Commutative Coherent Rings: Historical Perspective and Current Developments[1]

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Coherent ring theory constitutes a major area of research in commutative algebra. The wealth of results in this area, beside being of interest in their own right, contributed to the development of the general theory of commutative rings and influences research in other areas of algebra.

This article surveys the evolution of the theory of coherent rings since its beginning in 1960. The introduction describes the origin of the notion of coherence in algebra and the amalgamation of several research trends in diverse disciplines such as algebraic geometry, complex analysis, number theory and computer science, which gave rise to the definition of a coherent ring. The main body of this article concentrates on the description of the most significant topics of research in coherent ring theory and includes for each topic a mathematical history, statements of the main results and their contribution to the development of the general theory of commutative rings, and a probing of possible future research direction. The article concludes by briefly highlighting the impact of the research in coherent ring theory on other areas of algebra.

1. Introduction

For the past twenty to thirty years, research in commutative algebra has concentrated in two distinct, though occasionally intermingling, areas: Noetherian and non-Noetherian ring theory. Without attempting to capture the depth or the significance of the work done in either of these areas, one can roughly describe them as follows. Noetherian ring theory includes mainly the study of Noetherian regular and related rings and, more recently, computer-assisted commutative algebra, while non-Noetherian ring theory includes the study of rings with other finiteness properties rather than the ascending chain condition on ideals which characterizes Noetherian rings. Of the myriad of rings falling into this second category the most fascinating, the most extensively studied, and those having an impact on other areas of algebra are the so-called

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coherent rings.

The history of coherent algebraic objects starts in 1944, in the area of analytic manifolds, an area that we nowadays connect more with analysis than with algebra. In an attempt to investigate several global properties of analytic manifolds, CARTAN [16] introduced a notion he called 'a coherent system of pointwise ideals,' which imposed certain finiteness properties on the collection of all analytic functions in a neighborhood of every point of an analytic manifold. Without actually naming it, Cartan discovered the notion of an analytic coherent sheaf. In his 1953 paper, CARTAN [17] actually named his discovery and proceeded along with SERRE [83] to apply the newly-developed concept to change the definition and, consequently, revolutionize the theory of analytic manifolds.

Two years later, in 1955, SERRE [84] introduced the notion of a coherent algebraic sheaf. The definition is analogous to that of a coherent analytic sheaf and, like the former, its application to algebraic geometry had far-reaching consequences. SERRE, in his 1955-56 papers [84, 85], and later GROTHENDIECK and DIEUDONNE in 1961-1971 [38-41]. redefined algebraic varieties in terms of sheaves (and later in terms of schemes) and applied the machinery of coherent sheaves to obtain global properties of varieties from given local properties. These works produced a tremendous change in the field of algebraic geometry and, in fact, marked the beginning of modern research in the area.

Sheaves in algebraic geometry involve collections of rings and modules; hence, it seemed natural to expect that the definition of a coherent algebraic sheaf would give rise directly to an object in commutative algebra called a coherent ring or a coherent module but, in fact, this was not the case. One reason might be that, for the most part, the base rings considered in algebraic geometry are Noetherian. The coherence conditions imposed on the sheaf structure provided just the right kind of formal properties necessary for the development of the sheaf cohomology machinery. The implications of these conditions as a replacement and generalization of the Noetherian property of a ring itself has, therefore, been explored in a different context. Nevertheless, looking at both definitions, one can conclude in retrorespect that if an algebraic sheaf associated with a module over the spectrum of a ring is a coherent sheaf, then the module itself is a coherent module. Coherent rings (and coherent modules along with them) developed separately from algebraic geometry either in the form of specific rings suited for particular research purposes or as rings, not yet named, satisfying certain desirable properties. It was not until 1964 that BOURBAKI [13], noticing the striking similarity between the definitions, named these rings coherent rings.

The specific coherent rings that appear in commutative algebra form part of the collection of all the classical non-Noetherian rings. A class of such rings, whose beginnings date back to Hilbert, are the so-called Boolean rings, that is, rings R satisfying $x^2=x$ for every element x of R. Boolean rings were first connected with Hilbert's spectral theorem; then, through the works of M.H. STONE in 1936-1937 [86, 87], these rings became instrumental to the development of the notion of the spectrum of a commutative ring, and with the introduction of computers they found their uses in computer science as well. These versatile rings satisfy two useful finiteness properties related to their ideals. The relation $x(x-1)=0$ implies for every element x in R, the ideal $(0{:}x) = \{y \in R \mid xy = 0\}$ is generated by the element $x-1$, and for any two ideals I and J we have $I \cap J = IJ$. We conclude that a Boolean ring R satisfies the following properties:

(1) The intersection of two finitely generated ideals of R is a finitely generated ideal of R, and for every $x \in R$, $(0{:}x)$ is a finitely generated ideal of R.

These properties will eventually characterize the rings which will be called coherent rings.

