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GAUSSIAN PROPERTIES OF TOTAL RINGS OF QUOTIENTS

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ABSTRACT. In this paper we consider five possible extensions of the Prüfer domain notion to the case of commutative rings with zero divisors and relate the corresponding properties on a ring with the property of its total ring of quotients. We show that a Prüfer ring R satisfies one of the five conditions if and only if the total ring of quotients $Q(R)$ of R satisfies that same condition. We focus in particular on the Gaussian property of a ring.

1. INTRODUCTION

Prüfer domains are domains in which every non zero finitely generated ideal is invertible. There are a great number of equivalent characterizations of Prüfer domains, of which many have been extended to the case of rings with zero divisors, giving rise to different classes of rings. It is commonly accepted to define Prüfer rings as the rings in which every finitely generated regular ideal is invertible. In the present article we consider five possible extensions of the Prüfer domain notion to the case of commutative rings with zero divisors. More precisely, we consider the following Prüfer-like properties on a commutative ring R:

(1) R is semihereditary.
(2) The weak global dimension of R is at most one.
(3) R is an arithmetical ring.
(4) R is a Gaussian ring.
(5) R is a Prüfer ring.

In [8] and [9] it is proved that each one of the above conditions implies the next one, and examples are given to show that in general the implications cannot be reversed. Moreover, an investigation is carried out to see which conditions may be added to some of the preceding properties in order to reverse the implications.

In this article we push further the analysis of the five Prüfer-like conditions listed above through relating the property of a ring with the property of its total ring of quotients. In particular, in Section 3 we show that a Prüfer ring R satisfies one of the five conditions if and only if the total ring of quotients $Q(R)$ of R satisfies that same condition. This implies, that for a Prüfer ring with von Neumann total ring of quotients the five conditions are all equivalent. This is as far as one can go in requiring the equivalence of all the five conditions, when $Q(R)$ is not a field.

In Section 4 we generalize some results obtained by Tsang [19], by giving other characterizations of a local Gaussian ring R, and illustrating some properties of the annihilators of the elements of R.

In [9] the interest is primarily on the homological properties of the class of Gausssian rings. In particular, it is shown that if R is a coherent Gaussian ring, then the small finitistic dimension of R is at most one. In Section 5 we generalize this result by proving that it holds for any Gaussian ring.

In Section 6 we consider the problem of determining the possible values for the weak global dimension of a Gaussian ring. In [9] it is shown that for a coherent Gaussian ring the possible values for the weak global dimension of R are 0, 1, or ∞. We conjecture that the same is true for every Gaussian ring, and prove the conjecture in certain cases.We note that it follows from Osofsky [18] that arithmetical rings have weak global dimension at most one or ∞. We prove that the same holds for every Gaussian ring R which admits a maximal ideal $\mathfrak{m}$ such that the localization $R_{\mathfrak{m}}$ has nilpotent radical.

Throughout the paper R will always denote a commutative ring with identity and $Q(R)$ will denote the total ring of quotients of R.

2. Preliminaries

We recall the definitions of the five classes of rings mentioned in the introduction.

Definition 1. A ring R is called *semihereditary* if every finitely generated ideal of R is projective.

Since, a finitely generated ideal over a domain is invertible if and only if it is projective, the class of semihereditary rings provides an extension of the class of Prüfer domains to rings with zero divisors.

Definition 2. Denote by $w.gl.dimR$ the weak (or flat) global dimension of a ring R.Then $w.gl.dimR \leq 1$ if and only if every ideal of R is flat, or equivalently, if and only if every finitely generated ideal of R is flat.

Semihereditary rings have weak global dimension at most one. In fact, they are exactly the rings R with $w.gl.dimR \leq 1$, which are coherent (see [7]).

The class of rings of weak global dimension at most one can also be considered to be an extension of the class of Prüfer domains to rings with zero divisors. To see this, recall that for a ring R, $w.gl.dimR \leq 1$ if and only if every localization of R at a maximal ideal is a valuation domain (see [7]).

At this point it is worth mentioning the characterization of semihereditary rings given by Endo in [3].

Theorem 2.1. [3] *A ring R is semihereditary if an only if $w.gl.dimR \leq 1$ and $Q(R)$ is von Neumann regular.*

L. Fuchs [4] introduced the class of arithmetical rings. Arithmetical rings were also studied in [13], [14].

Definition 3. A ring R is *arithmetical* if the lattice of the ideals of R is distributive.

Arithmetical rings can be characterized by the property that in every localization at a prime (maximal) ideal, the lattice of the ideals is linearly ordered. Therefore the class of arithmetical rings provides another extension of the class of Prüfer

domains. Moreover, by the previous remarks, if $w.gl.dim R \leq 1$, then R is an arithmetical ring.

The focus of this article is on Gaussian rings introduced by Tsang in [19], which provide another class of rings extending the class of Prüfer domains to rings with zero divisors.

Definition 4. If R is a ring, and x is an indeterminate over R, the content $c(f)$ of a polynomial $f \in R[x]$ is the ideal of R generated by the coefficients of f. A polynomial $f \in R[x]$ is called a Gaussian polynomial if $c(fg) = c(f)c(g)$ for every polynomial $g \in R[x]$, and a ring R is called a *Gaussian* ring if every polynomial $f \in R[x]$ is Gaussian.

Tsang proved, among many other results, that if the content ideal of a polynomial f with coefficients in R is invertible, or more generally locally principal, then f is a Gaussian polynomial. Thus any arithmetical ring is a Gaussian ring. Tsang [19], and independently Gilmer [6] proved that a domain R is Gaussian if and only if it is Prüfer.

The Gaussian property is a local property, namely a ring R is Gaussian if and only if every localization of R at a prime (maximal) ideal is Gaussian. We will make frequent use of several equivalent characterizations of a local Gaussian ring, which we summarize in Theorem 2.2. The basic ideas behind the proofs go back to Tsang's unpublished Ph.D. Thesis [19]. We sketch some of the proofs here for the reader convenience.

