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REGULAR AND PRINCIPAL PROJECTIVE ENDOMORPHISM RINGS OF MIXED ABELIAN GROUPS

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1. Introduction

Let G be a mixed abelian group and let R = E(G) denote the endomorphism ring of G. This paper investigates the problem: under which conditions is the ring R is a von Neumann regular or a right principal projective (pp) ring?

The question as to when the ring of endomorphisms of an abelian group, or more generally of a module over a ring, satisfies some homological property has been studied by many authors. For a representative, but by no means complete, sample of papers on von Neumann regular, hereditary and pp endomorphism rings see ([A], [H-P], [Kr], [Kh], [L], [P], [F-R], [R-1], [R-2], [R-3]). In addition, a good discussion of properties of endomorphism rings of abelian groups can be found in Fuchs' book [F]. We remark that, for T a torsion group, our problem, posed in the first paragraph, has a nice solution. In [R-1] it is shown that, if $T = \oplus T_p$, where T_p denotes the p-component of T, then E(T) is (right or left) pp if and only if each T_p is either divisible or elementary ($pT_p = 0$). In [F-R] the authors prove that E(T) is von Neumann regular if and only if each T_p is elementary.

Of particular interest to us, in the mixed case, were the works of Rangaswamy [R-1], [R-2], [R-3] and of Fuchs and Rangaswamy [F-R], which investigated Baer, pp (called weak Baer in [R-1]) and von Neumann regular endomorphism rings. More specifically, in [R-1] it was shown that, for a large class of mixed abelian groups G with pp endomorphism rings, G can be embedded as a pure subgroup of ΠG_p containing $\oplus G_p$. Additionally each G_p must be an elementary p-group. Here G_p denotes the p-torsion subgroup of G. Reversing the outlook, we ask: if G is a mixed group with each G_p elementary, embedded as a pure subgroup of ΠG_p containing $\oplus G_p$, when is R a pp or a von Neumann regular ring? If the torsion-free rank of G is finite, we can obtain fairly definite answers to these questions in terms of a maximal independent torsion-free subset of G and an associated matrix algebra. This is done in Section 3, Theorems 3.5 and 3.8. In Section 4 we consider the case when $R = E(G)$ is commutative, or, equivalently, when rank $G_p \leq 1$ for all p. Our main result here is that, for S an infinite set of primes, each pure subgroup of $\Pi_{p\in S}Z(p)$ containing $\oplus_{p\in S}Z(p)$ that is of finite torsion-free rank has a von Neumann regular endomorphism ring. Here Z(p) is the cyclic group of order p.

2. Definitions and known results

Throughout the paper G will be a mixed abelian group and R will be the endomorphism ring of G. We regard G as a left $R = E(G)$ module. For a prime p, G_p will denote the p-torsion subgroup of G. Let $S = S_G = \{p \mid G_p \neq 0\}$. To avoid trivialities, we always will assume that, for each group G under consideration, the set S_G is infinite. A mixed group G is called S-local if $pG = G$ for all $p \notin S_G$. A ring R is (right) pp if every principal right ideal of R is projective, equivalently if the right annihilator in R of every element of R is generated by an idempotent.

It is well known that von Neumann regular rings are pp, but the converse does not necessarily hold, even for endomorphism rings ([R-1]). If $R = E(G)$ both the von Neumann regular and pp properties of R are reflected by the behavior of kernels and images.

Theorem 2.1 Let G be an abelian group with endomorphism ring R. Then:

(1) ([F-R]) R is von Neumann regular if and only if for every $\lambda \varepsilon R$ both $\ker \lambda$ and $\operatorname{im} \lambda$ are direct summands of G.

(2) ([R-1]) If, for every $\lambda \varepsilon R$, $\ker \lambda$ is a direct summand of G then R is pp. Indeed, if $e:G \to \ker \lambda$ is any projection, then the right annihilator of λ in R is eR.

(3) ([R-1]) Suppose G is a mixed S-local reduced group. Further suppose that $R = E(G)$ is pp. Then, for every $\lambda \varepsilon R$, $\ker \lambda$ is a direct summand of G.

The next theorem, slightly translated from [R-1], provided our main motivation for this paper. A sketch of the proof is included for the reader's convenience. Let Z/p^kZ denote the ring of integers mod p^k.

Theorem 2.2. Let G be a mixed reduced S-local group with $R = E(G)$ a pp ring. Then each G_p is elementary and G can be embedded as a pure subroup of $\Pi_{p\varepsilon S}G_p$ containing $\oplus_{p\varepsilon S}G_p$.

Proof. If $E(G)$ is pp and H is a summand of G then $E(H) \cong eE(G)e$ where e is a projection of G onto H. It follows that $E(H)$ is also pp. Since, for $k > 1$, $E(Z(p^k)) = Z/p^kZ$ is not pp, the group G can have no $Z(p^k)$ summand for $k > 1$. Hence each G_p must be an elementary p-group. Thus, for each $p \varepsilon S$, we have $G = G_p \oplus H_p$. Furthermore H_p must be p-divisible, otherwise there would be a nonzero map from H_p/pH_p into G_p which would provide a $\lambda \varepsilon R$ such that λR is not projective. Define $\iota : G \to \Pi_{p\varepsilon S}G_p$ by $\iota(g) = (\pi_p(g))$ where π_p is a projection of G onto G_p. It is not hard to check that $\ker \iota$ is a divisible subgroup of G; hence $\ker \iota = 0$ and ι is an embedding. Finally, one can check that $\oplus_{p\varepsilon S}G_p \leq \iota(G)$ and that $\iota(G)$ is pure in $\Pi_{p\varepsilon S}G_p$.