A Boolean ring is but one of a larger class of rings, all of which turn out to be coherent rings, called Von Neumann regular rings. The contribution made by the Von Neumann regular rings to commutative algebra,, although less dramatic than that of its subclass of Boolean rings, is nevertheless important. Interest in these rings increased with the development of homological algebra when it was noticed that their ring theoretic description, very similar to that of Boolean rings, is equivalent to a certain homological property, namely, Von Neumann regular rings are precisely those rings over which every module is flat. In particular, a Von Neumann regular ring R satisfies the following property:

(2) An arbitrary direct product of flat R modules is a flat R module.

This property, in its turn, will be an equivalent characterization of a coherent ring.

Several other classical non-Noetherian rings appear through the relation between commutative algebra and number theory. As early as 1847, KUMMER [57], in his studies of cyclotomic fields, introduced the definition of 'ideal numbers' which amount in modern language to defining valuations on a field $\mathbb{Q}(\alpha)$, where $\mathbb{Q}$ denotes the rational numbers and α is a p-th root of unity $\neq 1$ (p is an odd prime). Through the numerous works of Kronecker and Dedekind, this notion became fundamental to modern number theory. Over 80 years later, in 1931, KRULL [55] introduced and studied the general notion of a valuation and the resulting valuation ring. A valuation ring is a ring with no zero divisors, satisfying the property that for every two nonzero elements x and y in R, either $x/y \in R$ or $y/x \in R$. Valuation rings became valuable tools for investigating properties of rings in commutative algebra and of curves in algebraic geometry. Most valuation rings are not Noetherian rings, but they are all coherent rings. A generalization of a valuation ring which is also coherent is a Prüfer ring, that is, a ring with no zero divisors which is 'locally' a valuation ring. When allowing the ring to have zero divisors we obtain the so-called semihereditary and hereditary rings as generalizations of valuation rings. These rings, both of which are coherent, derive their importance from their homological characterizations as rings for which certain homological invariants (weak and global dimensions, respectively) are less or equal to one.

The introduction of coherence provided a new outlook on the area of classical non-Noetherian rings and the resulting vigorous research considerably enriched the subject.

The classical non-Noetherian rings described above do not exhaust the class of known coherent rings; for instance, Noetherian rings themselves provide one of the most important examples of coherent rings. Over a Noetherian ring, every submodule of a finitely generated module is finitely generated. Every finitely generated module is, therefore, finitely presented (that is, it is isomorphic to a quotient of two finitely generated modules). This makes the category of finitely generated modules over a Noetherian ring particularly easy to handle. A natural generalization is to consider rings R satisfying the following property:

(3) Every finitely generated submodule of a finitely presented R module is finitely presented.

The class of rings satisfying this property will be the class of coherent rings and, by analogy, the category of finitely presented modules over a coherent ring will acquire a special significance. It is, then, natural that an important direction of research in coherent ring theory consists of asking to what extent do results, which are known to hold for Noetherian rings, remain valid for coherent rings. In some of these investigations the theory that developed in order to obtain the answers had significance to the general theory of commutative rings.

The first paper to investigate the general notion of a coherent ring was CHASE [18], 1990. It is well known that over a Noetherian ring, an arbitrary direct product of injective modules is an injective module. Motivated by that result, Chase asked for what rings arbitrary direct products of projective (respectively flat) modules is a projective (respectively flat) module. The answer in the case where the modules are flat is that this result holds for precisely those rings which are coherent and, once again, coherent rings appear related to the algebraic objects which they most resemble, in an indirect way. Chase, without mentioning the term coherent, derived no less than seven equivalent definitions of a coherent ring, three of which appear in this introduction as properties (1), (2) and (3). The blend of purely homological (property (2)) and entirely ring theoretic (property (1)) approaches to coherence characterizes, from 1960 on, all research done in coherent ring theory.

In 1964, BOURBAKI [13] performed the official naming of a coherent ring, and although the relation to coherent sheaves is not pointed out, it appears implicitly in a number of exercises where the reader is asked to prove several properties of coherent rings analogous to the ones known to hold for algebraic coherent sheaves. The construction of new coherent rings as a 'flat directed union' of coherent rings appears among these properties. This construction provides the most naturally occurring non-Noetherian coherent ring in a

Noetherian setting, the ring $k[x_1, x_2, \ldots]$, where k is a field, $x_1, x_2, \ldots$ are variables over k, and $k[x_1, x_2, ..]$ is the ring consisting of the union of all polynomial rings in finitely many variables k and $[x_1, \ldots, x_n]$. Each polynomial ring in finitely many variables is a Noetherian ring, but their union $k[x_1, x_2, \ldots]$ is a coherent ring which is not Noetherian.

CHASE's paper [18], BOURBAKI's collection of exercises [13], and HARRIS's two notes [43, 44], written in 1966-67, form the foundation and beginning of research into coherent ring theory.

The year 1968 saw the beginning of an intensification of research in the theory of coherent rings, and an abundance of results started appearing in journals. The most significant of these results are described in this paper along with their particular mathematical history, the impact they had on the research in commutative algebra, and possible future research directions in each topic. This article also touches lightly on the relation between the research done in commutative coherent ring theory and that conducted in the variety of other coherent algebraic objects developed at about the same time, such as noncommutative coherent rings, coherent groups, coherent matrices, coherent functors, and coherent categories.