Theorem 2.2. [19] *Let $(R, \mathfrak{m})$ be a local ring with maximal ideal $\mathfrak{m}$. The following conditions are equivalent.*

(a) *R is a Gaussian ring.*
(b) *If I is a finitely generated ideal of R and $(0 : I)$ is the annihilator of I, then $I/I \cap (0 : I)$ is a cyclic R-module.*
(b') *Condition (b) for two generated ideals.*
(c) *For any two elements a, b in R, there exists d in the annihilator of (a, b) such that $(a, b) = (a, d)$ or $(a, b) = (b, d)$. Moreover, d can be chosen so that $b \in d + aR$, or $a \in d + bR$, respectively .*
(d) *For any two elements a, b in R, the following two properties hold:*
  (i) *$(a, b)^2 = (a^2)$ or (b^2),*
  (ii) *If $(a, b)^2 = (a^2)$ and $ab = 0$, then $b^2 = 0$*

Proof. Tsang proves the equivalence (a)⇔(b) and (a)⇔(b').

Condition (c) is easily seen to be just a reformulation of condition (b').

The implication (a)⇒(d) appears in Tsang's thesis and the equivalence (a)⇔(d) has been noted by Lucas in [16]. □

As a consequence of this theorem we obtain that if $(R, \mathfrak{m})$ is a Noetherian ring, then R is Gaussian if and only if $R/(0 : \mathfrak{m})$ is an arithmetical ring. Tsang also showed that the prime ideals of a local Gaussian ring $(R, \mathfrak{m})$ are totally ordered by inclusion, thus the nilradical is the unique minimal prime ideal of R. It follows that a local Gaussian ring modulo its nilradical is a valuation domain. In particular a reduced local Gaussian ring is a valuation domain.

Definition 5. R is a *Prüfer ring* if and only if every finitely generated regular ideal of R is invertible.

From the remarks following Definition 4 we conclude that Gaussian rings are Prüfer rings.

Gaussian rings were also considered in [1] and [2].

In summary (see also [8], [9]), we have the following implications among the five Prufer-like conditions considered in the introduction: $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$.

3. The total ring of quotients

In this section we prove that if the total ring of quotients of a Prüfer ring R is Gaussian or arithmetical, then the same holds for R. As a Corollary we obtain necessary and sufficient conditions on $Q(R)$ for reversing all the implications of the five Prüfer-like conditions considered in the introduction.

We proceed by developing several notions that will be used in our proofs, and recalling some definitions found in [10], [11], [12] or [15].

Definition 6 ([10], [11]). Let P be a prime ideal of R. The *large quotient ring* of R with respect to P, denoted by $R_{[P]}$, consists of the elements $x \in Q(R)$ such that $xs \in R$ for some element $s \in R \setminus P$.

For every ideal $I \in R$, I^* denotes the set of elements $x \in Q(R)$ such that $xs \in I$ for some element $s \in R \setminus P$.

Clearly $R \subseteq R_{[P]} \subseteq Q(R)$ and $Q(R_{[P]}) = Q(R)$. We also have, $I^* \supseteq IR_{[P]}$ and if P is a prime ideal of R, then P^* is a prime ideal of $R_{[P]}$.

Moreover, in [11, page 415] it is proved that the operation $*$ is a one-to-one inclusion preserving correspondence between prime ideals of R contained in P and prime ideals of $R_{[P]}$ contained in P^*.

Definition 7 ([10], [15]). Let P be a prime ideal of R. The *core* $C(P)$ of P is the set of all elements $b \in R$ with the property that for each regular element $r \in R$ there exists an element $s \in R \setminus P$ such that $bs/r \in R$. Thus
$C(P) = \{b \in R \mid b/r \in R_{[P]} \text{ for every regular element } r \in R\}$.

Let P be a prime ideal of R, using ideas of [10] and [15] we show:

Lemma 3.1. *If P consists of zero divisors, then $C(P) = R$. If P is a regular prime ideal, then $C(P)$ is an ideal of R contained in P, and it consists of zero divisors.*

Proof. If P consists of zero divisors, then every regular element r of R is not in P; hence for every $b \in R$, $b = br/r \in R$ implies $b \in C(P)$. If P is a regular ideal and $b \in C(P)$, pick a regular element $r \in P$. Then $bs/r \in R$ for some $s \in R \setminus P$ implies $bs \in rR \subseteq P$, hence $b \in P$. To prove that $C(P)$ consists of zero divisors, assume that a regular element $r \in C(P)$. Then, r^2 is regular and $rs/r^2 \in R$ for some $s \in R \setminus P$ implies $s \in rR \subseteq P$, a contradiction. $\square$

Another important notion when dealing with Prüfer rings is the notion of a Manis valuation recalled below.

Definition 8 ([10], [12], [15], [17]). Let K be a commutative ring. A (Manis) valuation on K is a pair (v, Γ) where Γ is a totally ordered abelian group and v is a map from K onto $\Gamma \cup \infty$ satisfying the following properties:
  (1) $v(xy) = v(x) + v(y)$.
  (2) $v(x + y) \geq \min\{v(x), v(y)\}$.

(3) $v(1) = 0$ and $v(0) = \infty$.

If (v, Γ) is a valuation on K, then $R_v = \{x \in K \mid v(x) \geq 0\}$ is a subring of K, and $P_v = \{x \in K \mid v(x) > 0\}$ is a prime ideal of R_v. Moreover, $A = \{x \in K \mid v(x) = \infty\}$ is a prime ideal both of R and K.

Definition 9. Let R be a ring with total ring of quotients $Q(R)$. If P is a prime ideal of R such that $R = R_v$ and $P = P_v$ for some valuation (v, Γ) on $Q(R)$, then the pair (R, P) is called a *Manis valuation ring*.

It is known that (R, P) is a Manis valuation ring if and only if for every $x \in Q(R) \setminus R$ there exists $y \in P$ such that $xy \in R \setminus P$. It follows that if (R, P) is a Manis valuation ring and r is a regular element of R, then $v(r) \neq \infty$. In fact, $v(r^{-1}) = -v(r)$.

