analysis Using The Zaks-Skula Constant Of Element Factorizations In Dedekind Domains, *J. Algebra* **159** (1993), 176-190.
[27] S.T. Chapman and W.W. Smith, Generalized Sets Of Length, *J. Algebra* **200** (1998), 449-471.
[28] S.T. Chapman and W.W. Smith, Factorization In Dedekind Domains With Finite Class Group, *Israel J. Math*. **71** (1990), 65-95.
[29] S.T. Chapmam and W.W. Smith, On The HFD, CHFD And k-HFD Properties In Dedekind Domains, *Comm. Algebra* **20** (1992), 1955-1987.
[30] S.T. Chapman and J. Coykendall, Half-Factorial Domains, A Survey, this volume.
[31] I.S. Cohen, On The Structure And Ideal Theory Of Complete Local Rings, *Trans. AMS* **59** (1946), 54-106.
[32] J. Coykendall, A Characterization Of Polynomial Rings With The Half-Factorial Property, Lecture Notes in Pure and Appl. Math. 189, (1997), 291-294.
[33] J. Coykendall, The Half-Factorial Property In Integral Extensions, *Comm. Algebra* **27** (1999), 3153-3159.
[34] J. Coykendall, Half-Factorial Domains In Quadratic Fields, preprint
[35] J. Coykendall, A Counterexample Relating To The Integral Closure Of A Half-Factorial Domain, preprint
[36] D.E. Dobbs and M. Fontana, Integral Overrings Of Two-Dimensional Going-Down Domains, *Proc. AMS* **115** (1992), 655-662.
[37] D.E. Dobbs, Recent Progress On Going-Down I, this volume.
[38] W. Fanggui and R.L. McCasland, On w-modules Over Stong Mori Domains, *Comm. Algebra* **25** (1997), 1285-1306.
[39] W. Fanggui and R.L. McCasland, On Strong Mori Domains, *J. Pure and Appl. Algebra* **135** (1999), 155-165.
[40] M. Fontana and J. A. Huckaba, Localizing Systems And Semistar Operations, this volume.
[41] M. Fontana, J. Huckaba and I. Papick, "Prufer Domains", Marcel Dekker (1997).
[42] S. Gabelli, On divisorial Ideals In Polynomial Rings Over Mori Domains, *Comm. Algebra* **15** (1987), 2349-2370.
[43] S. Gabelli and E. Houston, Coherentlike Conditions In Pullbacks, *Michigan Math. J.* **44** (1997), 99-123.
[44] S. Gabelli and E. Houston, Ideal Theory In Pullbacks, this volume.
[45] L. Gillman and M. Henriksen, Rings Of Continuous Functions In Which Every Finitely Generated Ideal Is Principal, *Trans. AMS* **82** (1956), 366-391.
[46] R. Gilmer, "Multiplicative Ideal Theory" Queens Papers in Pure Appl. Math. 12, 1968
[47] R. Gilmer, Two Constructions In Prufer Domains, *J. Reine Angew. Math*. **239** (1970), 153-162.
[48] R. Gilmer, Commutative Rings Of Dimension 0, this volume.
[49] R. Gilmer and W. Heinzer, Products Of Commutative Rings And Zero-Dimensionality, *Trans. AMS* **331** (1992), 663-680.
[50] R. Gilmer, W. Heinzer and M. Roitman, Finite Generation Of Powers Of Ideals, *Proc. AMS*, to appear.
[51] S. Glaz, Finite Conductor Rings, *Proc. of AMS,* to appear.
[52] S. Glaz, Finite Conductor Rings With Zero-Divisors, this volume.
[53] S. Glaz and W. Vasconcelos, Gaussian Polynomials, Lecture Notes in Pure and Applied Math. # 185, Marcel Dekker (1997), 325-337.

[54] S. Glaz and W. Vasconcelos, The Content Of Gaussian Polynomials, *J. Algebra* **202** (1998), 1-9.
[55] S. Glaz, "Commutative Coherent Rings" Springer Verlag Lecture Notes in Math. # 1371 (1989).
[56] S. Glaz, Regular Symmetric Algebras, *J. of Algebra* **112** (1988), 129-138.
[57] S. Glaz, On The Weak Dimension Of Coherent Group Rings, *Comm. Algebra* **15** (1987), 1841-1858.
[58] S. Glaz, Coherence, Regularity And Homological Dimensions Of Commutative Fixed Rings, Commutative Algebra, World Scientific (1994), 89-106.
[59] F. Halter-Koch, "Ideal Systems: An Introduction To Multiplicative Ideal Theory", Marcel Dekker (1998).
[60] F. Halter-Koch, Constructions Of Ideal Systems With Nice Noetherian Properties, this volume.
[61] J. Hausen and J.A. Johnson, A Note On Constructing E-Rings, *Publ. Math. Debrecen* **38** (1991), 33-38.
