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FLAT IDEALS III

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0. Introduction. We continue here the study of the structure of commutative rings in terms of the properties exhibited by its rank one flat modules. Particular classes of such modules are now emphasizeu -- e.g., flat epimorphisms of finite type -- or, the ring itself is conveniently restricted -- e.g., Krull domains or rings of dimension one. Thus, for flat epimorphisms of finite type we shall discuss a bound for the number of generators needed, and for Krull domains we examine the endomorphism ring of a flat rank one module with the intention of ascertaining when they are epimorphisms. For domains of dimension one and Noetherian spectrum it is shown that every rank one flat module is a directed union of projective submodules.

A closely allied topic is the nature of the integral closure of a coherent domain of Krull dimension one. There are several cases where this closure is again coherent, leading to

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a conjecture that it might always be coherent. We look at this unresolved question indirectly--that is, via the examination of several expected properties of large classes of stably (i.e., polynomially) coherent rings.

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1. Flat epimorphisms of finite type.

Throughout rings are commutative with identity. A flat module I of such a ring R is said to have rank one if $\wedge^2 I = 0$. When R is an integral domain with field of quotients K, this condition simply means that I is isomorphic to a flat submodule of K. Rather loosely, we shall refer to the submodules of K as ideals. When we want to emphasize that I is isomorphic to an (proper) ideal of R we shall say that I is a fractional ideal. As basic references on flat modules, we will use [B], [L].

We begin this discussion on flat ideals with added structures by giving a dimension dependent bound on the number of generators of flat epimorphisms of finite type. It is reminiscent of the Swan-Foster bound [S] on the number of generators for modules in terms of local data.

Let $R \xrightarrow{\varphi} S$ be a flat epimorphism. Following [L], this will imply that for each prime ideal P of S, $R_Q \xrightarrow{\simeq} S_Q$ for $Q = \varphi^{-1}(P)$.

Proposition 1.1. Let R be a ring with Noetherian prime spectrum of Krull dimention d. Let $R \xrightarrow{\varphi} S$ be a flat epimorphism of finite type. Then S can be generated (as an R-algebra) by d elements.

Proof: We may assume $R \subset S$. Let $S = R[u_1, \ldots, u_n]$ and let $I = \bigcap_{i=1}^{n} (R :_R u_i)$; then $IS = S$. Pick J a finitely generated subideal of I satisfying $JS = S$. Note that $\sqrt{J} = \sqrt{I}$, [L].

We will first show by induction on the Krull dimention of R that there exists $a_1, \ldots, a_d \in J$ satisfying:

(*) $$S = R + (a_1, \ldots, a_d)S$$

For the case where R is a zero dimensional ring, one can easily check by localization that φ is surjective. For the higher dimensional case, one can see that if equation (*) holds modulo NS, with N = nilradical of R, a similar equation will be valid in S. Thus, we assume R is a reduced ring and let a_1 be a regular element of J, the existence of it being assured by the fact that R has finitely many minimal prime ideals, $R \subset S$ and $JS = S$. Pass to the flat epimorphism $R' = R/(a_1) \longrightarrow S/a_1S = S'$. Note that the ideal $J' = J/(a_1)$ still satisfies $J'S' = S'$. By the induction hypothesis, there are $a_2', \ldots, a_d' \in J'$ such that $S' = R' + (a_2', \ldots, a_d')S'$, and hence $S = R + (a_1, \ldots, a_d)S$ where $a_i' = a_i (\operatorname{mod} a_1)$.

To complete the proof note that the equality (*) multiplied by J gives rise to the equality $JS = S = J + (a_1, \ldots, a_d)JS = J + (a_1, \ldots, a_d)S$ and thus we obtain an equation:

(**) $$1 = a + \sum_{j=1}^{d} a_j s_j \; ; \quad a \in J,\ s_j \in S.$$

We claim that $S = R[s_1, \ldots, s_d]$. Indeed, if $x \in S$, as x has a polynomial expression in the u_i's, there exists a power r, such that $xJ^r \subset R$. Raise the equation (**) to the r-th power and multiply it by x, to get a representation of x as a polynomial in the s_j's. ∎

Remark. In the sequel, we make use of the following fact: From the equality $S = IS = JS$, with J = finitely generated, and $\sqrt{J} = \sqrt{I}$ we wrote, rather redundantly, $S = R + JS$. If, moreover, $J = J_0 + J^2$, it follows immediately that $S = R + J_0S$ without necessarily assuming $\sqrt{J_0} = \sqrt{J}$.

Remark. The following question illustrates the need for determining the exact number of generators of a flat epimorphism $R \longrightarrow S$. Assume $S = R[\omega, z]$, and let J be the kernel of a map from the polynomial ring $R[x, y]$ to S. It follows from the exact sequence of modules of differentials [M], that J/J^2 is a free S-module of rank two. J can then be generated by three elements: If it cannot be generated by two elements, then, using Serre's construction, a rank two indecomposable projective module over $R[x, y]$ would arise. Later on we shall discuss examples of non-simply generated flat epimorphisms; we know, however, very little about deciding when J is generated by $rk(J/J^2)$ elements.

Remark. Note that the proof of (1.1) does not use the full strength of the Noetherian hypothesis on the prime spectrum of R. In particular, the result is valid for all domains of Krull dimension one and several other kinds of rings. This raises the question on whether the bound of (1.1) is valid for all rings.

(a) Special rings: We consider now some special cases where a lower bound is attainable.

Proposition 1.2. Let $R = D[T_1, \ldots, T_n]$, D a Noetherian ring of Krull dimension d, and let $R \longrightarrow S$ be a flat epimorphism of finite type. Then S is generated by $d+1$ elements.

Proof: With the notation of proposition 1.1, we will show by a similar inductive procedure the existence of elements $a_1, \ldots, a_{d+1} \in I$ satisfying $S = R + (a_1, \ldots, a_{d+1})S$, and conclude as in proposition 1.1.

We may assume that D is a reduced ring and that $R \subset S$. Let K be the total quotient ring of D. Since K is a product of fields, $R \otimes K$ is a product of U.F.D.'s, thus $S \otimes K = R \otimes K[u]$. This implies the existence of a regular element $t \in D$ satisfying $tI \subset (f)$, $f \in I$. If t is a unit then $I = (f)$ and we are done. Otherwise pass to the ring $R/(t)$, and apply induction to obtain $a_1, \ldots, a_d \in I$ such that $S/tS = R/(t) + (a_1, \ldots, a_d)S/tS$ or $S = R + (a_1, \ldots, a_d, t)S$. Multiply this equality by I to get $S = I + (a_1, \ldots, a_d, f)S$ and the conclusion follows. ∎

A different case is that of an affine regular Z-algebra, R. In this case a flat homomoprhism $R \longrightarrow S$ of finite type is simply written as $S = R[I^{-1}]$, where I is an invertible (proper) ideal of R. To estimate the number of generators of S, we use the following.