Lemma 3.2. *Let P be a proper prime ideal of R and assume $(R_{[P]}, P^*)$ is a Manis valuation ring with valuation (v, Γ). The following hold:*

 (1) *If the prime ideal P consists of zero divisors, then $R_{[P]}$ coincides with $Q(R)$ and (v, Γ) is a trivial valuation, namely $v(x) = 0$ for every $x \in R_{[P]} \setminus P^*$ and $v(y) = \infty$ for every $y \in P^*$,*
 (2) *If P is a regular prime ideal, then $C(P) = \{b \in R \mid v(x) = \infty\}$; hence $C(P) = v^{-1}(\infty) \cap R$ is a prime ideal of R.*

Proof. (1) Let $x \in Q(R)$, $x = a/r$ for some elements $a, r \in R$ with r regular. Then $r \notin P$, hence $xr \in R$ implies $x \in R_{[P]}$. It remains to show that $v(y) = \infty$ for every $y \in P^*$. Assume $v(y) = \gamma \in \Gamma$. Since v is a surjective map, there exists $x \in Q(R) = R_{[P]}$ such that $v(x) = -\gamma$. But $v(x) \geq 0$ for every $x \in R_{[P]}$, implies $\gamma = 0$ contradicting $y \in P^*$.

(2) If r is a regular element of P, then r is a regular element of P^*. Thus $0 < v(r) \neq \infty$, namely Γ is not the trivial group. Let $b \in C(P)$; then for every regular element $r \in R$, $b/r \in R_{[P]}$. So $v(b) \geq v(r)$ for every regular element $r \in R$. We show now that for every element $0 < \gamma \in \Gamma$ there is a regular element $r \in R$ such that $v(r) \geq \gamma$, and conclude that $v(b) = \infty$. Let $0 < \gamma \in \Gamma$; since v is a surjective map, there exists an element $x \in Q(R)$ such that $v(x) = -\gamma$. Since x is of the form a/r' for some element $a \in R$ and some regular element $r' \in R$, we have $v(r') = v(a) + \gamma \geq \gamma$.

Conversely, assume $b \in R$ is such that $v(b) = \infty$. Then, for every regular element $r \in R$, $v(b/r) = \infty$ and $v(r) \neq \infty$. So $b/r \in R_{[P]}$, namely $b \in C(P)$. $\square$

In [10, Theorem 13] Griffin characterizes Prüfer rings by means of fifteen equivalent conditions which are the generalizations of analogous conditions on Prüfer domains. We will use Griffin's condition stating that a ring R is Prüfer if and only if $(R_{[P]}, P^*)$ is a Manis valuation ring.

We are now in a position to prove our main result concerning the Gaussian property of a total ring of quotients.

Theorem 3.3. *Let R be a Prüfer ring. Then R is Gaussian if and only if $Q(R)$ is Gaussian.*

Proof. If R is a Gaussian ring, so is $Q(R)$ since it is a localization of R. To prove sufficiency it suffices to show that every localization of R at a maximal ideal P of R is Gaussian. If P is not regular, then $PQ(R)$ is a proper prime ideal of $Q(R)$, and

it is immediate to check that R_P is the localization of $Q(R)$ at $PQ(R)$. It follows that R_P is Gaussian, since by hypothesis $Q(R)$ is Gaussian.

Let P be a regular maximal ideal of R. By Tsang's characterization of a local Gaussian ring, Theorem 2.2, we have to prove that given two elements $a, b \in R$, the ideal $(a, b)R_P$ is of the form $(a, d)R_P$ or $(b, d)R_P$, for some element d which annihilates $(a, b)R_P$. Now R is a Prüfer ring, so by [10, Theorem 13] $(R_{[P]}, P^*)$ is a Manis valuation ring. If a, b are not both in the core of P, we can apply [10, Lemma 5] to conclude that $(a, b)R_P$ is a principal ideal of R_P, so it is generated by a or b and we are done. Thus, assume a, b are both in the core C of P. By Lemmas 3.2 and 3.1, C is a prime ideal of R and it consists of zero divisors. Thus R_C is Gaussian, since it is the localization of $Q(R)$ at the prime ideal $CQ(R)$. Therefore, by Tsang's Theorem 2.2 (c) we can assume that $(a, b)R_C = (a, d)R_C$, where d is an element of R which annihilates $(a, b)R_C$; moreover, we can choose d such that $b \in d + aR_C$.

Claim (a). Every element of R in the annihilator of $(a, b)R_C$, belongs to the annihilator of $(a, b)R_P$ in R_P .

In fact, let $d \in R$ be in the annihilator of $(a, b)R_C$. Then, there exists $t' \in R \setminus C$ such $t'da = 0 = t'db$. Since $v(t') \neq \infty$, we can choose $z \in Q(R)$ such that $v(z) = -v(t')$. Then $v(zt') = 0$, hence there exists $s \in R \setminus P$ such that $szt' \in R \setminus P$. So, $szt'da = 0 = szt'db$ implies $d(a, b)R_P = 0$ in R_P.

We can write $b = (r/t)a + d$, for some $r \in R$ and $t \in R \setminus C$.

We consider two cases:

First case: $v(r) \geq v(t)$. Arguing as above there are $z \in Q(R)$ and $s \in R \setminus P$ such that $szt \in R \setminus P$. Moreover, $v(zr) \geq 0$; so there is $s' \in R \setminus P$ such that $s'zr \in R$. Then $s_0 = s'szt \in R \setminus P$ and we have $s_0 b = (s'zr)sa + s'sztd$. This shows that $b \in (a, d)R_P$ and also $d \in (a, b)R_P$. So we conclude that $(a, b)R_P = (a, d)R_P$ with $d(a, b)R_P = 0$.

Second case: $v(r) < v(t)$. Let $y \in Q(R)$ be such that $v(y) = -v(r)$; then $v(yr) = 0$ and $v(yt) > 0$. So there exist $s, s' \in R \setminus P$ such that $syr \in R \setminus P$ and $s'yt \in R$. Then, $s_0 = s'syr \in R \setminus P$ and $s_0 a = (s'yt)sb - (s'yt)sd$. This shows that $a \in (b, cd)R_P$, where $c = s'yts \in R$ and $cd \in (a, b)R_P$. Hence, $(a, b)R_P = (b, cd)R_P$. Clearly cd belongs to the annihilator of $(a, b)R_C$ in R_C, hence, by Claim (a) cd annihilates $(a, b)R_P$.