More specifically, Section 2 presents one of the most studied questions in coherent ring theory, namely, the question of stable coherence, and its ramification and contribution to the classification of rings of global dimension two, to the construction of 'special format' cartesian squares of rings, and to the definition of the concept of non-Noetherian grade. Section 3 explores the known results in the study of the structure of the integral closure of a coherent domain and the branching out of the research done in this subject into investigating both overrings of domains, and the nature of prime ideals in polynomial rings. Section 4 concentrates on the investigation which was carried out in order to clarify the relation between the compactness of the minimal prime spectrum of a ring and the Von Neumann regularity of its total ring of quotients, and describes the contribution that this investigation had made to our understanding of classical non-Noetherian rings. Section 5 follows the evolution of the concept of non-Noetherian regularity and describes the enrichment of our knowledge of the homological properties of several well-known universal algebras that resulted from the investigation of this concept. Section 6 exhibits several ring constructions that were developed, generalized, examined, or re-examined as a direct consequence of the research done in coherent ring theory. Section 7 approaches the notion of uniform coherence, a concept which is closely related and has its origin in the concept of coherence, from a Noetherian ring theory point of view. Section 8 highlights the impact that the research in coherent ring theory has on the various areas of algebra.

2. HILBERT BASIS THEOREM

Hilbert Basis Theorem states that a polynomial ring in finitely many variables over a Noetherian ring is a Noetherian ring. It is natural, and follows the time-honored ring theorists' tradition, that among the first questions to be asked when studying coherence was whether this result is still valid when the

Noetherian hypothesis is replaced by the coherence hypothesis. The answer in general is no, and a ring R satisfying the property that the polynomial rings in finitely many variables over R are coherent rings along with R, was termed a *stable coherent ring*. The effort exerted in order to determine exactly which rings are stably coherent rings led, among other results, to a classification theory for rings of global dimension two, and to the conceptualization of a unifying notion of non-Noetherian grade. The search for an answer proceeded approximately in order of complexity of the ring.

Von Neumann regular rings are stably coherent rings. This result was proved independently by SOUBLIN [92], 1970, SABBAGH [78], 1974, VASCONCELOS [97], 1976 and FIELDHOUSE [25]. Valuation rings are stably coherent rings. This result was proved independently by LEBIHAN [58], 1971 and SABBAGH [78], 1974. More generally, semihereditary rings are stably coherent rings. This result was proved by VASCONCELOS [96, 97], 1975-1976. It follows that hereditary rings are stably coherent rings.

The ordering of complexity employed here has to do with two homological invariants of a ring R that measure how close its modules are to being flat or projective.

The flatness measurement, called the weak dimension, is denoted by $w.dim\, R$. $w.dim\, R = 0$ means that all R modules are flat; this is equivalent to R being a Von Neumann regular ring. $w.dim\, R = 1$ means that all ideals, but not all modules, of R are flat. Together with coherence, this property is equivalent to semihereditarity. $w.dim\, R = 2$ means that ideals of R are 'once removed' from being flat; that is, if I is an ideal of R, and we map onto I a free R module F, by a homomorphism ρ, then in the exact sequence $0 \to K \to F \xrightarrow{\rho} I \to 0$, with $K = \ker \rho$, we have that K is a flat R module. That is then the next order of complexity of a ring related to the weak dimension, and here Hilbert Basis Theorem breaks down. SOUBLIN [90], 1968, provided an example of a coherent ring R, of $w.dim\, R = 2$ for which $R[x]$ is not coherent.

EXAMPLE. $R = \prod_{\alpha \in N} S_\alpha$, where N denotes the natural numbers, and $S_\alpha \approx \mathbb{Q}[[t, u]]$, the power series ring in indeterminates t and u over the rational numbers $\mathbb{Q}$. R is a coherent ring (not a domain) of weak dimension two, and the polynomial ring in one variable over R is not a coherent ring.

The second invariant employed in the ordering of the ring complexity measures how close are the R modules to being projective. This invariant is called the global dimension of R and is denoted by $gl.dim\, R$. The relation between the first and second invariants is $w.dim\, R \leqslant gl.dim\, R$. Now $gl.dim\, R = 0$ means that every R module is projective. These rings are actually Noetherian (semisimple) and, thus Hilbert Basis Theorem holds for them. $gl.dim\, R = 1$ means that every ideal of R, but not every module of R, is projective. Those are exactly the hereditary rings and as mentioned above, are stably coherent rings. $gl.dim\, R = 2$ means that the ideals of R are 'once removed' from being projective, in the same sense as the ideals of a ring of weak dimension two are 'once removed' from being flat. Surprisingly, in view of the situation for coherent rings of $w.dim\ R = 2$, we have the result that coherent rings of global dimension two are stably coherent rings. This result was proved by VASCONCELOS [95, 97], 1973, 1976, and GREENBERG & VASCONCELOS [36], 1976. The proof makes use of a relation between a local ring of global dimension two and a special format cartesian square of rings. These cartesian squares were then investigated in their own right by GREENBERG [35,37], 1974, 1978, and used by VASCONCELOS [93, 94, 97], 1972, 1974, 1976, to obtain a classification of rings of global dimension two.