<u>Proposition 1.3</u>. Let R be an affine Z-algebra of Krull dimension $d > 1$. Let I be a projective ideal. Then there exists a power $J = I^n$ such that J/J^2 is generated by $d-1$ elements. In particular J is generated by d elements.

Note that I itself might in general require $d+1$ generators.

<u>Proof</u>: Consider the inductive series of identifications:

$$I/I^2 = L_{(1)} \cong \text{projective ideal of } R_{(1)} = R/I$$

$$L_{(1)}/L^2_{(1)} \cong L_{(2)} = \text{projective ideal of } R_2 = R_{(1)}/L_{(1)}$$

$$\vdots$$

$$L_{(d-2)}/L^2_{(d-2)} \cong L_{(d-1)} = \text{projective ideal of } R_{(d-1)} = R_{(d-2)}/L_{(d-2)}.$$

It might as well happen that $R_{(r)} = 0$ for $r < d-1$, in which case we would stop with $R_{(r-1)}$. We focus on the projective ideal $L_{(d-1)}$ of the one-dimensional ring $R_{(d-1)}$. By [C-V] there is a power ℓ such that $\otimes^{\ell} L_{(d-1)}$ is principal. Taking the ℓ-th tensor power of the last identification, we get

$\otimes^{\ell}(L_{(d-2)}/L^2_{(d-2)}) = \otimes^{\ell}(L_{(d-2)}\otimes R_{(d-2)}) = (\otimes^{\ell}L_{(d-2)})\otimes(R_{(d-2)}/L_{(d-2)})$

$\cong L^{\ell}_{(d-2)} \otimes R_{(d-2)}/L_{(d-2)} \cong L^{\ell}_{(d-2)}/L^{\ell+1}_{(d-2)}$ and $L^{\ell}_{(d-2)}/L^{\ell+1}_{(d-2)}$ being principal implies that $L^{\ell}_{(d-2)}$ can be generated by two elements. If a projective ideal L is generated by t elements, then L^{ℓ} is generated by t elements as well, and thus $L^{\ell}_{(d-3)}/L^{\ell+1}_{(d-3)}$ will be generated by three elements. Continuing the process we can eventually conclude that $I^{\ell}/I^{\ell+1}$ is generated by d-1 elements. The quotient $I^{\ell}/I^{2\ell}$ is generated by d-1 elements as well. Thus setting $J = I^{\ell}$ we obtain the desired conclusion. ∎

Corollary 1.4. Let R be an affine regular Z-algebra of Krull dimension $d > 1$. Let $R \longrightarrow S$ be a flat epimorphism of finite type. Then S can be generated by d-1 elements.

Proof: Let $S = R[I^{-1}]$ and let I/I^{n+1} be generated by $a_1, \ldots, a_{d-1}$. We claim that $S = R + (a_1, \ldots, a_{d-1})S$. To prove this equality suffices to check it at every localization at a prime P containing I. But at every such localization I^n is generated by the a_j's and $S = IS = I^nS$. The conclusion now follows. ∎

(b) Simple flat epimorphisms. We next examine flat epimorphisms of the type $S = R[z]$, where R is an integral domain. Richman [R] proved that if R is integrally closed and $z = a/b$, then (a, b) is an invertible ideal of R. We now give an 'explanation' of this result. Recall, first, that R is said to be seminormal if the canonical map $\mathrm{Pic}(R) \longrightarrow \mathrm{Pic}(R[T])$ is an isomorphism, that is, if every invertible ideal of $R[T]$, T = indeterminate, is extended from R.

Proposition 1.5. The following conditions are equivalent for an integral domain R.

i) R is seminormal.

ii) For each simple (principal) flat epimorphism $S = R[z]$, the R-fractional ideal $(1, z)$ is invertible.

Proof: i) $\longrightarrow$ ii): Let $S = R[z]$, $z = a/b$, be a flat epimorphism over R, and let I be the kernel of the surjection $R[T] \longrightarrow S$, sending T to z. I is an invertible ideal of $R[T]$, and its so called content, $c(I)$, that is, the ideal of R generated by the coefficients of the polynomials in I, is the unit ideal, [V] and [O-R]. Since by the definition of seminormality I is extended from R, there exists and invertible ideal J of R and an element f in $K[T]$, K being the field of quotients of R, such that $I = JfR[T]$, $bT - a \in I$. It follows that $f = \beta T - \alpha$ for $\beta, \alpha \in K$ and $bT - a = c(\beta T - \alpha)$. Thus $(1, z) = (\alpha, \beta)(\beta^{-1})$ is invertible if (α, β) is an invertible R-fractional ideal. But $I = JfR[T]$ implies $R = c(I) = J(\alpha, \beta)$ and we are done.

(ii) $\longrightarrow$ i) For the converse, we use the following characterization of seminormality [B-G], [G-H], [S]: A domain R is seminormal if whenever $x^2, x^3 \in R$ for x in the field of quotient K of R, we have $x \in R$. Assume R is not seminormal and pick $x \in K$ for which the preceding condition fails. Let I be the ideal of $R[T]$ generated by $x^2T^2 - 1$ and $x^3T - x^2$, that is the Schanuel's ideal of R [S]. I is an invertible ideal [loc. cit.] which is not extended. On the other hand, since the content of I is (1), $R[T]/I$ is a flat R-module [V], [O-R], necessarily a flat epimorphism since I contains linear polynomials. But $R[T]/I$ may be identified with $R[z]$ for $z = x^2/x^3$, it follows easily that (x^2, x^3) is not an invertible ideal. ∎

Remark. The restriction that R be a domain can be slightly relaxed if one uses an extended notion of seminormality [RU], i.e.: A ring R is seminormal if R is reduced and whenever $b, c \in R$ satisfy $b^3 = c^2$, there is an a in R with $a^2 = b$ and $a^3 = c$.

If b and c are regular elements of R satisfying $b^3 = c^2$, the Schanuel-like ideals of $R[T]$: $I = (cT - b, bT^2 - 1)$ and $J = (b^2, cT - b)$ satisfy $IJ = (cT - b)$. Thus in this case I is an invertible ideal. The ring $R[z]$, $z = b/c$ is a flat epimorphism of $R[V]$, [O-R]. It follows easily that (b, c) is invertible if and only if there is an element a in R with $a^2 = b$ and $a^3 = c$, and Proposition 1.5 is valid in this case.