So we have shown that given two elements $a, b \in R$, the ideal $(a, b)R_P$ is of the form $(a, d)R_P$ or $(b, d)R_P$, for some element d which annihilates $(a, b)R_P$. $\qquad\square$

We consider now the case of arithmetical rings. The following easy lemma will be useful.

Lemma 3.4. *Let P be a prime ideal of a ring R. Then, the total ring of quotients of the localization of R_P at the prime ideal P, is a localization of $Q(R)$ with respect to a multiplicative subset of R.*

Proof. Let T be the multiplicative set of the regular elements of R and let S be the multiplicative set $R \setminus P$. Let

$$U = \{a \in R \mid a/1 \text{ is a regular element of } R_P\}.$$

U is a multiplicative subset of R and it is immediate to check that $U \supseteq T$.

Consider the subset $UR_P = \{a/s \mid a \in U, s \in S\}$ of R_P. UR_P is the set of regular elements of R_P. In fact, r/s is a regular element of R_P if and only if $r/1 = (r/s)s$ is a regular element of R_P, since s is invertible in R_P.

The ring of quotients $Q(R_P)$ of R_P is the localization of R_P at the multiplicative set UR_P. Thus $Q(R_P)$ is the localization of R first at S and then at UR_P. So $Q(R_P)$ is also the localization of R at the multiplicative set US. As noted above U contains the set of regular elements of R; thus R_U is a localization of $R_T = Q(R)$. We conclude that $Q(R_P)$ is a localization of $Q(R)$

More precisely, $R_U = (R_T)_{(UR_T)}$ and $Q(R_P) = R_{(US)} = (R_U)_{(SR_U)} = (R_T)_{(USR_T)}$. $\square$

Proposition 3.5. *Let R be a Gaussian ring. Then R is arithmetical if and only if the total ring of quotients $Q(R)$ of R is arithmetical.*

Proof. Necessity is obvious, since $Q(R)$ is a localization of R. To prove sufficiency, it is enough to show that for every maximal ideal P of R, R_P is an arithmetical ring. R_P is a Gaussian ring and by Lemma 3.4, $Q(R_P)$ is a localization of the arithmetical ring $Q(R)$, hence it is arithmetical, too. Thus, without loss of generality, we can assume that R is a local ring. As noted by Tsang [19], the lattice of the prime ideals of a local Gaussian ring is linearly ordered; therefore the set of zero divisors $Z(R)$ of R is a prime ideal L and the total ring of quotients of R is R_L. Given two elements $a, b \in R$ we will prove that the ideal (a, b) is principal, and therefore conclude that R is arithmetical, as desired. If one of the two elements is regular, then the ideal (a, b) is principal, since R is a local Gaussian ring, hence a local Prüfer ring. Assume a, b are both zero divisors and consider the ideal $(a, b)R_L$ of R_L. Since, by hypothesis, $R_L = Q(R)$ is arithmetical, we have, say, $b = a(r'/r)$ where r is a regular element of R. Consider the ideals (a, r) and (b, r) in R. They are regular ideals, hence, by the previous remark they are principal. We conclude that $(a, r) = (b, r) = (r)$. Thus, $a = rc, b = rd$ for some elements c, d in R. We have $rb = r^2 d$, $ar' = r'rc$ and, using $rb = ar'$, we conclude that $r^2 d = r'rc$. The regularity of r implies that $b = rd = r'c$. Hence $(a, b) = (rc, r'c) = c(r, r')$. Again by the regularity of r and by the fact that R is a local Prüfer ring, the ideal (r, r') is principal. We conclude that (a, b) is also principal. $\square$

Theorem 3.6. *Let R be a Prüfer ring. Then R is arithmetical if and only if the total ring of quotients $Q(R)$ of R is arithmetical.*

Proof. The necessary condition follows by the fact that $Q(R)$ is a localization of R. For the converse, note that if $Q(R)$ is arithmetical, then it is also Gaussian. By Theorem 3.3 R is Gaussian. Thus, to conclude it is enough to appeal to Proposition 3.5. $\square$

We remark that Theorem 3.6 can also be deduced from a result of Griffin [10, Theorem 19].

We now make use of the results found so far to clarify the exact relation between each of the Prüfer-like conditions on the ring R, and the corresponding condition on its total ring of quotients $Q(R)$. The implication in one direction is summarized in the result below:

Theorem 3.7. *If R is a ring satisfying any of the five Prüfer-like conditions mentioned in the introduction, then $Q(R)$, the total ring of quotients of R, satisfies the same Prüfer-like condition.*

Proof. Conditions (1), (2), (3) and (4) are inherited by localizations, hence if R satisfies one of them, the same holds for $Q(R)$, since it is a localization of R. Moreover, any total ring of quotients is a Prüfer ring. $\square$

Concerning the converse of Theorem 3.7, we note that none of the five Prüfer-like conditions on the total ring of quotients of a ring R implies the same condition on the ring R. In fact, for condition (5), note that any total ring of quotients is a Prüfer ring while there are non Prüfer rings, even Noetherian ones. In [8] it is shown that if R is a local noetherian reduced ring which is not a domain, then $Q(R)$ is Von Neuman regular. By Endo's Theorem 2.1, $Q(R)$ is semihereditary, hence it satisfies all the five conditions above, while R is not a Prüfer ring, so its doesn't satisfy any of the five conditions.

In the examples below we show that the implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5)$ cannot be reversed, without additional conditions, even if all rings involved are total rings of quotients.

Example 3.8. A non Gaussian total ring of quotients.

Let $R = k[X,Y]/(X,Y)^3$ where k is a field, X, Y are indeterminates over k. R coincides with its total ring of quotients, so it is a Prüfer ring, but it is not Gaussian. In fact, the maximal ideal of R is finitely generated, but its square is not principal.

Example 3.9. A Gaussian total ring of quotients which is not arithmetical.