In [L] it is shown that if $R = E(G)$ and $\lambda \varepsilon R$ with $\operatorname{im} \lambda$ a direct summand of G then $R\lambda$ is projective. Thus, we have the left handed version of Theorem 2.1 (2): If im λ is a direct summand of G for each $\lambda \varepsilon R$ then R is left pp. Curiously, the left handed version of Theorem 2.1 (3) fails. In Example 4.3 we construct a mixed reduced S-local group G with endomorphism ring R such that R is commutative pp but not von Neumann regular. In view of Theorem 2.1 (1) and (3), we cannot have im λ a summand of G for each $\lambda \varepsilon R$.

3. The finite torsion-free rank case

Throughout this section G will be a mixed abelian group with $T(G) = \oplus_{p \varepsilon S} G_p < G < \Pi_{p \varepsilon S} G_p$ (we use the symbol < to denote a pure subgroup). Note that under these assumptions G/T(G) will be a divisible torsion-free group. We regard G/T(G) as a Q-vector space. We also assume that each G_p is elementary and regard each G_p as a vector space over the field Z/pZ. In view of Theorem 2.2 such groups G are the only candidates for mixed reduced S-local groups with pp endomorphism rings. We investigate the exact conditions under which such a group G will have a pp or a von Neumann regular endomorphism ring.

Henceforth, we will often drop reference to the (necessarily infinite) index set $S_G = \{p \mid G_p \neq 0\}$, if there is no need to refer to it, and just write $\oplus G_p$ or ΠG_p for the direct sum or product over all primes p in S. When discussing a mixed group G, all primes referred to are always assumed to be in the appropriate S_G.

To make our problem more tractable we work in the finite torsion-free rank case, that is we assume that the dimension of the Q-vector space G/T(G) is finite. If the torsion-free rank of G is equal to n we call a subset $X = \{x_1, ..., x_n\} \subset G$ such that $\{x_1+T(G), ..., x_n+T(G)\}$ is a Q-basis for G/T(G) a **maximal independent set** in G. If $X \subset G$ is a maximal independent set then $G < \Pi G_p$ has the following simple description: G is the set of all elements $g = (g_p) \varepsilon \Pi G_p$ such that there exists rational numbers $\alpha_1, ..., \alpha_n$ and a natural number k (depending on g), such that $p \geq k$ implies $g_p = \alpha_1 x_{1p} + ... + \alpha_n x_{np}$. Here x_{ip} denotes the p-component of $x_i \varepsilon \Pi G_p$. Note that, for any rational number $\alpha = a/b$ in reduced form, and for any prime p with $p > |b|$, α has an unique interpretation as an element of Z/pZ. Thus, for any fixed finite set of rationals $\{\alpha_1, ..., \alpha_n\}$, the expression $\alpha_1 x_{1p} + ... + \alpha_n x_{np}$ will make sense for almost all primes p.

If $R = E(G)$ then R_p, the p-torsion subgroup of R, can be naturally identified with $E(G_p)$. Since the restriction map $\lambda \to \lambda|_{T(G)}$ is a monomorphism from R into $E(T(G)) \cong \Pi E(G_p) \cong \Pi R_p$, we can regard $R \subset \Pi R_p$ ([R-1]). In this setting R is simply the subring of $\Pi R_p = \Pi E(G_p)$ consisting of all maps which send G into G. As a consequence, R is a pure subring of ΠR_p. The following lemma provides a useful criterion for deciding when an element of ΠR_p is an endomorphism of G.

Lemma 3.0. With notation as above, $\lambda \in \Pi R_p$ is an endomorphism of G if and only if $\lambda(X) \subset G$ for any maximal independent set $X \subset G$.

Proof. Suppose $X = \{x_1, ..., x_n\}$ is a maximal independent subset of G and $\lambda \in \Pi R_p$ with $\lambda(X) \subset G$. Since $\Pi R_p = \mathrm{End}(\Pi G_p)$ and $\oplus G_p$ is the torsion subgroup of both G and ΠG_p, it follows that, for any $g \in G$ of finite order, $\lambda(g) \in \oplus G_p \subset G$. Let $g \in G$ be an element of infinite order. Since $X = \{x_1, ..., x_n\}$ is a maximal independent set, there is a dependence relation $mg = \Sigma m_i x_i$, with m and the m_i's integers, $m \neq 0$. Thus $m\lambda(g) = \lambda(mg) = \Sigma m_i \lambda(x_i) \in G$. Since G is pure in ΠG_p it follows that $m\lambda(g) = mg'$, for some $g' \in G$. Thus $m(\lambda(g) - g') = 0$. Hence $(\lambda(g) - g') \in G$. Therefore, $\lambda(g) = g' + (\lambda(g) - g')$ is in G and the proof is complete.