It was noticed, relatively early, that induction on the number of variables used in the Noetherian case does not work in the coherent case. Rather, in most proofs of stable coherence, the variables were used as one entity $x = \{x_1, \ldots, x_n\}$. Whether the coherence of the polynomial ring in one variable over a ring R implies the stable coherence of R, is a question that is still open.

ALPHONSI's Ph. D. thesis [4], 1977, and subsequently published paper [5], 1980, had a significant impact on the study of stable coherence, and on the general theory of commutative rings. At that time, there were several existing definitions of non-Noetherian grade (see NORTHCOTT [65, 66], 1968, 1976 and BARGER [9, 10], 1970, 1972) all having their useful properties. In his work Alfonsi defined a unifying and more general non-Noetherian grade. This allowed him to prove, among other results, the most general version of Buchsbaum-Eisenbud exactness criteria for complexes over not necessarily Noetherian rings. This criteria he used to prove the following result:

THEOREM. *Let R be a coherent ring of finite weak dimension, then the polynomial ring $R[x_1, \ldots, x_n]$ is a coherent ring if and only if $R_P[x_1, \ldots, x_n]$ is a coherent ring for every prime ideal P of R.*

This result greatly simplified all existing proofs of stable coherence, which were lengthy and complex, GLAZ [32]. As an example, consider a Von Neumann regular ring R. It is known that in this case R_P, the localization of R at a prime ideal P, is a field for every prime ideal P of R, so the usual Hilbert Basis Theorem guarantees that $R_P[x_1, \ldots, x_n]$ are Noetherian, and therefore coherent, rings, and thus by Alfonsi's Theorem, so is $R[x_1, \ldots, x_n]$. The four original proofs of this fact are each technically difficult and at least a page long.

Using this theorem, Alfonsi sharpened Soublin's counterexample to the case of a coherent domain of weak dimension equal to two, namely one of the localizations of the ring in Soublin's original example.

3. THE INTEGRAL CLOSURE OF A COHERENT RING

For a domain R with field of quotients $Q(R)$, let $\overline{R} = \{s \in Q(R),\ f(s)=0$ for some polynomial $f(x)$ with coefficient of highest power of x in $f(x)$ equal of one$\}$. The ring $\overline{R}$ is called the integral closure of R. The relation between the properties of a domain R and that of $\bar{R}$ is traditionally utilized to gain more information on the ring itself. In particular, one can ask to what extent properties of R are reflected by $\bar{R}$.

If R is a Noetherian domain, we have accumulated a considerable amount of information in that direction.

KRULL [54], 1930, and AKIZUKI [3], 1935, proved that if R is a Noetherian domain of *Krull dim* $R = 1$, then $\bar{R}$ is a Prüfer Noetherian domain (so-called Dedekind domain). NAGATA [64], 1962, proved that if R is a Noetherian domain of *Krull dim* $R = 2$, then $\bar{R}$ is a Noetherian domain. For rings R, with *Krull dim* $R \geqslant 3$, MORI [62], 1953, and NAGATA [63], 1955, showed that $\bar{R}$ may not be Noetherian, but it retains some finiteness properties by being a so-called Krull domain.

The Krull dimension of a ring can, roughly speaking, be calculated as follows. Consider chains of prime ideal in R, $0 \subset P_1 \subset P_2 \subset \ldots \subset P_n$. If this is the longest possible such chain, then *Krulldim* $R = n$. The results mentioned above prove that the Noetheriannes property if preserved by the integral closure of Noetherian rings of Krull dimensions less than or equal to two but may fail to do so for higher Krull dimensions.

The investigations into the nature of $\bar{R}$, in case R is a coherent domain,is at its very beginning. It is known that *Krulldim* $R =$ *Krulldim*$\bar{R}$ (see KAPLANSKY [53]). Coherence is not necessary of this result. Also, one can show that if R is an integrally closed (that is, $R = \bar{R}$) coherent domain of Krull dimension one, then R is a Prüfer domain. With this observation in mind, Vasconcelos posed the following question:

QUESTION. Let R be a coherent domain of Krull dimension one. Is $\bar{R}$ a Prüfer domain (and, hence, necessarily coherent)?

The answer to this question is not yet known. The only known example of a nontrivial coherent domain of Krull dimension one, due to HOCHSTER [28], 1984, has Prüfer integral closure. The research into the nature of the integral closure of a coherent domain of Krull dimension one resulted in several instances of positive answers, and branched out into investigating, on one hand, overrings of domains (an overring of a domain R is a ring S with $R \subset S \subset Q(R)$) and, on the other hand, properties of prime ideals in polynomial rings.

PAPICK [69, 70, 71], 1978-1979, employed an overring approach to this question. In the process, the one dimensionality property is dropped, to be replaced by a stronger coherence assumption. Papick proved that if each proper overring of a domain R is a coherent ring, then $\bar{R}$ is a Prüfer domain. DOBBS [24], 1978, showed that if R is a going down domain the conclusion is still valid if we ask that every overring of R be merely locally coherent. Papick's investigation led to a thorough study of pairs of domains for which each intermediate domain is a coherent ring, PAPICK [72], 1981.