Let $R = k[X,Y]/(X,Y)^2$ where k is a field, X, Y are indeterminates over k. R coincides with its total ring of quotients; R is Gaussian, since it is local and the maximal ideal has square zero, but is is clearly not arithmetical.

Example 3.10. An arithmetical total ring of quotients with $w.gl.dim > 1$.

It is immediate to see that $R = k[X]/(X)^2$ where k is a field and X is an indeterminate over k satisfies the desired conditions.

Note that the rings considered in the preceding examples are all Noetherian. To find an example of a non semihereditary total quotient ring with $w.gl.dim \leq 1$ we need to leave the class of Noetherian rings, since any coherent ring with $w.gl.dim \leq 1$ is semihereditary.

Example 3.11. A non semihereditary total ring of quotients with $w.gl.dim \leq 1$.

Consider an example of a non semihereditary ring R with $w.gl.dim R \leq 1$, as for instance the example produced in [8]. The total ring of quotients of R, $Q(R)$, has $w.gl.dim Q(R) \leq 1$ and cannot be semihereditary, otherwise by Endo's characterization of semihereditary rings (Theorem 2.1), $Q(R)$ would be von Neumann regular and thus the ring R would be semihereditary.

At this point, using the results proved so far in this section, and other known results, we have a complete understanding of the effect that the assumption of one of the Prüfer-like conditions on the total ring of quotients of a ring R has on the ring itself.

Theorem 3.12. *Let R be a ring with total ring of quotients $Q(R)$. The following conditions hold:*

    (i) *R is a semihereditary ring if and only if R is a Prüfer ring and $Q(R)$ is a semihereditary ring.*

(ii) *R has weak global dimension at most one if and only if R is a Prüfer ring and $Q(R)$ has weak global dimension at most one.*

(iii) *R is an arithmetical ring if and only if R is a Prüfer ring and $Q(R)$ is an arithmetical ring.*

(iv) *R is a Gaussian ring if and only if R is a Prüfer ring and $Q(R)$ is a Gaussian ring.*

(v) *Let (n)= (1), (2),(3) or (4). Then R satisfies condition condition (n) if and only if R satisfies condition (n+1) and $Q(R)$ satisfies condition (n).*

Moreover, if the total ring of quotients of R is von Neumann regular, then all the five conditions above are equivalent on R.

Proof. (i) The necessary condition has been proved in Theorem 3.7. For the converse, note that, if a total ring of quotients is semihereditary, then it is von Neumann regular by Endo's Theorem 2.1. Moreover, by [10, Theorem 20], a ring R is semihereditary if and only if R is Prüfer and $Q(R)$ is von Neumann regular.

(ii) By Theorem 3.7 only the sufficiency has to be proved. Recall that in [9, Theorem 2.2] it is proved that a ring has weak dimension less or equal 1 if and only if it is a reduced Gaussian ring. Assume that $w.gl.dim Q(R) \leq 1$; then $Q(R)$ is Gaussian and reduced, so R is reduced, too. By Theorem 3.3 R is Gaussian, thus it has weak dimension less or equal 1. So (ii) follows.

(iii) Follows from Theorem 3.6 and (iv) is Theorem 3.3.

(v) Let (n)= (1), (2), (3) or (4). If R satisfies condition (n+1), then R is a Prüfer ring, so (v) follows by the previous conditions (i)-(iv).

By [10, Theorem 20] a total ring of quotients is von Neumann regular if and only if it is semihereditary. So the last statement follows by the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and part (v). □

As a consequence of Theorem 3.12 we note that the total ring of quotients of any ring R satisfying condition (n+1) but not condition (n), for every n=1,2,3,4, cannot be von Neumannn regular.

Example 3.13. The simplest example of a Prüfer ring R such that the total ring of quotients of R is not von Neumann is the ring $k[X]/(X^2)$, where k is a field and X is an indeterminate over k. In fact, R is even an arithmetical ring, it coincides with its total ring of quotients and it has infinite weak global dimension.

4. Local Gaussian rings

In this section we give another characterization of a local Gaussian ring, besides the ones obtained by Tsang in [19], and recalled in Section 2. Moreover, we consider the case of a local Gaussian ring $(R, \mathfrak{m})$ in which $\mathfrak{m}$ is the nilradical of R and we find properties of the annihilators of the elements of R.

Given two ideals I, J of R, let $(I : J) = \{x \in R \mid xJ \subseteq I\}$.

Theorem 4.1. *Let $(R, \mathfrak{m})$ be a local ring and let $D = \{x \in R \mid x^2 = 0\}$. Consider the following conditions:*

(1) *D is an ideal of R, $D^2 = 0$, and R/D is an arithmetical ring.*

(2) *For every $a \in R$, $aD \subseteq a(Ra \cap D)$.*

(3) *For every $a \in R$, $(0 : a)$ and D are comparable and $D \subseteq Ra + (0 : a)$.*

Then R is Gaussian if and only if R satisfies (1) and (2) or (1) and (3).

Proof. First note that $(3) \Rightarrow (2)$. In fact, if $D \subseteq (0\colon a)$, then $aD = 0$, hence (2) is trivially satisfied. Otherwise we have $(0\colon a) \subsetneq D$. Let $d \in D$; by (3) $d = ya + c$ for some $y \in R$ and $c \in (0\colon a)$. Thus $ya \in D$ and $ad = a(ya) \in a(Ra \cap D)$.

We prove that a Gaussian ring satisfies (1), (2), and (3).

(1) Given two elements $x, y \in D$, then $(x, y)^2 = 0$ by Theorem 2.2 (d); so D is an ideal whose square is zero. Consider a two generated ideal $(a + D, b + D)$ of R/D. By Theorem 2.2 (c), we may assume $(a, b) = (a, d)$ with d in the annihilator of $(a, b) = (a, d)$; hence $da = 0 = d^2$. It follows that $(a + D, b + D) = (a) + D$ and R/D is arithmetical.