Fix a maximal independent set $X \subset G$. Let $X = \{x_1+T, ..., x_n+T\}$ be the basis of $G = G/T(G)$ obtained from X. Each $\lambda \in R$ induces a Q-linear transformation λ on G. Define a ring anti-homomorphism $\mu = \mu_X : R \to M_n(Q)$ by $\mu(\lambda) = [\mathrm{mat}(\lambda)_X]^t$, where $[\mathrm{mat}(\lambda)_X]^t$ denotes the transpose of the matrix of the induced map λ with respect to the basis X. (We are using the transpose instead of the matrix itself simply for notational convenience in what follows.) In the following lemma, we record three relevant properties related to the mapping μ.

Lemma 3.1. With notation as above

(a) Let $\lambda = (\lambda_p) \in \Pi R_p$. Then $\lambda \in R$ if and only if:

§ There exists a n x n rational matrix α such that, for almost all p

$$\begin{bmatrix} \lambda_p(x_{1p}) \\ \lambda_p(x_{np}) \end{bmatrix} = (\alpha) \begin{bmatrix} x_{1p} \\ x_{np} \end{bmatrix}.$$

Moreover, if condition (§) is satisified, then the rational matrix α will be precisely $\mu(\lambda)$.

(b) We have $\ker \mu = \mathrm{Hom}(G,T)$ and $\mathrm{im}\, \mu$ is a Q-subalgebra of $M_n(Q)$.

(c) If X' is another maximal independent subset of G and $\mu' = \mu'_{X'}$ is the map from R into $M_n(Q)$ defined for X' as μ is for X, then $\mathrm{im}\, \mu'$ and $\mathrm{im}\, \mu$ are conjugate Q-subalgebras of $M_n(Q)$.

Proof. (a) Suppose $\lambda = (\lambda_p) \varepsilon R$. For $g \varepsilon G$ let $g = g + T$. Since $\mu(\lambda)$ is the transpose of the matrix of λ with respect to X we have the rational matrix equation :

$$\begin{bmatrix} \lambda(x_1) \\ \lambda(x_n) \end{bmatrix} = \mu(\lambda) \begin{bmatrix} x_1 \\ x_n \end{bmatrix}$$

This equation, viewed componentwise, gives (§) for p greater than or equal to some suitably large k. In particular k will be chosen large enough so that $\mu(\lambda)$ can be interpreted as an nxn matrix over Z/pZ. Conversely, if $\lambda = (\lambda_p) \varepsilon \Pi R_p$ satisfies (§) then it is not hard to check that $\lambda(X) \subset G$. Hence, by Lemma 3.0, $\lambda \varepsilon R = E(G)$. Note that in this case $\alpha = \mu(\lambda)$.

(b) By definition of μ we have $\mu(\lambda) = 0$ if and only if λ is the zero map. The latter condition is equivalent to the requirement $\lambda(G) \subset T$. Thus, $\ker \mu = \mathrm{Hom}(G,T)$ and our first claim is proved. To prove the second claim, since μ is a ring anti-homomorphism, we only need show that $\mathrm{im}\, \mu$ is a Q-subspace of $M_n(Q)$. This is equivalent to showing that $\mathrm{im}\, \mu$ is divisible. Since T(R), the torsion subgroup of R, is contained in Hom(G,T) and R/T(R) is divisible, it follows that R/Hom(G,T) is also divisible. Thus, $\mathrm{im}\, \mu$, which is anti-isomorphic to R/Hom(G,T), is divisible.

(c) If X' is another maximal independent subset of G then $X' = \{x_1'+T, ..., x_n'+T\}$ is another Q-basis for G/T. Let β be the invertible nxn rational matrix defined by the matrix equation $X' = \beta X$. For $\lambda \varepsilon R$ we have $\mu'(\lambda) = \beta\, \mu(\lambda)\, \beta^{-1}$. Hence conjugation by β produces an isomorphism from $\mathrm{im}\, \mu$ to $\mathrm{im}\, \mu'$.

For a maximal independent set $X \subset G$ let X_p be the Z/pZ-subspace of G_p spanned by $\{x_{1p}, ..., x_{np}\}$. The set X is called strongly spanning if $X_p = G_p$ for almost all primes p. The set X is called strongly independent if $\{x_{1p}, ..., x_{np}\}$ is an independent subset of the Z/pZ-vector space G_p for almost all primes p.

We remind the reader that all primes referred to are always supposed to be in the set S_G.

Lemma 3.2. For maximal independent sets X,X' of G there exists $k > 0$ such that $p \geq k$ implies $X_p = X'_p$. Thus X is strongly independent (respectively strongly spanning) if and only if X' is.

Proof. Let $\beta \in M_n(Q)$ be the coordinate change matrix from X to X', as in the proof of Lemma 3.1 (c). If k is chosen large enough so that $p \geq k$ implies that β can be interpreted as a matrix in $M_n(Z/pZ)$ and also such that $\det \beta \neq 0$ in Z/pZ, then $p \geq k$ implies $X_p = X_p'$.

The next two Propositions relate the strong independence and spanning properties of a maximal independent set X to the associated map μ_X.

Proposition 3.3. Let X be a maximal independent set of G and let $\mu = \mu_X$. Then X is strongly independent if and only if: (1) μ is onto and (2) there exists $k > 0$ such that $p \geq k$ implies $X_p \neq 0$.