An example. To find examples of flat epimorphisms of finite type that are not simply generated one may proceed in the manner pointed out by Akiba [A]. Let $A = C[x, y, z]$ where $y^2z + yz^2 = x^3 - xz^2$ and let B be the localization of A at the origin. Let $P = (x, y)$; according to Tate [T], $(0, 0)$ is a non-torsion point of the elliptic curve $y^2 + y = x^3 - x$. Let S be the P-transform of A. According to the Zariski's theorem [N], S is a flat epimorphic image of finite type over A. Suppose $S = A[v]$, $v = a/b$, $a, b \in A$. From Proposition 1.5 (a, b) is then an invertible ideal of the graded ring A. The isomorphism of divisor class groups $Cl(A) = Cl(B)$ [F, (10.3)] show that (a, b) is principal, and thus one can write $S = A[1/u]$, $u \in A$. This shows that (u) has for radical the ideal P. But in the elliptic curve this would translate as saying that P is a torsion-point, contradicting the choice made.

2. Krull domains.

Let R be an integral domain of field of quotients K. In any classification of the flat submodules of K, two classes stand out: The flat fractional ideals and the flat epimorphisms. That these classes should play basic but independent roles rests on their relative simplicity, plus the fact that they are quite apart from each other, as the following remarks show. Suppose the fractional ideal I and the flat epimorphism S of R are isomorphic. Let

$$\varphi : S \longrightarrow I$$

denote the isomorphism and put $x = \varphi(1)$. Then, for $a/b \in S$ we have $\varphi(b \cdot a/b) = b\varphi(a/b) = a\varphi(1) = ax$. Thus $\varphi(a/b) = (a/b)x$, and $I = Sx$. In particular, if $s \in S$ then $s^n x \in I$ for all n. If R satisfies some form of 'complete integral closure' restriction--e.g., R is a Krull domain, as we shall assume often, one would have $s \in R$ and consequently I would be principal.

In this section, we study the question of whether for a rank one flat module I over a Krull domain R, the ring of endomorphisms $S = \mathrm{Hom}_R(I, I)$ is a flat epimorphism. All cases discussed here lead to the suspicion that it might always be so. At least such endomorphism rings share a large number of properties of flat epimorphisms.

Our basic tool is the notion of semi-divisoriality and the following result [G-V].

Theorem 2.1. Let I be a flat fractional ideal of the Krull domain R. Then I is finitely generated.

Remark. Let R be a Krull domain, let K be its field of quotients and let I be a flat submodule of K. The Rees ring $B = R[It] = \mathrm{Symm}(I)$ is integrally closed and $L = It + I^2t^2 + \cdots \simeq I \otimes B$ is a flat ideal of B. L is finitely generated iff I is finitely generated. Thus B is a Krull domain iff I is finitely generated.

We will briefly recall the notion of semi-divisoriality For any finitely generated ideal J with generators $a_1, \dots, a_n$, consider the exact sequence.

(1) $$0 \longrightarrow R \xrightarrow{\phi} R^n \longrightarrow C \longrightarrow 0$$

where $\phi(1) = (a_1, \dots, a_n)$, and $C = \mathrm{Coker}(\phi)$. If $J^{-1} = \{x \in K \mid xJ \subset R\} = R$, C is a torsion free module. A torsion free module E will be called semidivisorial if $E \otimes C$ is torsion free for every finitely generated ideal J with $J^{-1} = R$. A submodule E of the field of quotient of R is semidivisorial iff E is a directed union of divisorial ideals [S-V]; and two semidivisorial modules are equal iff they have the same localization at each height one prime of R.

We look now at the endomorphisms of a rank one flat ideal $I \subset K$; $S = I : I = \mathrm{Hom}_R(I, I)$.

Proposition 2.2. S is semidivisorial.

Proof: For a finitely generated ideal J, $J^{-1} = R$, consider the corresponding sequence (1) $\otimes I$:

$$0 \longrightarrow I \xrightarrow{\phi_J} I^n \longrightarrow I \otimes C \longrightarrow 0$$

$I \otimes C$ is a torsion-free R-module. Apply $\mathrm{Hom}_R(I, -)$ to get

$$0 \longrightarrow S \xrightarrow{\phi'_J} S^n \longrightarrow \mathrm{Hom}_A(I, I \otimes C).$$

Since $\mathrm{Hom}_R(I, I \otimes C)$ is torsion-free and $\mathrm{Coker}(\phi'_J)$ embeds in $\mathrm{Hom}_R(I, I \otimes C)$, we conclude that S is semidivisorial. ∎

Proposition 2.3. Let I be a flat ideal $(I \subset K)$ of a Krull domain and let $S = I : I$. For any multiplicative system T of R we have $T^{-1}S = T^{-1}I : T^{-1}I$.

Proof: Denote $T^{-1}I : T^{-1}I = S^T$ and observe $T^{-1}S \subset S^T$. Since $T^{-1}S$ and S^T are both semidivisorial ideals and R is a Krull domain, to show equality it suffices to check at the height one prime ideals of R.

Let P be a height one prime ideal. If $PI \neq I$, then, [S-V], I_P is R_P-principal and thus $(T^{-1}I)_P = (T^{-1}R)_P$ also. Thus

$$(S^T)_P = (T^{-1}I : T^{-1}I)_P \subset T^{-1}I_P : T^{-1}I_P = T^{-1}R_P \subset (T^{-1}S)_P$$

If, on the other hand, $PI = I$, $P^{-1}I = I$ and $P^{-1} \subset S$. Then $S_P = K$ and again $(S^T)_P \subset (T^{-1}S)_P$. ∎

Remark. The same proof applies if T is a multiplicative system of S: Just notice that since $I \otimes S = I$, I can also be viewed as a flat S-ideal and that $T^{-1}I$ is a flat R-ideal.

Proposition 2.4. Let R be an integral domain. Suppose that the flat ideal I is finitely generated over S. Then S is a flat epimorphism of R.

Proof: Let Q be a prime ideal of S and put $P = Q \cap R$. Since $QI \neq I$, $PI \neq I$ as well. Thus I_P is principal [S-V] and $S_P \subset I_P : I_P = A_P$. ∎

Corollary 2.5. Let I be a flat submodule of K. If $I^2 \subset I$, then $S = \mathrm{Hom}_R(I, I)$ is a flat epimorphic image of R.

Proof: Since S is semidivisorial, S is a Krull domain As $I \otimes_R S = I$, and $I^2 \subset I$, I can be viewed as a flat ideal of the Krull domain S. By (2.1) I is finitely generated (over S) and (2.4) applies. ∎

A similar situation arises when $I_P = K$ for only finitely many height one primes. Let $P(R)$ denote the set of height one prime ideals of R and let I be a flat ideal. Let

$$D_\infty(I) = \{ P \in P(R) \mid I_P = K \}.$$

Corollary 2.6. If $D_\infty(I)$ is finite, then $S = I : I$ is a flat epimorphism.