(2) Let $a \in R$. If $aD = 0$, then condition (2) is clearly satisfied. Assume $ad \neq 0$ for some $d \in D$. Then by (1) $a \notin D$ and by Theorem 2.2 (c), $(a, d) = (a, c)$ or $(a, d) = (d, c)$ for some c in the annihilator of (a, d); in particular $c^2 = 0 = ac$. It cannot be $(a, d) = (d, c)$ otherwise $a \in D$. So by Theorem 2.2 (c) we have $d = \lambda a + c$; then $ad = a^2 \lambda + ac = a(a\lambda)$ and $a\lambda = d - c$ yields $ad \in a(D \cap aR)$.

(3) Let $a \in R$; if $a^2 = 0$, then $aD = 0$, hence $D \subseteq (0\colon a)$. If $a^2 \neq 0$, let $c \in (0\colon a)$. By Theorem 2.2 (d), $ac = 0$ implies $a^2 = 0$ or $c^2 = 0$; so $c^2 = 0$ and $(0\colon a) \subseteq D$. For the second part, let $d \in D$. If $ad = 0$, then $d \in (0\colon a)$. Assume $ad \neq 0$; by (2) $ad = a(ay)$ with $y \in R$ and $ay \in D$. So $d - ay \in (0\colon a)$ and thus $d \in Ra + (0\colon a)$.

Conversely, assume that the local ring R satisfies (1) and (2). Let $a, b \in R$. By condition (1) (a, b) is principal modulo D. Recalling that over a local ring a finitely generated ideal $(a_1, a_2, \ldots, a_n)$ is principal if and only if it is generated by one of the a_i's, we may assume $(a, b) = (a, d)$ for some $d \in D$. By (2) $ad = a(ay)$ for some $y \in R$ such that $ay \in D$. Then, $a(d - ay) = 0$ and $(d - ay)^2 = -2ayd = 0$, since $D^2 = 0$. Letting $c = d - ay$ we clearly have $(a, d) = (a, c)$ and c annihilates (a, d). So Tsang characterization in Theorem 2.2 (c) is verified and R is Gaussian.

If R satisfies (1) and (3), then R satisfies also (1) and (2) so R is Gaussian. □

Example 4.2. This example shows that condition (1) is not enough to conclude that R is Gaussian.

Let k be a field X, Y two indeterminates over k. Let

$$R = k[X, Y]/(X^3, Y^2, X^2 Y)$$

and denote by x, y the images of X, Y in R. It is immediate to check that $D = (x^2, y)$, so $D^2 = 0$ and $R/D \cong k[X]/(X^2)$ is arithmetical. But R is not Gaussian, since the annihilator of the maximal ideal (x, y) of R is $(x, y)^2$ and $R/(x, y)^2$ is not arithmetical. So by Tsang's characterization of local noetherian Gaussian rings we conclude that R is not Gaussian.

In all what follows we let $D = \{x \in R \mid x^2 = 0\}$.

Lemma 4.3. *Assume that $(R, \mathfrak{m})$ is a local Gaussian ring. The following hold:*

(1) *If $a \in \mathfrak{m} \setminus D$, then $(0\colon a) \subseteq D$.*
(2) *If $\mathfrak{m}$ is a nil ideal, then, for every element $a \in \mathfrak{m} \setminus D$, we have $(D\colon a) \supsetneq D$.*
(3) *If $\mathfrak{m}$ is a nil ideal and $D\mathfrak{m} = 0$, then $\mathfrak{m}^4 = 0$*

Proof. (1) Let $ab = 0$. Since $a^2 \neq 0$, we conclude by Theorem 2.2 (d) that $b^2 = 0$, hence $b \in D$.

(2) Let $a \in \mathfrak{m} \setminus D$. Since $\mathfrak{m}$ is a nil ideal there exists a minimum integer $n > 1$ such that $0 \neq a^n \in D$. Then $a^{n-1} \in (D\colon a)$ and $a^{n-1} \notin D$.

(3) We first note that the condition $D\mathfrak{m} = 0$ implies that for every $a \in \mathfrak{m} \setminus D$, $(0\colon a) = D$. In fact, by hypothesis $Da = 0$ and by (1), $(0\colon a) \subseteq D$. Let n be the

minimum integer such that $0 \neq a^n \in D$, then $n > 1$ and $(0\colon a^n) = \mathfrak{m}$, because $a^n\mathfrak{m} \subseteq D\mathfrak{m} = 0$. Since $a^{n-1} \notin D$ we have $(0\colon a^{n-1}) = D$. Hence $\mathfrak{m} = (0\colon a^n) = ((0\colon a^{n-1})\colon a) = (D\colon a)$. Thus for every $a \in \mathfrak{m}$, $a\mathfrak{m} \subseteq D$, so $\mathfrak{m}^2 \subseteq D$ and we conclude that $\mathfrak{m}^4 = 0$. $\qquad\square$

Lemma 4.4. *Assume that $(R, \mathfrak{m})$ is a local Gaussian ring with nil radical $\mathfrak{m}$. If $\mathfrak{m}$ is not nilpotent, then $\mathfrak{m} = \mathfrak{m}^2 + D$ and $\mathfrak{m}^2 = \mathfrak{m}^3$.*

Proof. Consider the ring R/D. By Theorem 4.1, R/D is arithmetical and its maximal ideal $\mathfrak{m}/D$ is not nilpotent. In fact, if $(\mathfrak{m}/D)^n = 0$, for some n, then $\mathfrak{m}^n \subseteq D$ and so $\mathfrak{m}^{2n} = 0$ contrary to our hypothesis. Thus, by [5, X, 6 pag.357], $\mathfrak{m}/D$ is idempotent, namely $\mathfrak{m}^2 + D = \mathfrak{m}$. Multiplying this equality by $\mathfrak{m}$ and by D we obtain $\mathfrak{m}^2 = \mathfrak{m}^3 + \mathfrak{m}D$ and $\mathfrak{m}D = \mathfrak{m}^2 D$. So $\mathfrak{m}D \subseteq \mathfrak{m}^3$ and $\mathfrak{m}^3 + \mathfrak{m}D = \mathfrak{m}^3$ implies $\mathfrak{m}^2 = \mathfrak{m}^3$. $\qquad\square$