Proof. Suppose X is strongly independent. Choose k such that $p \geq k$ implies that $\{x_{1p}, ..., x_{np}\}$ is independent. Plainly, for $p \geq k$, $X_p \neq 0$. Let $\alpha \in M_n(Q)$. Choose $k' \geq k$ such for $p \geq k'$ the matrix α can be interpreted as a matrix in $M_n(Z/pZ)$. For $p \geq k'$ the subspace X_p is free on $\{x_{1p}, ..., x_{np}\}$. Therefore the matrix α induces an endomorphism on X_p. This endomorphism can be extended to an endomorphism λ_p of G_p by sending a complementing summand of X_p in G_p to zero. For $p < k'$ let λ_p be an arbitrary endomorphism of G_p. Then, by Lemma 3.1 (a), $\lambda = (\lambda_p)$ is an endormorphism of G. By definition of the map μ, $\mu(\lambda) = \alpha$.

Conversely, suppose μ is onto and there exists k_0 with $X_p \neq 0$ for $p \geq k_0$. For $1 \leq i,j \leq n$ let E_{ij} be the rational matrix with 1 in the (ij) position and 0's elsewhere. Choose $\lambda_{ij} \varepsilon R$ with $\mu(\lambda_{ij}) = E_{ij}$ and choose positive integers $k_{ij} = k(\lambda_{ij})$ as in the proof of Lemma 3.1 (a). Take $k = \max\{k_0, k_{ij} \mid 1 \le i,j \le n\}$ and suppose p is a fixed prime with $p \ge k$. By assumption there exists at least one j such that $x_{jp} \ne 0$. Now suppose $\Sigma c_t x_{tp} = 0$ for some scalars $\{c_1, ..., c_n\}$ in Z/pZ. For any $1 \le i \le n$, applying Lemma 3.1 (a), $0 = \lambda_{ij}(\Sigma c_t x_{tp}) = [c_1, ..., c_n] E_{ij} [x_{tp}] = c_i x_{jp}$. Here $[x_{tp}]$ is the nx1 column matrix consisting of the ordered set $\{x_{1p}, ..., x_{np}\}$ and $[c_1, ..., c_n]$ is the 1xn row matrix consisting of the ordered set $\{c_1, ..., c_n\}$. Since $x_{jp} \ne 0$ it follows that $c_i = 0$. Thus $p \ge k$ implies $\{x_{1p}, ..., x_{np}\}$ is independent.

Proposition 3.4. Let G be a group with maximal independent set X and $\mu = \mu_X$. Then the following are equivalent:

(1) X is strongly spanning

(2) $\ker \mu = \oplus R_p$

(3) For some k, $R \cap [\Pi_{p\ge k} R_p]$ is countable.

Proof. (1) → (2) Let $\lambda = (\lambda_p) \varepsilon \ker \mu$. Then, for all $p \ge k(\lambda)$, λ_p induces the zero endomorphism on X_p. But for some $k \ge k(\lambda)$ and all $p \ge k$, $X_p = G_p$. Thus, for all $p \ge k$, $\lambda_p = 0$ or, equivalently, $\lambda \varepsilon \oplus R_p$.

(2) → (3) For each p write $G_p = X_p \oplus Y_p$. Then $\Pi E(Y_p)$ can naturally be regarded as a subring of R and viewed in this way $\Pi E(Y_p) \le \ker \mu$. Since $\ker \mu = \oplus R_p$ it follows that there exists k such that $Y_p = 0$ for $p \ge k$. Thus, for $p \ge k$, $R_p = E(G_p) = E(X_p)$ is finite. Let μ^* be the restriction of μ to the ideal $R^* = R \cap [\Pi_{p\ge k} R_p]$. Then $\ker \mu^* = \oplus_{p\ge k} R_p$ is countable. Moreover $\operatorname{im} \mu^* = \operatorname{im} \mu \subset M_n(Q)$ is also countable. Hence R^* is countable, as desired.

(3) → (1) As in the proof of (2) → (3) we have $\Pi_{p\ge k} E(Y_p) \subset R^*$. Since R^* is a countable ring, for some $k' \ge k$ and all $p \ge k'$, Y_p must be zero. Thus $p \ge k'$ implies that $G_p = X_p$. Hence X is strongly spanning and the proof is complete.

Let $A = \text{image } \mu_X$ for X a maximal independent subset of G. By Lemma 3.2 (c) this subalgebra of $M_n(Q)$ is an isomorphism invariant of G.

For the rest of this section we consider the pp and von Neumann regular properties of R in terms of the pp and von Neumann regular properties of A.

Theorem 3.5. Let G be a mixed group of finite torsion-free rank with $\oplus G_p < G < \Pi G_p$ and suppose that X is strongly spanning for some maximal independent set $X \subset G$. Then :

(a) R is von Neumann regular if and only if A is von Neumann regular.

(b) If R is pp then A is left pp.

Proof. (a) Clearly if R is von Neumann regular then A is. For any ring R and ideal K, if both K and R/K satisfy the defining condition for von Neumann regularity, then so does R. Suppose X is strongly spanning. Then, by Proposition 3.4, $\ker \mu_X = \oplus R_p$. Since each $R_p \cong E(G_p)$ is von Neumann regular, so is $\ker \mu_X$. Hence if $A \cong R/\ker \mu_X$ is von Neumann regular then R is von Neumann regular.