Proof: Let M be a prime ideal of S and let $\underline{m} = R \cap M$. Suffices to show that $R_{\underline{m}} = S_{\underline{m}}$ for any such M. According to (2.3) we may assume that R and S are local rings. If

$D_\infty(I) = \{P_1, \ldots, P_n\}$ if follows that $R = S \cap R_{P_1} \cap \ldots \cap R_{P_n}$ since $R_P = S_P$ for the other elements of $P(R)$. We then conclude that $R = S$ [H. p. 311]. ∎

Remarks: 1) If S is a finitely generated R-algebra, say $S = R[u_1, \ldots, u_n]$, then $D_\infty(I)$ consists precisely of the elements of $P(R)$ containing the ideal $L = \bigcap(R :_R u_i)$; and hence it is a finite set.

2) When $D_\infty(I)$ is cofinite, i.e., $D_\infty(I) = P(R) - \{P_1,\ldots,P_n\}$. Pick $0 \neq d \in R$ satisfying $dI_{P_i} \subset R_{P_i}$, $i = 1, \ldots n$; then setting $J = dI$ we have $J^2 \subset J$ and proposition 2.5 applies.

Another setting leading to the flatness of S is that of a Krull domain R with the property that at each localization by a maximal ideal $\underline{m}$, $Cl(R_{\underline{m}})$ is a torsion group. Note that this condition is weaker than the requirement $Cl(R)$ is a torsion group.

Proposition 2.7. With R as above, let I be a flat ideal of R. Then $S = I : I$ is a flat epimorphism of R.

Proof: We may assume by (2.3) that $(R, \underline{m})$ is a local ring and that $\underline{m}S \neq S$. If R and S differ there is a height one prime P with $R_P \neq S_P = K$. Thus $PI = I$. Because the class of P is torsion in $Pic(R)$, there is an element $d \in P$ and an integer n so that P^n and (d) represent the same divisor. The semidivisorial ideals I and dI agree at all primes of height one $\neq P$. At P, $(dI)_P = d_P I_P = P_P^n I_P = I_P$. Thus $dI = I$ and $1/d \in S$, contradicting $\underline{m}S \neq S$. ∎

Let I be an (integral) ideal of the domain R. Write I^{-n} for $(I^n)^{-1}$, and let $I^{-\infty} = \bigcup_n I^{-n}$. We call $I^{-\infty}$ the I-transform of R [N]. Put $I^{-\infty} = S = S(R:I)$. Since S is the direct limit of divisorial ideals, it is semidivisorial If R is a Krull domain, S is also a Krull domain. It also follows that if $P_1, \ldots, P_n$ are the prime ideals of height one that contain I, then $S(R:I) = S(R:I')$, where $I' = Q_1 \cap \ldots \cap Q_n$, and Q_i is the P_i-primary component of I.

The introduction of these transforms arises out of the following need: Let I be a flat ideal of R and put $S = I:I$. To show that S is a flat epimorphism (R Krull domain) we may, as in (2.3) assume that $(R, \underline{m})$ is a local ring and $\underline{m}S \neq S$. Under these conditions we must have $R = S$. We check the equality of these semidivisorial ideals at the height one primes. If at such a prime P, $I = PI$ then S will contain the P-transform of R. In particular to contradict $S \neq \underline{m}S$ it suffices to show that these transforms are themselves flat epimorphisms. Considerable information about these objects are, of course, connected with the Hilbert's 14th problem, especially the theorem of Zariski [N] stating that ideal-transforms of two-dimensional normal affine domains are flat epimorphisms.

Putting together these remarks, we have

Propositions 2.8. Let R be a two dimensional normal affine domain and let I be a flat ideal. Then $S = I:I$ is a flat epimorphism.

3. Rings of dimension one.

We provide here a discription of the rank one flat modules over one dimensional domains with a Noetherian prime spectrum. It is essentially the result which is well known for Dedekind domains.

Theorem 3.1. Let R be an integral domain with Noetherian prime spectrum of Krull dimension one. Let I be a rank one flat module over R. Then I is a directed union of projective submodules.

Proof. Let K denote the field of quotients of R. Since $I \otimes K \cong K$, we identify I with a submodule of K. We begin by observing that we may assume $R \subset I$. Indeed, let $0 \neq x \in R \cap I$; then $R \subset Ix^{-1} \cong I$.

Let $T = I/R$. T is a torsion module and, since R has Krull dimension one and Noetherian prime spectrum, in the same manner as for Dedekind domains, it follows that

$$T = \coprod_{P \in \mathrm{Spec}(R)} T(P),$$

where $T(P)$ is the P-primary component of T, that is, each element of $T(P)$ is annihilated by an ideal with radical P.

Since R has Krull dimension one, for each maximal ideal P, $I_P = K$ or, $I_P \cong R_P$ [GV]. Thus, with respect to each such prime ideal, two possibilities arise:

(1) $T(P) = K/R_P$: and let (X) denote the collection of such primes.

(2) $T(P) = R_P a^{-1}/R_P$, with $a \in R$: and let (Y) denote the collection of such primes.

From (X) and (Y) we now construct the needed invertible ideals.

Lemma 3.2. Let R be a ring of Krull dimension one, and let J be a finitely generated ideal containing a regular element. There exists a projective ideal J_0 such that $\sqrt{J} = \sqrt{J_0}$.

Proof: Let x be a regular element in J. Since $B = R/(x)$ has Krull dimension zero, B_{red} is a von Neumann ring. Thus JB_{red} is generated by one idempotent e', $e \in J$. We may even pick e to be idempotent modulo (x). Let $J_0 = (x,e)$. It is clear that $\sqrt{J} = \sqrt{J_0}$, and an easy localization checking shows J_0 to be locally principal, hence invertible. ∎

Before we proceed with the proof of (3.1) we point out a consequence where the Noetherian hypothesis is not used.

Corollary 3.3: Let R be an integral domain of Krull dimension one and let $R \longrightarrow S$ be a flat epimorphism. Then S is a directed union of invertible ideals i.e., S is a directed union of simple flat epimorphisms.

Proof: For each finite set $u_1, \ldots, u_n$ of elements of S, let $I = \bigcap (R \underset{R}{:} u_i)$. It is still the case that $IS = S$ [L]. Let J be a finitely generated subideal of I such that $JS = S$. Let J_0 be the invertible ideal provided by Lemma 3.2. Then $u_i \in J_0^{-1}$; it is also clear that $R[J_0^{-1}] \subset S$. Since each $R[J_0^{-1}]$ is a directed union of invertible ideals the first assertion follows. The remainder is immediate from Theorem 3.1. ∎

We now look at the prime ideals in (Y). Let then $T(P) = R_P a^{-1}/R_P$.

Lemma 3.4. $L(P) = (aR_P) \cap R$ is an invertible ideal of R.