Lemma 4.5. *Assume that $(R, \mathfrak{m})$ is a local Gaussian ring with nil radical $\mathfrak{m}$. If $\mathfrak{m}$ is not nilpotent, then there exists an element $d \in D$ such that $D \subsetneq (0\colon d) \subsetneq \mathfrak{m}$.*

Proof. By Lemma 4.3 (3), there exists $d_1 \in D$ such that $(0\colon d_1) \subsetneq \mathfrak{m}$. Let $a \in \mathfrak{m}$, $a \notin (0\colon d_1)$; then $0 \neq ad_1 \in D$ and $a \notin D$. Now, $(0\colon ad_1) = ((0\colon d_1)\colon a) \supseteq (D\colon a)$. Then by Lemma 4.3 (2), $(0\colon ad_1) \supsetneq D$. Assume, by way of contradiction, that $(0\colon ad_1) = \mathfrak{m}$, for every $a \in \mathfrak{m}$, $a \notin (0\colon d_1)$. Consider the local ring $\overline{R} = R/(0\colon d_1)$ with maximal ideal $\overline{\mathfrak{m}} = \mathfrak{m}/(0\colon d_1)$. $\overline{R}$ is such that the annihilator of every $\overline{0} \neq \overline{a} \in \overline{\mathfrak{m}}$ is $\overline{\mathfrak{m}}$; hence $\overline{\mathfrak{m}}^2 = \overline{0}$, that is $\mathfrak{m}^2 + (0\colon d_1) \subseteq (0\colon d_1)$. Thus $\mathfrak{m}^2 + D \subseteq (0\colon d_1)$ and by Lemma 4.4 we conclude that $\mathfrak{m} \subseteq (0\colon d_1)$, a contradiction. $\qquad\square$

5. THE FINITISTIC PROJECTIVE DIMENSION OF A GAUSSIAN RING

Denote by $\mod R$ the class of R-modules with a projective resolution consisting of finitely generated projective modules and by $p.d_R M$ the projective dimension of the R-module M. Recall that the small finitistic dimension of R is defined by

$$fP.dim R = \sup\{p.d_R M \mid p.d_R M < \infty, M \in \mod R\}$$

In [9, Theorem 3.2] it is proved that if R is a coherent Gaussian ring, then $fP.dim R \leq 1$. In the proof of that theorem the coherence of R is used only in quoting [7, Corollary 3.1.4] which is formulated for a coherent ring. We show how to adapt the proof of [9, Theorem 3.2] without assuming the coherence of R.

The following lemma generalizes [7, Theorem 3.1.2].

Lemma 5.1. *Let R be a ring and let I be an ideal contained in the Jacobson radical of R. If $M \in \mod R$ and $\operatorname{Tor}_p^R(R/I, M) = 0$ for every $p \geq 1$, then:*

$$p.d_R M = p.d_{R/I}(M/IM).$$

Proof. The proof of [7, Theorem 3.1.2] is by induction and relies on the fact that the syzygies of a finitely presented module over a coherent ring are again finitely presented. If a module $M \in \mod R$, then its syzygies modules are again in $\mod R$; thus the same proof of [7, Theorem 3.1.2] carries out in our hypotheses. $\qquad\square$

Lemma 5.2. *Let R be a ring and let I be an ideal contained in the Jacobson radical of R. Then*

$$fP.dim R \leq fP.dim R/I + w.dim_R R/I.$$

Proof. The proof is exactly the same as the proof of [7, Theorem 3.1.3]. In fact, assuming $M \in \text{mod}\,R$ instead of M finitely presented the proof goes on without the hypothesis of coherence of the ring. $\qquad\square$

Proposition 5.3. *Let R be a Gaussian ring, then $fP.dim R \leq 1$.*

Proof. First we assume that R is local with maximal ideal $\mathfrak{m}$. In case $\mathfrak{m}$ consists of zero divisors, then the proof of [9, Theorem 3.2, Case 1] shows that $fP.dim R = 0$. If $\mathfrak{m}$ contains a regular element a, the proof of Case 2 in [9, Theorem 3.2] shows that $fP.dim(R/aR) = 0$. So by Lemma 5.2, $fP.dim R \leq w.dim_R R/aR$ and, since a is a regular element, we have $w.dim_R R/aR \leq p.d_R R/aR \leq 1$.

Assume R is not local. For every R-module M and every maximal ideal $\mathfrak{m}$ of R we have $p.d_{R_\mathfrak{m}} M_\mathfrak{m} \leq p.d_R M$; moreover, if $M \in \text{mod}\,R$, then clearly $M_\mathfrak{m} \in \text{mod}\,R_\mathfrak{m}$. Thus if $M \in \text{mod}\,R$ has finite projective dimension it follows that $p.d_{R_\mathfrak{m}} M_\mathfrak{m} \leq fP.dim R_\mathfrak{m} \leq 1$. Since for every module $M \in \text{mod}\,R$ the weak and the projective dimension coincide and $w.dim_R M = \sup\{w.dim_{R_m} M_\mathfrak{m} \mid \mathfrak{m} \in \text{Max}\,R\}$l, we conclude that $fP.dim R \leq 1$ as desired. $\qquad\square$

6. The weak global dimension of a Gaussian ring.

In [9] it is proved that the weak global dimension of a coherent Gaussian ring is either infinite or at most one. We first note that the same conclusion holds in the more general case of a Prüfer coherent ring.

Proposition 6.1. *Let R be a coherent Prüfer ring. Then $w.gl.dim R = 0, 1$ or ∞.*

Proof. Assume that $w.gl.dim R = n < \infty$. Then, every finitely generated ideal of R has finite projective dimension, that is R is a regular ring. By [7, Corollary 6.2.4], $Q(R)$ is von Neumann regular, thus by Theorem 3.12 the five Prüfer-like conditions are equivalent on R. We conclude that $w.gl.dim R \leq 1$. $\qquad\square$

We would like to extend the above result to an arbitrary Gaussian ring.