(b) Let $r_B(b)$ [respectively $l_B(b)$] denote the right [respectively left] annihilator of an element b in a ring B. Assume that R is pp and let $\mu(\lambda) \varepsilon A$. By definition $r_R(\lambda) = eR$ for some idempotent $e \varepsilon R$. We claim that $\mu(e)$ is an idempotent generator for $l_A(\mu(\lambda))$. Plainly $\mu(e)$ is an idempotent with $\mu(e)\,\mu(\lambda) = 0$. Suppose $\mu(\delta) \varepsilon A$ with $\mu(\delta)\,\mu(\lambda) = 0$. Then $\lambda\delta \varepsilon \ker \mu = \oplus R_p$. Thus $(\lambda\delta)_p = 0$ for all but finitely many p. Define $\hat{\delta} \varepsilon R$ by $\hat{\delta}_p = \delta_p$ if $(\lambda\delta)_p = 0$ and $\hat{\delta}_p = 0$ otherwise. Then $(\hat{\delta} - \delta) \varepsilon \oplus R_p = \ker \mu$, so $\mu(\hat{\delta}) = \mu(\delta)$. Since $\hat{\delta} \varepsilon r_R(\lambda)$, $e\hat{\delta} = \hat{\delta}$. Thus,
$\mu(\delta) = \mu(\hat{\delta}) = \mu(e\hat{\delta}) = \mu(\hat{\delta})\,\mu(e) = \mu(\delta)\,\mu(e)$, which completes the proof of the claim.

Corollary 3.6. Let G be a mixed group of finite torsion-free rank with $\oplus G_p < G < \Pi G_p$. If X is strongly spanning and strongly independent then $R = E(G)$ is von Neumann regular.

Proof. If X is strongly independent then, by Proposition 3.3, image $\mu_X = M_n(Q)$, which is a Von Neuman regular ring. Hence, by Theorem 3.5, R is von Neumann regular.

We need one more lemma before we can give our main result. For each p let $G_p = X_p \oplus Y_p$ be a fixed direct sum decomposition. Denote $G \cap \Pi X_p$ by G_X and let $Y = \oplus Y_p$.

Lemma 3.7. With notation as above
(a) $G = G_X \oplus Y$ and
(b) R is pp if and only if $E(G_X)$ is pp.

Proof. Part (a) is a straightforward exercise. The "only if" part of (b) follows since $E(G_X) \cong uRu$ where u is a projection of G onto G_X. Conversely, suppose $E(G_X)$ is pp. Let $\lambda \in R = E(G)$. To show that R is pp it is enough to show that $r_R(\lambda)$ is generated by an idempotent. To verify this latter condition it suffices to construct an element $e \in R$ such that, for each p, e_p is an idempotent projection $e_p : G_p \to \ker \lambda_p$. Then e will be a projection of G onto $\ker \lambda$ and, by Theorem 2.1 (2), it follows that $r_R(\lambda) = eR$.

Since $\lambda \in E(G)$ the induced map $\lambda \in E(G)$ is represented by the matrix $\mu_X(\lambda)$. By Lemma 3.1 (a) that there exists a $k > 0$ such that, for all $1 \le i \le n$ and all $p \ge k$, $\lambda_p(x_{ip})$ is a fixed rational combination of $\{x_{1p}, ..., x_{np}\}$. Let $\{x_1', ..., x_n'\}$ be a new maximal independent set, where each x_i' is defined by $x_{ip}' = 0$ for $p < k$ and $x_{ip}' = x_{ip}$ for $p \ge k$. Note that if X_p' is the subspace of G_p generated by $\{x_{1p}', ..., x_{np}'\}$ and $G_{X'} = G \cap \Pi X_p'$ then $\oplus_{p<k} X_p \oplus G_{X'} = G \cap \Pi X_p = G_X$. Thus, since $E(G_X)$ is pp, so is $E(G_{X'})$. Furthermore, by the construction of X', $\lambda_p(X'_p) \subset X_p'$ for all primes p. Thus $\lambda(G_{X'}) \subset G_{X'}$.

By abuse of notation, rename X' by X and write $G = G_X \oplus Y$ as in part (a). With our new X the map $\lambda \in \mathrm{Hom}(G_X \oplus Y, G_X \oplus Y)$ is represented by a 2x2 matrix $\begin{bmatrix} \lambda_1 & f \\ 0 & \lambda_2 \end{bmatrix}$ where $\lambda_1 \in E(G_X)$, $\lambda_2 \in E(Y)$ and $f \in \mathrm{Hom}(Y,G_X)$.
Note that, since Y is torsion, $\mathrm{Hom}(Y,G_X) = \mathrm{Hom}(Y,T(G_X)) = \mathrm{Hom}(Y,\oplus X_p)$.