Proof: Assume $L(P) \neq R$. By definition, L(P) is a P-primary ideal. Since R has Noetherian spectrum, there exists a finitely generated ideal J, $J \subset L(P)$, $\sqrt{J} = P$. It is now clear that $L(P) = (a, J)$ and that it is locally principal.

To put I together, consider two collections of invertible submodules of I:

(X') For each $P \in X$, construct J(P) as in Lemma 3.2, and form all $J(P)^{-n}$, $n = 1, 2, \ldots$.

(Y') For each $P \in Y$, let L(P) be the ideal constructed in Lemma 3.4.

Now form the directed system of invertible ideals obtained by considering finite propducts of ideals from (X') and (Y'). Call I' its directed union. Since I and I' are locally the same, theorem 3.1 has been proved. ∎

Remark. The theorem will still be true over reduced rings of dimension one and Noetherian prime spectrum. It is not valid, however, for all non-reduced rings (cf. example in Lazard [L]), on the other hand, it is tempting to conjecture a similar statement for all domains of dimension one without the Noetherian hypothesis.

4. Coherent rings.

In this section we discuss several relationships among expected properties of large classes of coherent domains. The methods, employed in our investigation resemble tangentially, those used to study rank one flat modules, with the interplay between finiteness and divisoriality being a key ingredient.

(a) R is stably coherent -- that is, the polynomial rings $R[T_1, \ldots, T_n]$, $n \geq 0$, are coherent.

It is known that there exist coherent rings which are not stably coherent [So]. Although Soublin's example-- $Q[x, y]^{(N)}$ -- is not a domain, Gruson has shown that one of its localizations--they are all domains--is not stably coherent.

Several instances of stably coherent rings were uncovered through the following feature:

Conjecture C_1: Let R be a commutative ring. If $R[T_1]$ is coherent, then R is stably coherent.

(b_n) Uppers of 0 are finitely generated--that is, if P is a prime ideal of $R[T_1, \ldots, T_n]$, $n \geq 0$, and $P \cap R = 0$, then P is finitely generated (thus(b_1) refers to the case of one indeterminate).

Example 4.1. There are several classes on non-Noetherian rings for which (a) and (b_n) hold. Let R be an integral domain that is either (i) Prüfer, or (ii) of global dimension two. Then $B = R[T_1, \ldots, T_n]$ is coherent [VA]. As for the finiteness of uppers, one has more generally

Lemma. Let E be a finitely generated B-module which is R-torsion free. Then E is finitely presented.

Proof: (i): R is Prüfer. In this case E is R-flat and the assertion follows from [R-G] (II.3.4.6).

(ii): R has global dimension two. Here the set U of primes of B where E is R-flat is open (cf. [RG]), and covers finitely many primes of R (cf. [VA], (Chap. 8)). As in [VA], we may reduce to the local case and then make use of the fact that local rings of global dimension two are presentable as pull-backs of Noetherian and Prüfer rings.

Note that if S is a finite type integral domain over a Prüfer or a global dimension two ring (a) and (b_n) are still valid, so is the case if $S = R/Q$, where R is a ring of global dimension two and Q is a prime ideal of R, as a closer look into the structure of a ring of global dimention two will reveal.

Example 4.2. Some uppers of 0 are finitely generated, because of the inherent presence of flatness and regardless of the coherence of the ring.

Thus, if $\underline{m}$ is a maximal ideal which is an upper of 0, $\underline{m}$ can be generated by n+1 elements. To prove this, note that in this case $R[T_1, \ldots, T_n]/\underline{m}$ is a flat R-algebra of finite type and hence $\underline{m}$ is finitely generated [R-G]. Using the Nullstellensatz it follows that $m/\underline{m}^2$ is generated by n elements and therefore, $\underline{m}$ can be generated by n+1 elements.

In case $n = 1$ and R is integrally closed we can improve the result and state that $\underline{m}$ is indeed principal.

Note that $c(\underline{m})$ - the content of $\underline{m}$ = R and therefore we have a polynomial f in $\underline{m}$ satisfying $c(f) = R$. Let g be a nonzero polynomial in $\underline{m}$ of least degree, then there exist an element a in R and a polynomial $h \in R[T]$ such that $af = h \cdot g$. Using the content formula [G], and the integrality condition of R, we obtain the invertibility of the ideal $J = c(g)$, and the equality $\underline{m} = J^{-1}g\,R[T]$. Let a_0 be the constant term of g, we claim that $J^{-1}a_0 = R$. Suppose otherwise, and let P be a prime ideal of R containing $J^{-1}a_0$. Then $(\underline{m}, T) \subset (P, T)$, contradicting the maximality of $\underline{m}$. It follows that J, and hence $\underline{m}$, are principal ideals.

<u>Conjecture</u> C_2: (a) ⇏ (b_n).

Here one can focus on the case of a single indeterminate. We will show that it is the case that many uppers of 0-- regardless of the stably coherent hypothesis--are finitely generated. Unfortunately those are not the uppers that come in later in connection to integral closure questions.

<u>Theorem 4.3</u>. Let R be a coherent domain and let P be an upper of zero in $R[T] = C$ containing a polynomial f with $c(f)^{-1} = R$. Then P is finitely generated.

Before actually proving theorem 4.3, we will embark in a brief discussion and a series of Lemmas that are interesting for their own sake.

We denote by C_n the R-submodule of C consisting of all polynomials of degree at most n. For an ideal I of C denote by $I_n = I \cap C_n$, by $\ell(I)$ the ideal of leading coefficients of I, by $\ell_0(I)$ the ideal of tail (constant) coefficients of I. A class of uppers of 0 that plays an important role throughout our investigation is

$P_R\left(\frac{a}{b}\right) = P\left(\frac{a}{b}\right) = \{f \in C[T],\ f\left(\frac{a}{b}\right) = 0\}$ for elements a and b in R. Note that $\ell\left(P\left(\frac{a}{b}\right)\right) = \ell_0\left(P\left(\frac{b}{a}\right)\right)$ and conversely.

We start by exhibiting a relation between a general upper of 0, P and uppers of the type $P\left(\frac{a}{b}\right)$.

Let $g = a_nT^n + \cdots + a_0$ be a polynomial of minimal degree in P and $R[T]/P = R[u]$. We can view P in $R[u][T]$ as the ideal consisting of all polynomials $f \in R[T]$, satisfying $f(u) = 0$. Let $v = a_nu$, then v is integral over R and $B = R[v] \subset R[u]$ is a finite free R-module generated by $1, v, \ldots, v^{n-1}$ and hence a coherent ring along with R. Consider the map $R[T] \longrightarrow B[T] \xrightarrow{\varphi} R[u]$ where $\varphi(T) = u$. and let $Q = \ker\varphi$. Since $a_nT - v$ lies in Q we have: $R[u][T] \supset Q = \{h \in B[T]/h(u) = 0\}$, and $Q \cap R[T] = P$. $u = \frac{v}{a_n}$ lies in the quotient field of B hence Q is $P_B\left(\frac{v}{a_n}\right)$.