GLAZ & VASCONCELOS [28], 1984, approached the question by trying to exploit the relation between the Prüfer property of $\overline{R}$, and the content of certain prime ideals in the polynomial ring in one variable over R. (If P is a prime ideal in $R[x]$-the polynomial ring in one variable over R, the content of P, $c(P)$ is equal to the ideal of R generated by the coefficients of all the polynomials in P). The prime ideals in question, so-called uppers of zero satisfy that $P \cap R = 0$. Among other results, Glaz and Vasconcelos proved that if R is a coherent domain possessing a canonical module and satisfying that the uppers of zero over R are finitely generated, then $\overline{R}$ is a Prüfer domain. In the process they thoroughly explored the question of finite generation of uppers of zero over coherent domains. The investigation into the nature of uppers of zero and related ideals continues. See, for example, HOUSTON & ZAFRULLAH [50], and HAMANN, HOUSTON & JOHNSON [42].

4. MIN(R) AND THE TOTAL RING OF QUOTIENTS OF R.

For a ring R, $Min(R)=\{P$, P is a minimal prime ideal of R in the sense that no other prime ideal of R is contained in $P\}$, and $Q(R)=\{a/b,\ a,b \in R$, and b is a nonzero divisor in $R\}$. $Min(R)$ is called the space of minimal prime ideals of R, and $Q(R)$ is called the total ring of quotients of R. Note that for a domain R, $Min(R) = \{0\}$ and $Q(R)$ is the field of quotients of R. In 1965, HENRIKSEN and JERISON [46] published a paper exploring several properties of $Min(R)$. They modestly stated: 'We are focusing attention on the space of minimal prime ideals because of its special role to the case of rings of continuous functions,' and indeed, at that time it was hard to foresee the importance of this set to the study of classical non-Noetherian rings. The set of all prime ideals of a ring R, the so-called spectrum of R, denoted by $Spec(R)$ can be topologized in several ways. The most useful of these topologies is the so-called Zariski topology. In the Zariski topology $Min(R)$ becomes a subspace of $Spec(R)$. Among other results, Henriksen and Jerison conjectured that for a reduced ring R (that is, a ring R with no nilpotent elements) $Min(R)$ is compact if and only if $Q(R)$ is a Von Neumann regular ring.

In 1967-1968, OLIVIER [67, 68] discovered relations between the compactness of $Min(R)$, the Von Neumann regularity of $Q(R)$, and the behavior of a certain universal algebra of R, namely the maximal flat epimorphic extension of a reduced ring R.

It is only with the publication of QUENTEL'S paper [73] in 1971 that the exact relation between the compactness of $Min(R)$ and the Von Neumann regularity of $Q(R)$ was clarified, strangely enough, due to several errors in this otherwise excellent paper. This clarification contributed significantly to our understanding of the classical non-Noetherian rings.

QUENTEL provided a counterexample to Henriksen and Jerison's conjecture by exhibiting a reduced ring R with compact $Min(R)$, whose total ring of quotients (equal, in this case, to R itself) is not a Von Neumann regular ring. This example, basically correctable and strikingly clever, nevertheless represented a challenge inherent in the difficulties and minor errors of its exposition. A correct version appears in [32]. I believe a similarly intriguing challenge motivated several people when confronting the error in the proof of one of Quentel's results, providing an exact relation between the compactness of $Min(R)$ and the Von Neumann regularity of $Q(R)$. Quentel corrected this error in an erratum [75], published a year later, by weakening the original result. Unfortunately, many of the important results in the rest of his paper relied heavily on the corrected theorem, resulting in mathematical confusion. Faced with this challenge, several other approached to the topic appeared. One such work, MATLIS [60], 1983, attacks the problem from a completely different perspective. Further clarifications by different methods can be found in VASCONCELOS [97] and in GLAZ [32]. The interesting phenomenon in which an excellent piece of work containing some errors can, by virtue of the right combination between the originality of the work and the nature of the errors, revive an entire area of research, is not new in mathematics. For another example of this phenomenon see HERMANN's paper [47] and SEIDENBERG's 'correction' papers [80, 81, 82].

The combined efforts of all those distinct approaches yielded the following theorem:

THEOREM. *Let R be a ring. The following conditions are equivalent.*
(1) *$Min(R)$ is compact, and every principal ideal of R is flat.*
(2) *Every principal ideal of R is projective.*
(3) *$Q(R)$ is Von Neumann regular, and every principal ideal of R is flat.*

MATLIS [60] and QUENTEL [73] derived other characterizations of the compactness of $Min(R)$, as well as of the Von Neumann regularity of $Q(R)$.

One of the consequences of this theorem is the proof that a coherent ring of finite weak dimension has a Von Neumann regular total ring of quotients. This result yielded, in its turn, most useful characterizations of semihereditary and hereditary rings.