We will construct our desired idempotent e in similar matrix form $e = \begin{bmatrix} e_1 & g \\ 0 & e_2 \end{bmatrix}$. By assumption $E(G_X)$ is pp. Thus, there exists $e_1 \in E(G_X)$ such that $e_1 E(G_X) = r_{E(G_X)}(\lambda_1)$. For each p let

$U_p = \ker \lambda_{2p} \cap f_p^{-1}(\mathrm{im}\ \lambda_{1p})$, a subspace of Y_p. Write $Y_p = U_p \oplus V_p$ where

V_p is any complementing subspace. Define $e_2 \varepsilon E(Y) = E(\oplus Y_p)$ componentwise by taking e_{2p} to be projection of Y_p onto U_p with kernel V_p. Define $g \varepsilon \mathrm{Hom}(Y, G_X) = \mathrm{Hom}(\oplus Y_p, \oplus X_p)$ componentwise by $g_p(V_p) = 0$ and, for $u_p \varepsilon U_p$, $g_p(u_p) = (1 - e_{1p})(x_p)$, where $x_p \varepsilon X_p$ satisfies $f_p(u_p) = -\lambda_{1p}(x_p)$. Each map g_p is well defined since, if $f_p(u_p) = -\lambda_{1p}(x_p')$, then $(x_p - x_p') \varepsilon \ker \lambda_{1p}$, hence $(1 - e_{1p})(x_p - x_p') = 0$.

Let $e = \begin{bmatrix} e_1 & g \\ 0 & e_2 \end{bmatrix}$. Then $e \varepsilon E(G)$ and simple matrix computations show that $e^2 = e$ and that $\lambda e = 0$. It remains to show that, for each fixed prime p, $e_p = \begin{bmatrix} e_{1p} & g_p \\ 0 & e_{2p} \end{bmatrix}$ is a projection of G_p onto $\ker \lambda_p$ and not onto some proper subspace of $\ker \lambda_p$. Let $a_p \varepsilon \ker \lambda_p$ for some fixed p. To complete the proof we will show that $e_p a_p = a_p$. Write $a_p = x_p \oplus y_p \varepsilon X_p \oplus Y_p$. Then $0 = \lambda_p(a_p) = [\lambda_{1p}(x_p) + f_p(y_p)] \oplus \lambda_{2p}(y_p)$. It follows that $\lambda_{2p}(y_p) = [\lambda_{1p}(x_p) + f_p(y_p)] = 0$. Hence $y_p \varepsilon \ker \lambda_{2p} \cap f_p^{-1}(\mathrm{im}\, \lambda_{1p}) = U_p$. By definition $e_p(a_p) = [e_{1p}(x_p) + g_p(y_p)] \oplus e_{2p}(y_p) = [e_{1p}(x_p) + (1 - e_{1p})(x_p)] \oplus y_p = x_p \oplus y_p = a_p$. The proof is now complete.

Theorem 3.8. Let $\oplus G_p < G < \Pi G_p$. Suppose that G is of finite torsion-free rank with each G_p elementary and let $A = \text{image } \mu_X$ where $X \subset G$ is any maximal independent set. Then:

(a) If A is von Neumann regular then R is pp.

(b) If R is pp (respectively von Neumann regular) then A is left pp (respectively von Neumann regular).

Proof. (a) Suppose that A is von Neumann regular. Write $G = G_X \oplus Y$ as in Lemma 3.7 (a). By Lemma 3.7 (b), to show that R is pp it is enough to show that $R' = E(G_X)$ is pp. By the definition of G_X, $X \subset G_X$ is a maximal independent set which is strongly spanning. Let μ' be the restriction of the map μ_X to the subring $R' \subset R$. The map μ' coincides with the map from R' to $M_n(Q)$ defined with respect to $X \subset G_X$ as in the paragraph preceding Lemma 3.1. Moreover $\mathrm{im}\, \mu' = \mathrm{im}\, \mu = A$ is von Neumann regular, so, by Theorem 3.5 (a), the ring R' is von Neumann regular. Hence R' is pp and the proof is complete.

(b) Let μ' be as in part (a) and suppose that R is pp. By Lemma 3.7 (b) and the fact that $A = \text{im}\ \mu'$, we may assume that X is strongly spanning. Therefore we can apply Theorem 3.5 (b) to conclude that A is left pp. The second statement of part (b) is clear.

Corollary 3.9. Let $\oplus G_p < G < \Pi G_p$ be as in Theorem 3.8. Suppose that X is strongly independent for some maximal independent set $X \subset G$. Then R is pp.

Proof. By Proposition 3.3, if X is strongly independent then μ_X is onto. Hence $A = M_n(Q)$ for n = torsion-free rank G. Theorem 3.8 (a) now applies to show that R is pp.

Corollary 3.10. Let $\oplus G_p < G < \Pi G_p$ with each G_p elementary and with torsion-free rank G = 1. Then R = E(G) is pp.

Proof. In this case A = Q is von Neumann regular, so, by Theorem 3.8, R is pp.

We will now construct an example to show that even if A is both left and right pp R need not be pp. Thus Theorem 3.8 (a) cannot be strengthened. The example also shows that a reduced S-local group of torsion-free rank two need not have a pp endomorphism ring (see Corollary 3.10).