<u>Lemma</u> <u>4.4</u>. With the notation above we have that $P_k = Q_k \cap (R[T])$ for every $k > 0$ and $\ell(Q) \cap R = \ell(P)$. In particular P_k is a finite R-module for every k.

<u>Proof</u>: If $P = P\left(\frac{a}{b}\right)$, P_k is the module of relations of the finitely generated, and hence finitely presented, ideal $(a, b)^k$. Since clearly $P_k = Q_k \cap (R[T])_k$, the last assertion is clear.

We have $\ell(P) \subset \ell(Q) \cap R$. To prove the converse let $a \in \ell(Q) \cap R$, then there is a polynomial h in Q with leading coefficient equal to a, which can be assumed to have degree $s \geq n$. Write:

$$h = aT^s + \left(\sum_{i=0}^{n-1} b_{i,s-1}v^i\right)T^{s-1} + \cdots + \left(\sum_{i=0}^{n-1} b_{i,0}v^i\right),\ b_{i,j} \in R.$$

Then $h(u) = 0$ implies:

$$0 = au^s + \left(\sum_{i=0}^{n-1} b_{i,s-1}\, a_n^i u^i\right)u^{s-1} + \cdots + \left(\sum_{i=0}^{n-1} b_{i,0}\, a_n^i u^i\right), \text{ and}$$

$$0 = b_{n-1,s-1}\, a_n^{n-1}\, u^{n+s-2} + (b_{n-2,s-1}\, a_n^{n-2} + b_{n-1,s-2}\, a_n^{n-1})$$

$u^{n+s-3} + \ldots (a + b_{1,s-1}a_n + b_{2,s-2}a_n^2 + \ldots)$

by subtracting suitable multiples of $g(u) = 0$ from the above equality we obtain an element of the form $a + e a_n$, $e \in R$ in $\ell(P)$ and hence $a \in \ell(P)$. ∎

Note that if $c(g)$ is invertible $P = c(g)^{-1} g R[T]$, as $c(g)^{-1} g R[T] \subset P$ by the torsion freeness of $R[T]/P$ and the other inclusion can be verified locally and follows from the fact that $PK = gK[T]$, where K is the field of quotients of R. This equality does not depend on the coherence of R or the number of variables $T = \{T_1, \ldots, T_n\}$.

Lemma 4.5. Let R be a coherent domain, P an upper of 0 in $C = R[T]$ containing a polynomial f with $c(f) = R$. If $\deg f = n$, then $P = (f, P_n)$. In particular P is finitely generated.

Proof: We may assume R is local with maximal ideal $\underline{m}$. By considering a faithfully flat change of rings $R \longrightarrow R(Y) = R[Y]_{\underline{m}R[Y]}$ which is compatible with P and P_k, for every k, we can assume the residue class field $R/\underline{m}$ is infinite. In this case, we consider an automorphism of $R[T]$, $T \mapsto T + a$ for an element a in R, and assume the constant term b of f is invertible. Let F be a polynomial in P of degree higher than n and constant term r. We have $F - b^{-1}rf = TG$ for $G \in R[T]$. If $P \neq (T)$ then G lies in P and we proceed by induction on degree. ∎

Lemma 4.6. Let R be an integrally closed domain, P an upper of 0 in $C = R[T_1, \ldots, T_n]$ containing a polynomial f with $c(f) = R$, then P is invertible.

Proof: Let g be a nonzero polynomial of least degree in P. $PK = gK[T_1, \ldots, T_n]$, thus there exist a nonzero element a in R and a polynomial h in $R[T_1, \ldots, T_n]$ such that $af = gh$.

Thus $(a) = c(gh)$. Using the content formula [G] we obtain: $c(hg)c(g)^m = [c(g)c(h)]c(g)^m$, where $m = \deg h$, hence $[c(g)c(h)]a^{-1} \subset R$ and $c(g)c(h) = (a)$. We conclude that $c(g)$ is invertible and $P = c(g)^{-1}g\, R[T_1 \ldots T_n]$.

The existence of a polynomial f in P having $c(f) = R$ is critical to the finiteness of P in the general case. As an example, let $R = k + xk[x, y]$, where k is a field of characteristic 0, and let $P = \ker(R[T] \xrightarrow{\varpi} R[1/y])$, where $\varpi(T) = 1/y$. P is generated by elements of the form $aT - b$, a and b in $R[N]$. Note that $a \in (xy) : x = (xy, xy^2, \ldots)$, hence P is not finitely generated although R is integrally closed [G]. Note that R is not coherent.

Following [GL] [G-V] we denote by $\mathcal{O}(R) = \{p \in \operatorname{Spec}(R),$ p minimal over $(a) : b$ for some $a, b \in R\}$.

Lemma 4.7. Let R be a coherent domain, P an upper of 0, and g a nonzero polynomial in P of least degree n. If $c(g)$ is invertible at each $p \in \mathcal{O}(R)$, then $P = c(g)^{-1}g\, R[T]$.

Proof: We have that $c(g)^{-1}g\, R[T] \subset P$. Let F be a polynomial in P of degree $r \geq n$. Define $L = \{a \in R,$ $aF \in C_{r-n}\, c(g)^{-1}g\}$. L is a finitely generated ideal of R, and by hypothesis $L_p = R_p$ for every $p \in \mathcal{O}(R)$. Thus $L^{-1} = R[G\text{-}V]$. On the other hand $C_{r-n}c(g)^{-1}g$ is R-isomorphic to $r - n + 1$ copies of $c(g)^{-1}$ and thus a reflexive R module. Taking double duals in $LF \subseteq C_{r-n}c(g)^{-1}g$ we obtain $F \in C_{r-n}c(g)^{-1}g \subset c(g)^{-1}g\, R[T]$. ∎

Corollary 4.8. Let R be a coherent integrally closed domain and let P be an upper of 0, then P is finitely generated.

Proof: Note that P satisfies the hypothesis of lemma 4.7, since if g is a polynomial of least degree in P then $(c(g)c(g)^{-1})^{-1} = R$. ∎

We now prove Theorem 4.3:

Proof: First note that P_k are reflexive, finitely generated, R-modules for every k, since $(P_k^{-1})^{-1} \subset C_k$. Let F be a polynomial in P of degree $s \geq n$, where $n = \deg f$, and put $L = \{a \in R/aF \in C_{s-n} f \oplus P_n\}$. L is a finitely generated ideal of R and by Lemma 4.5 $L_p = R_p$ for every $p \in \mathcal{P}(R)$. Thus $L^{-1} = R$, and since $C_{s-n}f \oplus P_n$ is a reflexive R module we conclude that $F \in (f, P_n)$. ∎

We remark that if R is a coherent domain with $\mathcal{P}(R)$ compact and P is an upper of 0 in R[T] such that $c(P) \not\subset p$ for any $p \in \mathcal{P}(R)$, then P is finitely generated. An example of a ring satisfying the above condition is a coherent H-domain [G-V].