COROLLARY 1. *Let R be a ring. The following conditions are equivalent:*
(1) *R is a semihereditary ring.*
(2) *R is a coherent ring and w.dimR $\leqslant 1$.*
(3) *$Q(R)$ is a Von Neumann regular ring, and R_m is a valuation domain for every maximal ideal m of R.*

COROLLARY 2. *A ring R is hereditary if and only if $Q(R)$ is hereditary, and any ideal of that R is not contained in any minimal prime ideal of R is projective.*

The proof of Corollary 1 us due to MCRAE [61] for $(1)\leftrightarrow(2)$, and QUENTEL [73] for $(1)\leftrightarrow(3)$. The original proof of Corollary 2 is due to MAROT [59], while proofs using the results of this section can be found in VASCONCELOS [97] and GLAZ [32].

The interplay between the compactness of $Min(R)$ and other properties of the ring R continues to be explored through various approaches. See, for example, HUTSON [51], 1988.

5. COHERENT REGULAR RINGS

A commutative ring R is called a *regular ring* if every finitely generated ideal of R has finite projective dimension, that is, every finitely generated ideal of R is 'finitely many places' removed from being projective, in the sense described in Section 2. For a Noetherian ring R this definition coincides with the classical definition of regularity, namely that every localization of R at a prime ideal is a unique factorization domain.

There is a strong relation between the condition of regularity of a ring R and the behavior of the weak and projective dimensions of modules over R. For a Noetherian local ring R, Serre proved that R is a regular ring if and only if $w.dim\, R < \infty$. In this case we also have that $w.dim\, R = Krull\, dim\, R$. For a coherent ring R the situation is not quite as tame. There are local, coherent regular rings of infinite weak dimension.

EXAMPLE. $R = k[[x_1, x_2, \ldots]]$, the power series ring in infinitely many variables over a field k, is a local, coherent regular ring of infinite weak dimension.

Nevertheless, any coherent ring of finite weak dimension is a regular ring and, therefore, the class of coherent regular rings includes all classical non-Noetherian rings, in particular Von Neumann regular, semihereditary, and hereditary rings.

The notion of regularity has been extended from Noetherian rings to coherent rings with a considerable degree of success.

BERTIN [11], 1971, extended the definition of regularity from Noetherian to local coherent rings. He also proved that such rings are integrally closed.

QUENTEL [74], 1971, found necessary and sufficient conditions for a local, coherent regular ring to be a unique factorization domain. Contrary to the Noetherian case, this is not always true.

VASCONCELOS [97], 1976, proved that a local coherent regular ring is a greatest common divisor domain. He also dropped the 'local' condition from the definition of regularity.

When working with the notion of coherent regularity it is convenient to separate, in a sense, the finiteness condition of coherence from the homological condition on the ideals of the ring; thus, GLAZ [29], 1982, arrived at the present definition. In addition, in GLAZ [31], a result is proved, which resembles Serre's condition for local Noetherian regularity, namely that for a local coherent regular ring R, $depth\, R = w.dim\, R$. The similarity of this result to Serre's condition is derived from the fact that, for a local ring R, depth of R is a notion related to non-Noetherian grade which, for a local Noetherian regular ring, coincides with the Krull dimension of the ring.

Applying the results of section 4 we see that the total ring of quotients of a coherent regular ring is a Von Neumann regular ring.

Research into the regularity of coherent rings branched out naturally into two, not necessarily mutually exclusive, directions. One direction explores the structural properties of a coherent regular ring. The results described above comprise the total amount of knowledge known in the subject. Given that our understanding of the structural properties of Noetherian regular rings is considerably broader, and it is still a vigorous subject of research, we see that much more can be accomplished in that direction. One very basic topic of interest will be the determination of the structure and properties of complete coherent regular rings.

The other direction taken in the study of regularity for coherent rings deals with the following set up: Given a ring R and an R algebra S, find conditions for the extension $R \to S$ to ascend or descent coherent regularity. Here 'ascent' (respectively 'descent') of a property (P) means: if R (respectively S) satisfies (P), so does S (respectively R). In particular, explore the exact relation between the weak dimension of R and that of S, and determine necessary and sufficient conditions for ascent or descent of Von Neumann regularity, semihereditarity, and hereditary. The algebras S considered so far are: polynomial rings, group rings, symmetric algebras, and various localizations of polynomial rings.

GLAZ [30], 1988, proved that the polynomial rings over a stably coherent regular ring are coherent regular rings.

Using this result GLAZ [29, 30], 1987-1988, derived necessary and sufficient conditions for the group ring RG of an abelian group G over a coherent ring R to be coherent regular. Through a relation between the weak dimension of R, that of RG and the rank of G, conditions were found for RG to be von Neumann regular (see also AUSLANDER [7], 1957) or semihereditary, GLAZ [29], 1987. The conditions found in GLAZ [29] for the group ring RG to be a coherent ring tie up with a topic of interest in noncommutative ring theory, namely the determination of condition which will yield the coherence of RG, and related algebras, given that R or G or both are not commutative. See, for example, CHOO, LAM, and LUFT [19], 1973, BIERI and STREBEL [12], 1979, ABERG [1], 1982, and DICKS & SCHOFIELD [22], 1988.

GLAZ [30], 1988, exhibited conditions for $S_R(M)$, the symmetric algebra of a rank one flat module M over a Noetherian ring R to be coherent regular.

In [31] and [33] GLAZ explored the coherence regularity, in particular Von Neumann regularity, semihereditary, and hereditary of two well-known localizations of the polynomial ring in one variable over a ring R, namely the rings $R<x>$ and $R(x)$.