[62] W. Heinzer and C. Huneke, Gaussian Polynomials And Content Ideals, *Proc. AMS* **125** (1997), 739-745.
[63] W. Heinzer and C. Huneke, The Dedekind-Mertens Lemma And The Content Of Polynomials, *Proc. AMS* **126** (1998), 1305-1309.
[64] W. Heinzer and M. Roitman, Generalized Local Rings And Finite Generation Of Powers Of Ideals, this volume.
[65] W. Heinzer and S. Wiegand, Prime Ideals In Polynomial Rings Over One Dimensional Domains, *Trans. AMS* **347** (1995), 639-650.
[66] R. Heitmann, PID's With Specified Residue Class Fields, *Duke Math. J*. **41** (1974), 565-582.
[67] M. Hochster, Prime Ideal Structure In Commutative Rings, *Trans. AMS* **137** (1969), 43-60.
[68] J.A. Huckaba and I. Papick, Connecting Trace Properties, this volume.
[69] W. Kucharz, Invertible Ideals In Real Holomorphy Rings, *J. Reine Angew. Math*. **395** (1989), 171-185.
[70] W. Kucharz, Generating Ideals In Real Holomorphy Rings, *J. Algebra* **144** (1991), 1-7.
[71] M. Larsen, W. Lewis and T. Shores, Elementary Divisor Rings And Finitely Presented Modules, *Trans. AMS* **187** (1974), 231-248.
[72] K.A. Loper, On Prufer Non-D-Rings, *J. Pure Appl. Algebra* **96** (1994), 271-278.
[73] K.A. Loper, Constructing Examples Of Integral Domains By Intersecting Valuation Domains, this volume.
[74] T. Lucas, The Radical Trace Property And Primary Ideals, *J. Algebra* **184** (1996), 1093-1112.
[75] D. Michel and J. Steffan, Repartition Des Ideaux Premiers Parmi Les Classes D'Ideaux Dans Un Anneau De Dedekind Et Equidecomposition, *J. Algebra* **98** (1986), 82-94.
[76] A. Mimouni, Pullbacks And Coherent-Like Properties, Lecture Notes in pure and applied Math. # 205, Marcel Dekker (1999), 437-459.
[77] J.L. Mott, Groups of Divisibility: A Unifying Concept For Integral Domains And Partially Ordered Groups, "Lattice-Ordered Groups", Kluwer Academic Publishers (1989), 80-104.
[78] J.L. Mott and M. Schexnayder, Exact Sequences Of Semi-Value Groups, J. *Reine Angew. Math.* **283/284** (1976), 388-401.
[79] G. Picavet and M. Picavet-L'Hermitte, Anneaux t-clos, *Comm. Algebra* **23** (1995), 2643-2677.
[80] G. Picavet and M. Picavet-L'Hermitte, T-Closedness, this volume.
[81] R.S. Pierce, E-Modules, Abelian Group Theory, *Contemporary Math*. **87** (1989), 221-240.
[82] J. Querre, Ideaux Divisoriels d'un Anneau de Polynomes, *J. Algebra* **64** (1980), 270-284.
[83] L. Reis, L.G. Roberts and B. Singh, On Weak Subintegrality, *J. Pure Appl. Algebra* **114** (1996), 93-109.
[84] L.G. Roberts and B. Singh, Invertible Modules And Generic Subintegrality, *J. Pure Appl. Algebra* **95** (1994), 331-351.
[85] M. Roitman, On The Complete Integral Closure Of A Mori Domain, *J. Pure Appl. Algebra*

Kluwer Academic Publishers, Math. and Its Appl. 520, 459-476, 2000
66 (1990), 315-328.
[86] M. Roitman, On Finite Generation Of Powers Of Ideals, preprint
[87] D.E. Rush, Picard Groups In Abelian Group Rings, *J. Pure Appl. Algebra* **26** (1982), 101-114.
[88] R. Swan, n-generator Ideals In Prufer Domains, *Pacific J. Math*. **111** (1984), 433-446.
[89] W. Vasconcelos, "The Rings Of Dimension Two", Lecture Notes in Pure and Appl. Math. # 22, Marcel Dekker (1976).
[90] C. Vinsonhaler, E-Rings And Related Structures, this volume.
[91] R. Warfield Jr. Decomposability Of Finitely Presented Modules, *Proc. AMS* **25** (1970), 167-172.
[92] R. Wiegand and S. Wiegand, The Maximal Ideal Space Of A Noetherian Ring, *J. Pure Appl*. *Algebra* **8** (1976), 129-141.
[93] R. Wiegand and S. Wiegand, Prime Ideals And Decompositions Of Modules, this volume.
[94] M. Zafrullah, Putting T-Invertibility To Use, this volume.
[95] A. Zaks, Half-Factorial Domains, *Bulletin of AMS* **82** (1976), 721-723.
[96] A. Zaks, Half-Factorial Domains, *Israel Journal of Math*. **37** (1980), 281-302.