(c) Coherent Rees algebras - that is, for any finitely generated ideal I, the corresponding Rees algebra $\mathcal{R}(I) = \Sigma I^n t^n$ is coherent.

The usefulness of this property lies in the fact that some versions of the Artin-Rees lemma become available. If the ideal I is generated by the elements $a_1, \ldots, a_n$, we have a presentation $R[T, \ldots, T_n] \longrightarrow \mathcal{R}(I)$, $T_i \longrightarrow a_i t$. The kernel, P, is an upper of 0. Thus

Propositon 4.9. (a) + (b_n) ⇒ (c).

(c) Finite 0-th cohomology - that is, for each finitely generated ideal I the ring $R^I = \bigcup_n (I^n : I^n)$ is a finitely generated R-module.

The reason for the terminology comes from the fact that if $X = \mathrm{Proj}(\mathcal{R}(I))$, then $H^0(X, \mathcal{O}_X) = R^I$.

Proposition 4.10. (c) ⇒ (d).

Proof: Let I be a finitely generated ideal of the integral domain R and let L be the irrelevant ideal $\mathcal{R}(I)_+$ of the Rees ring $\mathcal{R}(I)$. The ring $C = \mathrm{Hom}_{\mathcal{R}(I)}(L, L)$ consists of all polynomials $f = \Sigma a_i t^i$, $a_i \in K$ = field of fractions of R, such that $a_i I^s \subset I^{s+i}$ for all $s > 0$.

If suffices to show that if C is a finitely generated $\mathcal{R}(I)$-module, then $R^I = \bigcup_n (I^n : I^n) = I^s : I^s$ for $s \gg 0$. For that, suppose $a_i t^{r_i}$, $i = 1, \ldots, m$ are homogeneous generators of C as a $\mathcal{R}(I)$-module, and denote $r = \sup\{r_i\}$. We show that it is enough to take $s > r$. Pick $w \in I^{s+1} : I^{s+1}$ and let $x \in I^s$. Note that $wxt^s \in C$ since

$$wxI^u = x(xI)I^{u-1} \subset I^{s+1}I^{u-1} = I^{s+u}$$

for $u > 0$. We may thus write

$$wx = \Sigma a_i w_i, \quad w_i \in I^{s-r_i}.$$

But then $a_i w_i \in I^s$, and $wx \in I^s$, as desired. ∎

Denote by $R_n^I = (I^n : I^n)$. We will connect the finite generation of uppers of 0 of the type $P\left(\frac{a}{b}\right)$ and the finiteness of R^I.

Proposition 4.11. Let R be a coherent domain. $I = (a, b)$ an ideal of R, then $R^I = R_n^I$ for some n iff $P\left(\frac{a}{b}\right)$ is finitely generated.

Proof: Assume $R^I = R_n^I$ with $n > 0$ minimal such that equality holds. Sine $P\left(\frac{a}{b}\right)_k$ is finite for every k, a Hilbert basis theorem argument ensures the finite generation of $P\left(\frac{a}{b}\right)$ provided $\ell\left(P\left(\frac{a}{b}\right)\right)$ is finitely generated. We claim $\left(R_k^I \frac{b}{a}\right) \cap R \supset \ell\left(P\left(\frac{a}{b}\right)_k\right) \supset \left(R_{k-1}^I \frac{b}{a}\right) \cap R$ for every k and thus $\ell\left(P\left(\frac{a}{b}\right)\right) = \ell\left(P\left(\frac{a}{b}\right)_{n+1}\right)$ is finite. To prove the claim consider a polynomial $f = a_k T^k + \cdots + a_0 \in P\left(\frac{a}{b}\right)$. Then $a_k a^k + \cdots + a_0 b^k = 0$ and thus $a_k \frac{a}{b} (a, b)^k \subset (a, b)^k$

and $a_k \in \left(R^I_{k-1}\frac{b}{a}\right) \cap R$. Conversely let $a_{k-1} \in \left(R^I_{k-1}\frac{b}{a}\right) \cap R$ then $a_{k-1} = u\frac{b}{a}$ with $u \in R^I_{k-1}$. Thus $u\,a^{k-1} = \sum_{i\geq 0} r_i a^i b^{k-i-1}$ and $a_{k-1}a^k = u\frac{b}{a}\,a^k = \sum_{i\geq 0} r_i a^i b^{k-i}$ thus $a_{k-1} \in \ell\left(P\left(\frac{a}{b}\right)_k\right)$.

For the converse we note that since $\ell_0\left(P\left(\frac{a}{b}\right)\right) = \ell\left(P\left(\frac{b}{a}\right)\right)$ we have $P\left(\frac{b}{a}\right)$ finitely generated along with $P\left(\frac{a}{b}\right)$. Let n be an integer greater than the degrees of all polynomials in a generating set for $P\left(\frac{a}{b}\right)$ and $P\left(\frac{b}{a}\right)$. Then $\ell\left(P\left(\frac{a}{b}\right)\right) = \left(R^I\frac{b}{a}\right) \cap R = \left(R^I_n\frac{b}{a}\right) \cap R$ and $\ell\left(P\left(\frac{b}{a}\right)\right) = \left(R^I\frac{a}{b}\right) \cap R = \left(R^I_n\frac{a}{b}\right) \cap R$.

Let $x \in R^I_{n+1}$. We will show that $x\,a^j\,b^{n-j} \subset I^n$ using induction on $n-j$, and thus conclude that $R^I = R^I_n$.

For $j = 0$, we have $x\,I^{n+1} \subset I^{n+1}$ and $x\,a^{n+1} = \sum_{i=0}^{n+1} r_i\; a^i b^{n+1-i}$ and $x\frac{b}{a}\,a^{n+1} = \sum_{i>0} r_i a^{i-1} b^{n+2-i} + r_0\frac{b}{a}b^{n+1}$.

Thus $r_0\frac{b}{a}b^{n+1} = x\,b\,a^n - \sum_{i>0} r_i a^{i-1} b^{n+2-i}$ and $r_0\frac{b}{a}\,I^{n+1} \subset I^{n+1}$.