6. RING CONSTRUCTIONS

The special format cartesian square of rings, mentioned in section 2, is but one of the ring constructions developed and explored as a direct consequence of research into coherent rings.

Another class of such ring constructions is the class of rings of the type $D + M$. These rings arise frequently in algebra, especially in connection with counterexamples. The original definition required a valuation domain T with maximal ideal M, $K = T/M \subset T$, and D a subring of K. Then the ring $D + M$ is equal to $\phi^{-1}(D)$ where $\phi: T \to T/M$ is the natural map $\phi(t) = t + M$. If K is actually contained in T, the notation $D + M$ is justified by the structure of $\phi^{-1}(D)$. An account of the basic properties of this construction can be found in GILMER [27]. The introduction of coherence resulted in a renewed interest in the properties of these rings and, consequently, in their generalization. It started as a not-so-simple question: When is a ring of the original $D + M$ type a coherent ring? This question was answered in 1976 by DOBBS & PAPICK [23]. The definition of the original construction was then extended to include any domain T and any field K which is a retract of T, that is, $K \subset T$ and there is a map $\psi: T \to K$ satisfying $\psi(t) = t$ for every t in K. The properties of these newly defined rings, including coherence, were explored by BREWER & RUTTER [14], 1976. Another ramification of the original definition was to consider rings of the type $R = D + xK[x]$ or $= D + xD_S[x]$, where K is a field, D is a subring of K with field of quotients equal to K, S is a multiplicatively closed subset of D, and x is an indeterminate. The properties of these rings were explored by COSTA, MOTT & ZAFRULLAH [21] in 1978, and ZAFRULLAH [98]. The property of coherent regularity and the behavior of the weak dimension of these ring constructions have not yet been explored.

Let R be a ring and let M be an R module, *the trivial extension of R by M*, denoted by $R \alpha M$ is the set $R \times M$ with natural addition and multiplication defined by $(r, m)(r', m') = (rr', rm' + r'm)$ for all $r, r' \in R$ and $m\, m' \in M$. This construction was described and used by NAGATA [64] in 1962. In 1975 VASCONCELOS [96] defined yet another useful measure of the coherence of a ring R, the so-called λ *dimension of R*. In his book [97], describing in more detail the machinery of the λ dimension, he posed the following problem: 'Exhibit all positive integers as λ dimensions of commutative rings.' This problem prompted a renewed investigation into the coherence and homological properties of trivial ring extensions and resulted in the following attractive answer provided by ROOS [77] in 1931:

THEOREM. *Let R be a local Noetherian Gorenstein ring with maximal ideal m. Let $E(R/m)$ be the injective envelope of R/m, then*

$\lambda\, dim\, R\, \alpha\, E(R/m) = Krull\, dim\, R.$

There exists a local, Noetherian, Gorenstein ring of any given Krull dimension, for example $R = k[[x_1, \ldots, x_n]]$, the power series ring in n variables over a field k is a local, Noetherian, Gorenstein ring of $Krulldim\, R = n$. Therefore, $E(R/m)$ the injective envelope of R/m (which in some sense is the smallest injective module containing R/m) satisfies $\lambda\, dim\, R\, \alpha\, E(R/m) = n$, and Vasconcelos' problem is solved. A category theory approach to trivial ring extensions, including their coherence properties, can be found in FOSSUM, GRIFFITH & REITEN [26], 1975.

Let R be a coherent ring and let $R[[x]]$ be the power series ring in one variable over R. Contrary to the existing situation for a polynomial ring over R, $R[[x]]$ need not be a coherent ring, even if the ring R is Von Neumann regular. The following example is due to SOUBLIN [92], 1970:

EXAMPLE. Let R be the ring of all stationary sequences of rational numbers with natural addition and multiplication. R is a Von Neumann regular ring, and $R[[x]]$ is not a coherent ring.

In 1977, BREWER, RUTTER & WATKINS [15] determined necessary and sufficient conditions for a power series ring over a Von Neumann regular ring, to be a coherent ring. JØNDRUP & SMALL [52], 1974, and VASCONCELOS [96], 1975, proved independently that if R is a valuation domain of rank $R > 1$, then $R[[x]]$ is not a coherent ring. No other result concerning the coherence of a power series ring is known. It will be of interest, for example, to explore the situation for valuation domains of rank one.

7. UNIFORMLY COHERENT RINGS

The notion of uniform coherence was first introduced by SOUBLIN [88] in 1968.

DEFINITION. A commutative ring R is called a *uniformly coherent ring* if there exists a map $\phi:\mathbb{N} \to \mathbb{N}$, where $\mathbb{N}$ denotes the natural numbers, such that for every $n \in \mathbb{N}$, and any nonzero homomorphism $f: R^n \to R$, the kernel of f is generated by $\phi(n)$ elements. The map ϕ is called the *uniformity map* of R.

Considering maps f from the free module R^n to R, we see that $Im\ f$ is an ideal of R generated by n elements, and conversely, given I an ideal of R generated by n elements, there is a map f from the free module over R, R^n onto I. Thus, the uniformity condition of the definition guarantees that for every ideal I of R generated by n elements, the modules of relations between the generators of I, that is, the modules $ker f$ for the various maps: $f: R^n \to I$ have a finite and bounded number of generators that depends on n alone (and not on I or on f).