Since $w.gl.dim R = \sup\{w.gl.dim R_\mathfrak{m} \mid \mathfrak{m} \in \text{Max}\,R\}$, it is enough to prove that a local Gaussian ring has $w.gl.dim = 0, 1$ or ∞. Moreover, by [9, Theorem 2.2], every reduced Gaussian ring has weak global dimension at most one. Thus, we can consider only non reduced local Gaussian rings. Furthermore, recalling that the prime ideals in a local Gaussian ring R are linearly ordered, the nilradical $\mathfrak{n}$ of R is a prime ideal and $w.gl.dim R \geq w.gl.dim R_\mathfrak{n}$. So we can restrict our investigation to the case of a local Gaussian ring $(R, \mathfrak{m})$ such that the non zero maximal ideal $\mathfrak{m}$ coincides with the nilradical of R.

We consider first the case in which the maximal ideal is nilpotent.

Lemma 6.2. *Let $(R, \mathfrak{m})$ be a local ring which is not a field.*
Then $w.dim_R(R/\mathfrak{m}) = w.dim_R \mathfrak{m} + 1$.

Proof. Consider the exact sequence $0 \to \mathfrak{m} \to R \to R/\mathfrak{m} \to 0$. Then $w.dim_R(R/\mathfrak{m}) = w.dim_R \mathfrak{m} + 1$ or $R/\mathfrak{m}$ is flat. Assume by way of contradiction that $R/\mathfrak{m}$ is flat; then $\mathfrak{m}$ is pure in R. Hence $a\mathfrak{m} = aR \cap \mathfrak{m} = aR$, for every $a \in \mathfrak{m}$. By Nakayama's Lemma, $a\mathfrak{m} = aR$ implies $a = 0$, a contradiction. $\qquad\square$

Proposition 6.3. *Let $(R, \mathfrak{m})$ be a local ring with non zero nilpotent maximal ideal. Then $w.dim_R \mathfrak{m} = \infty$.*

Proof. Let n be the nilpotency index of $\mathfrak{m}$. We prove that for every $1 \leq k < n$, $w.dim_R \mathfrak{m}^{n-k} = w.dim_R \mathfrak{m} + 1$. So for $k = n - 1$ we get $w.dim_R \mathfrak{m} = w.dim_R \mathfrak{m} + 1$ which yields $w.dim_R \mathfrak{m} = \infty$. Let $k = 1$. Then $\mathfrak{m}^{n-1} \mathfrak{m} = 0$ so $0 \neq \mathfrak{m}^{n-1}$ is isomorphic to a direct sum of copies of $R/\mathfrak{m}$. By Lemma 6.2, we conclude that $w.dim_R \mathfrak{m}^{n-1} = w.dim_R \mathfrak{m} + 1$. Let $1 \leq h < n$ be the maximum integer such that $w.dim_R \mathfrak{m}^{n-k} = w.dim_R \mathfrak{m} + 1$ for every $k \leq h$ and assume, by way of contradiction, that $h < n - 1$. Consider the exact sequence

$$(*) \qquad 0 \to \mathfrak{m}^{n-h} \to \mathfrak{m}^{n-(h+1)} \to \frac{\mathfrak{m}^{n-(h+1)}}{\mathfrak{m}^{n-h}} \to 0.$$

The term $\mathfrak{m}^{n-(h+1)}/\mathfrak{m}^{n-h}$ is a non-zero semi-simple, thus its weak dimension is equal to $w.dim_R \mathfrak{m} + 1$. By assumption $w.dim_R \mathfrak{m}^{n-h} = w.dim_R \mathfrak{m} + 1$. Thus, from the long exact sequence associated to $(*)$ by tensoring with an arbitrary module X, we infer that $w.dim_R \mathfrak{m}^{n-(h+1)} = w.dim_R \mathfrak{m} + 1$; contradicting the maximality of h. $\square$

Theorem 6.4. *Let R be a Gaussian ring admitting a maximal ideal $\mathfrak{m}$ such that the nilradical of the localization $R_\mathfrak{m}$ is a non-zero nilpotent ideal. Then $w.gl.dim R = \infty$*

Proof. Let $\mathfrak{m}$ be a maximal ideal of R such that $R_\mathfrak{m}$ has a non-zero nilpotent nilradical. Since $R_\mathfrak{m}$ is a Gaussian ring, the nilradical of $R_\mathfrak{m}$ is a prime ideal, hence of the form $\mathfrak{n}R_\mathfrak{m}$ for some prime ideal $\mathfrak{n}$ of R. Thus, the maximal ideal of the localization of R at $\mathfrak{n}$ is non-zero and nilpotent. By Proposition 6.3, $w.gl.dim R_\mathfrak{n} = \infty$. Since $w.gl.dim R \geq w.gl.dim R_S$ for every localization R_S of R we get the desired conclusion. $\square$

We were not able to prove that in general the weak global dimension of any Gaussian ring is either 0, 1, or ∞. This is true for every arithmetical ring; in fact, Osofsky in [18] proved that an arithmetical local ring with zero divisors has infinite weak global dimension. Thus if R is an arithmetical ring such that every localization of R at a maximal ideal is a domain, then by [7] $w.gl.dim R \leq 1$; otherwise there is a localization of R with infinite weak global dimension and the same holds true for R. With this evidence we formulate the following conjecture.

Conjecture The weak global dimension of a Gaussian ring R is 0, 1, or ∞.

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Changes made in "Gaussian Properties of Total Rings of Quotients" by S. Bazzoni and S. Glaz, after considering the referee's comments.

We made all the changes and corrections as suggested by the referee, except for the first comment which reads:

"Page 3: In the proof of the "Moreover" statement of Theorem 2.2 (c), it is only shown that there exists a d in the annihilator of (a,b) such that $(a,b) = (a,d)$ and $d = ra - b$ for some r in R. What I believe is needed here (and certainly true, in any case) is to show that for any d in the annihilator of (a,b) such that $(a,b) = (a,d)$ there exists an r in R such that $d = ra - b$.This is how the part (c) is used in the proof of Theorem 3.3."

Our comment :
We slightly modified condition (c) of Theorem 2.2, because the original formulation was misleading. In the proof of Theorem 3.3 we made a slight change that clarifies the fact that we are using only the existence of one such element d.