Example 3.11. Partition the set of all primes into two disjoint infinite subsets P_1 and P_2. For each prime p define an elementary p-group G_p by $G_p = Z(p)x_{1p} \oplus Z(p)x_{2p}$ for $p \in P_1$, $G_p = Z(p)x_{1p}$ for $p \in P_2$. Let $x_1 = (x_{1p})$ and $x_2 = (x_{2p})$, where $x_{2p} = 0$ for $p \in P_2$. Take G to be the (unique) pure subgroup of ΠG_p generated by $\oplus G_p$ and $\{x_1, x_2\}$. Then for $\lambda \in E(G)$ there must exist a positive integer k and a 2x2 rational matrix α such that $p \geq k$ implies $\begin{bmatrix}\lambda(x_{1p})\\ \lambda(x_{2p})\end{bmatrix} = \alpha \begin{bmatrix}x_{1p}\\ x_{2p}\end{bmatrix}$. Since $x_{2p} = 0$ for all $p \in P_2$ it follows that α must upper triangular. Conversely, one can check that for any upper triangular matrix β there exists $\lambda \in E(G)$ with matrix $\mu(\lambda) = \beta$. Thus, A is the subalgebra of $M_2(Q)$ consisting of the upper triangular matrices. Note that A is left and right pp.

To see that R itself is not pp consider the map $\lambda = (\lambda_p)$, where $\lambda_p \varepsilon E(G_p)$ is given by : $\lambda_p(x_{1p}) = x_{2p}$, $\lambda_p(x_{2p}) = 0$, for $p \varepsilon P_1$ and $\lambda_p = 0$ for $p \varepsilon P_2$. Then $\lambda(x_1) = x_2$ and $\lambda(x_2) = 0$, so, by Lemma 3.0, $\lambda \varepsilon E(G)$. Suppose $r_R(\lambda) = eR$ for some idempotent $e = (e_p) \varepsilon R$. For all $p \varepsilon P_2$, since $\lambda_p = 0$, we must have $e_p = 1$. Moreover, since $e \varepsilon R$, there exists a positive integer k and fixed rationals q_1, q_2 such that, for $p \geq k$, $e_p(x_{1p}) = q_1 x_{1p} + q_2 x_{2p}$. Thus, for $p \geq k$ with $p \varepsilon P_2$, $x_{1p} = e_p(x_{1p}) = q_1 x_{1p} + q_2 0$. Since the set P_2 is infinite it follows that $q_1 = 1$. But then, for $p \geq k$ and $p \varepsilon P_1$, $\lambda_p e_p(x_{1p}) = \lambda_p(x_{1p} + q_2 x_{2p}) = x_{2p} \neq 0$, a contradiction.

We end this section with an example to show that the conclusion of Corollary 3.10 cannot be strengthened from pp to von Neumann regular.

Example 3.12. For each prime p let $G_p = Z(p)x_p \oplus Z(p)y_p$. Define G to be the pure subgroup of ΠG_p generated by $\oplus G_p$ and $x = (x_p)$. Since the torsion-free rank of G is one, $A = Q$. Thus A is a von Neumann regular ring. The map $\lambda = (\lambda_p)$ belongs to $R = E(G)$ if and only if there exists $k > 0$ and $\alpha \varepsilon Q$ with $\lambda_p(x_p) = \alpha x_p$ for all $p \geq k$. Thus, R consists of all $\lambda = (\lambda_p)$ such that for $p \geq k(\lambda)$ the map $\lambda_p \varepsilon End(Z(p)x_p \oplus Z(p)y_p)$ is represented by a matrix $\begin{bmatrix} \alpha & a_p \\ 0 & b_p \end{bmatrix}$ for $\alpha \varepsilon Q$ and $a_p, b_p \varepsilon Z/pZ$. Let $\sigma \varepsilon R$ be such that each σ_p has matrix representation $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. A matrix computation shows that $\sigma \lambda \sigma \neq \sigma$ for any $\lambda \varepsilon R$ and, therefore, that R is not von Neumann regular. (Compare this example with Theorem 4.1.)

4. The Commutative Case.

Let $\oplus G_{p\varepsilon S} < G < \Pi G_{p\varepsilon S}$ with each G_p elementary. As before, to avoid trivialities, we assume S to be infinite. In this section we examine the case where E(G) is a commutative ring. From the general description of E(G) we conclude that this occurs precisely when, for every $p \in S$, rank $G_p = 1$. Thus, we are looking at pure subgroups G of $\Pi_{p\varepsilon S} Z(p)$ containing $\oplus_{p\varepsilon S} Z(p)$. If the torsion-free rank of G is finite we obtain the following result.

Theorem 4.1. Let G be a pure subgroup of $\Pi_{p\varepsilon S}Z(p)$ containing $\oplus Z(p)_{p\varepsilon S}$ with torsion-free rank $G = n < \infty$. Then $R = E(G)$ is a von Neumann regular ring.

Proof. Let X be a maximal independent set and, as in Lemma 3.7, write $G = G_X \oplus Y$. Since each $G_p = Z(p)$ the group Y is simply the direct sum of the G_p's over the set $S' = \{p \varepsilon S \mid X_p = 0\}$ and $G_X = G \cap \Pi_{p\varepsilon S \setminus S'}G_p$. Both of these summands are fully invariant and, hence, there is a ring direct sum decomposition $R = R' \oplus R''$ where $R' = E(G_X)$ and $R'' = E(Y)$. Since $R'' \cong \Pi_{p\varepsilon S'}Z/pZ$ is von Neumann regular, to show that R is von Neumann regular we need only show that R' is von Neumann regular. Moreover $X \subset G_X$ is strongly spanning so, by Theorem 3.5 (a), to show that R' is von Neumann regular it is enough to show that $A = \text{image } \mu_X$ is von Neumann regular. But A is a commutative subring of $M_n(Q)$, and therefore of Krull dimension zero. Further $X \subset G_X$ is strongly spanning so $\ker \mu_X = \oplus R'_p$. Hence A is isomorphic to $R'/\oplus R'_p$ which is a ring with no nilpotent elements. Thus A is a commutative ring with no nilpotent elements for which every prime ideal is minimal, and therefore is von Neumann regular [G].