In [88, 91] SOUBLIN exhibited several of the basic properties of uniformly coherent rings, in particular the close relation of this notion to coherence. Thus, a uniformly coherent ring is a coherent ring, while for a ring R to be uniformly coherent, it is necessary and sufficient that the ring R^N, where N denotes the natural numbers, is coherent.

In 1972 QUENTEL [76] exposed the exact relation between the uniform coherence of a Noetherian ring and that of its localizations, namely:

THEOREM. *A Noetherian ring R is a uniformly coherent ring if and only if all the localizations of R at maximal ideals are uniformly coherent rings admitting a common uniformity map.*

Quentel also provided an example of a Noetherian ring which is not uniformly coherent. In 1916, MACAULAY [2] constructed prime ideals P^m in $k[x,y,z]$, the polynomial ring in three variables over a field k, which need at least m generators. Proving that the number of generators of prime ideals in a Noetherian uniformly coherent ring is bounded, Quentel showed that $k[x,y,z]$ is not a uniformly coherent ring.

In conjunction with the above theorem, the question as to which Noetherian rings are uniformly coherent was settled between 1978 and 1983.

THEOREM. *A local Noetherian ring R is a uniformly coherent ring if and only if Krull dim R* $\leqslant 2$.

The proof that a local Noetherian ring of Krull dimension two is a uniformly coherent ring is due to SALLY [79], 1978. The converse was proved independently by GOTO [34], 1983, and by KUMAR & NISHIMURA [56], 1983.

Which coherent rings are uniformly coherent? The answer to this question is not yet known, although it is clear that it has to differ from the answer given in the Noetherian case. SOUBLIN [88], 1968, had shown that a local ring of global dimension less or equal to two is a uniformly coherent ring; thus, any valuation domain of global dimension less or equal to two is a uniformly coherent ring. It is known [97] that a valuation domain V has global dimension less or equal to two if and only if every ideal of V can be generated by less or equal to χ_0 generators. Let n be a given positive integer, and take V to be a valuation overring of $k[x_1, \ldots, x_n]$, the polynomial ring in n variables over a finite field k. Then V is a valuation domain of *Krull dim* $V = n$, and the ideals of V are generated by less or equal to χ_0 generators; therefore, V is uniformly coherent. We conclude that there exists a uniformly coherent ring in any given Krull dimension.

8. CONCLUDING REMARKS

Just as the introduction of coherent sheaves revolutionized algebraic geometry and, thus, contributed to a development which is Noetherian in nature, so do other coherent objects appear in different, and sometimes unexpected, algebraic settings, and contribute to a deeper understanding of the area.

It is my hope that this paper has succeeded in conveying the magnitude of the impact that research into coherent rings had on the development of commutative algebra. This direct contribution to enrichment of knowledge, broadening of concepts, and inspiration in defining related objects is not the only way that the notion of coherence influences research in commutative algebra. Coherent rings often become relevant in an indirect way, that is, when the original investigation involved, to start with, only Noetherian or general rings. Such an instance occurs when an investigated algebra over a Noetherian ring ceases to be Noetherian. One example of this kind, described in section 5, GLAZ [30], is the coherence of the ring $S_R(M)$, the symmetric algebra of a rank one flat module over a Noetherian ring R of finite Krull dimension. It is only through the coherence of this ring that properties like its regularity can be properly explored and understood. In general, it seems that rings 'larger than Noetherian' play an increasingly important role in constructions of Noetherian commutative algebra proper. A typical example is the recent works of HOCHSTER & HUNEKE [48, 49] on the absolute integral closure (in the sense of ARTIN [6]) of a Noetherian domain. An investigation into the coherence properties of these ring constructions will, no doubt, shed light on aspects of Noetherian ring theory. An example where coherence appears in a general setting can be found in HEINZER & PAPICK [45]. This paper explores properties of contracted ideals. A crucial step in their investigation, indeed the one that, to quote the authors, 'sets the stage for the succeeding results.' is the discovery that the rings involved are coherent.

The notion of coherence is not restricted to algebraic geometry and commutative algebra. As mentioned in section 5, noncommutative coherent rings are explored in the context of group rings and related algebras. Here one should also mention the free associative algebras explored by P.M. COHN [20] in the framework of the, necessarily coherent, free ideal rings. Though the (commutative) polynomial ring in n variables $x_1, \ldots, x_n$ over a field k is a Noetherian ring, its noncommutative counterpart, the free associative algebra $k < x_1, \ldots, x_n >$ is merely coherent. Coherent groups were introduced by BIERI & STREBEL [12], and coherent matrices were defined by CHOO, LAM & LUFT [19]. In [8] AUSLANDER defined and explored coherent functors, while FOSSUM, GRIFFITH & REITEN [26] provide a good source for learning some of the work done in coherent categories. The study of coherent objects conducted in each of these diverse areas serves as an intermutual source of inspiration and, thus, contributes to the advancement of all areas of algebra.

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