We have $r_i \in \left(R^I_{n+1}\frac{a}{b}\right) \cap R = \left(R^I_n\frac{a}{b}\right) \cap R$ and write $r_0 = y\frac{a}{b}$ with $y \in R^I_n$. Then $r_0\frac{b}{a} = y \in R^I_n$ and $\left(r_0\frac{b}{a}b^n\right)b \in I^n_b$. We conclude that $xa^n b \in I^n b$ and $x\,a^n \in I^n$. Assume $x\,a^j\,b^{n-j} \in I^n$ but $xa^{j-1}b^{n+1-j} \notin I^n$. $xa^j b^{n-j} = \sum_{i=0}^{n} r_i a^i b^{n-i}$ so

$xa^{j-1}b^{n+1-j} = \sum_{i>0} r_i a^{i-1} b^{n+1-i} + r_0\frac{b}{a}\,b^n$ and thus $r_0\frac{b}{a}b^n \notin I^n$.

and $r_0\frac{b}{a} \notin R^I_n$. On the other hand $r_0\frac{b}{a}b^{n+1} \in I^{n+1}$. It follows that $r_0 \in \left(R^I_{n-1}\frac{a}{b}\right) \cap R = \left(R^I_n\frac{a}{b}\right) \cap R$. ∎

<u>Corollary 4.12</u>. Let R be a coherent domain with Prüfer integral closure, then all uppers of 0 over R are finitely generated.

<u>Proof</u>: We have that $R^I = R^I_n$ for some n, for every finitely generated ideal I of R [C-V]. It follows that uppers of 0 of the type $P\left(\frac{a}{b}\right)$ are finitely generated. Let

$g = a_n T^n + \cdots + a_0$ be a nonzero polynomial of minimal degree of an upper P of O. With the notation of lemma 4.4 we construct the ring $B = R[v]$. B is a coherent domain with Prüfer integral closure, hence Q is finitely generated and by lemma 4.4 so is P. ∎

The last of the properties of coherent rings to be considered is:

(e) The integral closure of R is a Prüfer domain.

Conjecture C_3: Each coherent ring of Krull dimension one satisfies (e).

Proposition 4.13. For an integral domain R the following conditions are equivalent:

(i) The integral closure R' of R is a Prüfer ring.

(ii) For $a, b \in R$ the upper $P(\frac{a}{b})$ has unit content.

(iii) For $a, b \in R$ there exists an integer $n > 0$ such that with $I = (a, b)$, I^n is an invertible ideal of $\mathrm{Hom}_R(I^n, I^n)$.

Proof: The equivalence (i) <=> (iii) is proved in [C-V], while (i) <=> (ii) follows from [C-V], [G] and [E-S]. ∎

There are not many cases where (C_3) has been established. One amusing one was pointed out to us by Gruson several years ago. We use the following terminology. Given a coherent domain R, K is called a canonical module if the functor $\mathrm{Hom}_R(\cdot, K)$ is a self-dualizing functor on the category of finitely generated torsion-free R-modules.

Proposition 4.14. Let R be a coherent domain admitting a canonical module K. If uppers of 0 in R[T] are finitely generated, then the integral closure of R is Prüfer.

Proof: We show that the condition (ii) of Proposition (4.13) holds. Let a, b be elements of R. From Proposition

(4.11) we know that there is an integer $n > 0$ such that $R^I = I^n : I^n$. It is clear that $\mathrm{Hom}_R(R^I, K)$ is a canonical module for R^I. We may thus assume, changing the notation, that R admits a canonical module and $I^n : I^n = R$. for all n. Assume also that R is local and m is the minimal number of generators of K. Put $z = a/b$; it follows easily that $R = (R + Rz^{-1}) \cap (R + Rz^{m+1})$. By duality, we have $K = (K \cap Kz^{-m-1}) + (K \cap Kz)$. Take a generating set of K, $u_1, \ldots, u_m$, with $u_1, \ldots, u_r \in Kz^{-m-1}$ and $u_{r+1}, \ldots, u_m \in Kz$. We have the system of equations:

$$
\begin{aligned}
z^{m+1}u_1 &= a_{11}u_1 + \cdots + a_{1m}u_m \\
&\vdots \\
z^{m+1}u_r &= a_{r1}u_1 + \cdots + a_{rm}u_m \\
z^{-1}u_{r+1} &= a_{r+1,1}u_1 + \cdots + a_{r+1,m}u_m \\
&\vdots \\
z^{-1}u_m &= a_{m1}u_1 + \cdots + a_{mm}u_m,
\end{aligned}
$$

which implies the relation

$$
\begin{vmatrix}
a_{11}-z^{m+1} & \cdots & & & a_{1m} \\
\vdots & & & & \vdots \\
a_{r1} & \cdots & a_{rr}-z^{m+1} & \cdots & a_{r,m} \\
a_{r+1,1} & \cdots & a_{r+1,r} & a_{r+1,r+1}-z^{-1} \cdots & a_{r+1,m} \\
\vdots & & & & \\
a_{m1} & \cdots & & & a_{mm}-z^{-m}
\end{vmatrix} = 0
$$

that is, $p(z) = 0$ where p is a polynomial in $R[T, T^{-1}]$. But the coefficient of $T^{(r+1)m+2r}$ in p is $(-1)^m$. Multiplying p by a high power of T we obtain an element in the upper $P(\frac{a}{b})$, of unit content. The assertion now follows from Proposition 4.13. ■

Remark. It is not known whether the existence of the canonical module itself suffices to derive the finiteness of the uppers $P(\frac{a}{b})$.

Remark. It should be noted that the "slightest" finiteness condition imposed on R (in addition to coherence and dimension one) makes R a Noetherian ring. For example: ascending chain condition on divisorial ideals, or the property of being an H-domain.

We would like to conclude with the interesting example given by M. Hochester of a nontrivial one dimensional coherent domain with Prüfer integral closure.

Let k be a field and x_i, y_i, z_i indeterminates over k.

Let $A_1 = k[[x_1^2, x_1^3]] \simeq k[[z_1, y_1]]/(y_1^2 - z_1^3)$

$\vdots$

$A_n = k[[x_n^2, x_n^{3^n}]] \simeq k[[z_n, y_n]]/(y_n^2 - z_n^{3^n})$

$\varphi_{n,n+1}: A_n \longrightarrow A_{n+1}$ defined by $\varphi_{n,n+1}(\bar{z}_n) = \bar{z}_{n+1}^3$, $\varphi_{n,n+1}(\bar{y}_n) = \bar{y}_{n+1}$. For $n < m$ $\varphi_{n,m}: A_n \longrightarrow A_m$ is $\varphi_{n,m} = \varphi_{n,n+1} \circ \cdots \circ \varphi_{m-1,m}$; $\varphi_{n,n}$ = identity. Then each A_n is a one dimentional Noetherian local domain, with A_m flat over A_n for $n \leq m$. In fact A_n is free over A_{n-1} on three generators. $A = \varinjlim A_n$ is a one dimensional, coherent domain [B]. It is not Noetherian, but $\bar{A} = \varinjlim \bar{A}_n$ is Prüfer.

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