If the torsion-free rank of G is not finite the conclusion of Theorem 4.1 does not necessarily hold: If $\oplus Z(p) < G < \Pi Z(p)$ then $R = E(G)$ need not be pp and, even if pp, need not be von Neumann regular. Before providing examples, we need a short discussion.

Let $r = (r_p) \varepsilon \Pi_{p\varepsilon S}Z/pZ$. Define $\text{spt}(r) = \{p \varepsilon S \mid r_p \neq 0\}$ and $\text{csp}(r) = S \setminus \text{spt}(r)$. Call such an element r finite if $\text{spt}(r)$ is a finite set and cofinite if $\text{spt}(r)$ is cofinite in S. A subring $R \subset \Pi_{p\varepsilon S}Z/pZ$ is called finite-cofinite if every element $r \varepsilon R$ is either finite or cofinite.

Lemma 4.2. Let R be a subring of $\Pi_{p\varepsilon S}Z/pZ$. If R is finite-cofinite then R is a pp ring.

Proof. For an element $x \in R$ we have that $r_R(x) = R \cap \Pi_{p\varepsilon \text{csp}(x)}Z/pZ$. Since R is finite-cofinite, and since finite direct sums of R_p's are R-summands of R, it follows that, for all $x \varepsilon R$, $r_R(x)$ is idempotent generated.

Example 4.3. We will construct a ring R containing $\oplus Z/pZ$ (taken over the set of all primes) and contained as a pure subring in $\Pi Z/pZ$, with the property that R is finite-cofinite and $R/(\oplus Z/pZ) \simeq Q[x]$, the polynomial ring in one variable x over Q. By Lemma 4.2, R will be a pp ring. But R cannot be von Neumann regular since it has a ring homomorphic image which is not von Neumann regular. Furthermore, since R is a pure subring of $\Pi Z/pZ$, $R \cong E(G)$, where G is the additive group (R+) [R-1]. Such a ring will provide an example of an endomorphism ring $R = E(G)$ for a group G with $\oplus Z(p) < G < \Pi Z(p)$ such that R is pp but not von Neumann regular.

The construction: List all the non-constant polynomials in $Z[t]$, $\{f_i = f_i(t)\}_{i=1}^{\infty}$ and let $\{p_i\}$ be the set of primes enumerated in their natural order. Pick an integer $k(1)$ such that for every $i \geq k(1)$, Z/p_iZ contains a nonzero element which is not a root of f_1. Let $c_{k(1)} \in Z/p_{k(1)}Z$ be such an element. Now pick $k(2) \geq k(1)$ such that $i \geq k(2)$ implies that each Z/p_iZ contains a nonzero element which is not a root of $f_1 \cdot f_2$. Let $0 \neq c_{k(2)} \in Z/p_{k(2)}Z$ with $c_{k(2)}$ not a root of $f_1 \cdot f_2$. Now choose, for $k(1) < i < k(2)$, an element $c_i \in Z/p_iZ$ to be an arbitrary nonzero non-root of f_1. For $k(3) \geq k(2)$ let $0 \neq c_{k(3)} \in Z/p_{k(3)}Z$ be a non-root of $f_1 \cdot f_2 \cdot f_3$ and let, for $k(2) < i < k(3)$, $0 \neq c_i \in Z/p_iZ$ be non-roots of $f_1 \cdot f_2$. Continue this way to construct an element $c = (c_i) \in \Pi Z/p_iZ$ satisfying that if $f \in Z[t]$ is a non-constant polynomial there is an integer $k = k(f)$ such that for $i \geq k$ $f(c_i) \neq 0$. Denote $T = \oplus_p Z/pZ$. Then $(\Pi_p Z/pZ)/T$ is a Q-algebra containing the element $x = c + T$ which is transcendental over Q.

Let $R \supset T$ be the pure subring of $\Pi Z/pZ$ such that $R/T = Q[x] \subset (\Pi Z/pZ)/T$. It is not hard to check that R is finite-cofinite and, thus, satisfies all of our requirements.

A similar construction provides a ring $R = E(G) < \Pi Z/pZ$ which is not pp.

Example 4.4. Partition the set of all primes into two infinite disjoint sets P_1 and P_2. Using the method of Example 4.3 we can construct an element $c = (c_p) \in \Pi Z/pZ$ such that $x = c + T$ is transcendental over Q, and $\mathrm{spt}(c) = P_1$. If $R \supset T$ is the subring of $\Pi Z/pZ$ such that $R/T = Q[x]$ then R is pure in $\Pi Z/pZ$ and $E(R+) \cong R$. If $e \in R$ is an idempotent then $e + T$ is an idempotent in Q[x] and thus e + T is either 0 or 1. Therefore the element e is either finite or cofinite. It follows that $r_R(c) = R \cap \Pi_{p \varepsilon P_2} Z/pZ$ cannot be generated by an idempotent.

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