manuscripta math. 22, 325 - 341 (1977)

FLAT IDEALS II

Sarah Glaz and Wolmer V. Vasconcelos*

This paper is concerned with the relationships that hold between finiteness and divisibility properties of flat ideals of integral domains. Brought out often is the role of the arithmetic of a ring in the finiteness of its flat ideals.

INTRODUCTION

The sources for the problems discussed here arise in two areas: i) The divisibility problem for a pair {x,I}, where x is an element of the field of quotients of an integral domain A and I is a flat ideal - that is, when is x an element of I; ii) Flat ideals in polynomial rings resemble certain prime ideals encountered in the construction of Rees' rings - an useful device in the study of integral closure of Noetherian rings. In the first two sections we naively start our discussion of when a flat ideal I of an integral domain A is a direct limit of invertible ideals by looking at the ideal $I.I^{-1}$. It often results that I is a directed union of finitely

* Supported in part by NSF grant GP-33133X.

generated divisorial ideals. More interesting however is that whenever $I.I^{-1}$ passes a certain threshold then it must equal A - that is, I must be invertible. This will always be the case in Krull domains. Section 3 discusses a class of rings--H-domains--where the analysis of $I.I^{-1}$ is more easily dealt with. They show a resemblance to Krull domains in that divisibility questions can be usually determined at the primes associated to principal ideals. In section 4 the analysis of the finiteness of flat ideals in polynomial rings, in terms of the coefficient ideal, is carried out. The last section contains a number of unresolved questions where the elusive completeness of flat ideals plays a significant role.

1. FINITENESS

In the sequel A will denote an integral domain with field of quotients K. A-submodules of K will be referred to as ideals. Often we shall leave to the reader two tasks: i) The translation of the results where the timely presence of regular elements suffice, and ii) not to be confused by the possible ambiguity of usage of the word ideal here.

We recall that an ideal I of A is said to be *fractional* if there is $0 \neq d \in A$ with $dI \subset A$. For a fractional ideal I, $\mathrm{Hom}_A(I,A)$ may be identified with $I^{-1} = \{x \in K \mid xI \subset A\}$. I will be called *reflexive* or *divisorial* if $(I^{-1})^{-1} = I$.

(1.1) PROPOSITION. *Let I be a flat ideal of A. I is faithfully flat if and only if I is locally finitely generated.*

Proof. Follows directly from [5], since a flat ideal of a local ring $A,\underline{m}$ is principal if and only if $I \neq \underline{m}I$ - the version of Nakayama's lemma valid here.

The simplest case of finiteness of flat ideals occurs when I is projective. They are then shown to be invertible, that is, satisfy the equation $I.I^{-1} = A$, from which the finiteness ensues. More generally,

(1.2) PROPOSITION. *Let I be a flat ideal and let J be a nonzero ideal such that $I.J = L$ is finitely generated. Then I is finitely generated.*

Proof. Let $b_1,\ldots,b_n$ be generators for L. We may assume that $b_i = a_i c_i$ with $a_i \in I$, $c_i \in J$. We claim that $I = (a_1,\ldots,a_n)$. It suffices to assume A local with maximal ideal $\underline{m}$. If $a_i \in \underline{m}I$ for all a's we would have $L = \underline{m}L$, contradicting Nakayama's lemma.

2. SEMI-DIVISORIAL IDEALS

In this section we consider more robust estimates of $I.I^{-1}$ or rather unsuitable sizes for this ideal. These lead to partial answers to the divisibility question and various finiteness statements.

Let $J = (a_1,\ldots,a_n)$ be a finitely generated ideal of A and consider the exact sequence

$$0 \longrightarrow A \xrightarrow{\phi_J} A^n \longrightarrow C \longrightarrow 0$$

where $\phi_J(x) = (xa_1, \ldots, xa_n)$, $C = \mathrm{Coker}(\phi_J)$.

Let E be a torsionfree A-module and let a be an element of $E \otimes_A K$.

(2.1) PROPOSITION. *The following conditions are equivalent:*

(1) If $Ja \subset E$ then $a \in E$.

(2) $C \otimes_A E$ is a torsionfree A-module.

Proof. Suppose $C \otimes E$ is torsionfree and let $a \in K \otimes E$ with $a_i a \in E$. Let d be a nonzero element of A such that $da \in E$. Let c be the class of $(a_1 a, \ldots, a_n a)$ in $C \otimes E$; then $dc = 0$ and $a \in E$.

Conversely, if the class c of $(b_1, \ldots, b_n)$ in $C \otimes E$ is a torsion element, there are nonzero elements $b \in A$, $r \in E$, such that $b_i b = a_i r$ for all i. Then $Ja \subset E$, $a = r/b$, and we conclude that c is trivial.

(2.2) DEFINITION. A torsionfree A-module E *is semi-divisorial if the equivalent conditions of* (2.1) *hold for all ideals* J *such that* $J^{-1} = A$.

(2.3) COROLLARY. The following modules are semi-divisorial:

(a) I = *divisorial ideal of* A.

(b) E = *flat module*.

(c) $G = F \otimes E$, *where* F *is a flat module and* E *is semi-divisorial*.

(d) H = *direct limit of semi-divisorial modules*.

As an application of these ideas, let A be a Krull domain [1] and let I be a flat fractional ideal.

(2.4) THEOREM. I *is finitely generated*.

Proof. We must show that I is invertible. Since I^{-1} is a divisorial ideal, $I.I^{-1}$ $(= I \otimes I^{-1})$ is semi-divisorial by (2.3c).

Note that $(I.I^{-1})^{-1} = A$: If $x \in (I.I^{-1})^{-1}$, then $xI.I^{-1} \subset A$ means that $xI^{-1} \subset I^{-1}$. Since a Krull domain is completely integrally closed, $x \in A$. But in a Krull domain any ideal L, viz. $L = I.I^{-1}$, such that $L^{-1} = A$ contains a finitely generated subideal J with $J^{-1} = A$. But $J.1 \subset L = I.I^{-1}$ leads by the semi-divisoriality of L to $1 \in L$ and I is invertible as desired.

(2.5) COROLLARY. *Let A be a Krull domain and let E be a flat A-lattice. Then E is finitely generated.*

Proof. Just notice that if E is a lattice of rank r, then $\wedge^r E$, the rth exterior power of E, is a flat fractional ideal of A. The conclusion follows as in [5].

Again let A be a Krull domain and let I be a submodule of K.

(2.6) PROPOSITION. *I is semi-divisorial if and only if it is a directed union of divisorial ideals.*

Proof. Let M,N be subideals of I. Then

$$(M,N).(M,N)^{-1}.((M,N)^{-1})^{-1} \subset I$$

Since $((M,N).(M,N)^{-1})^{-1} = A$, as above we get a finitely generated ideal $J \subset (M,N).(M,N)^{-1}$, $J^{-1} = A$. We conclude then $((M,N)^{-1})^{-1} \subset I$.

REMARK. For a Krull domain it is easily verified that a submodule L of K is semi-divisorial if and only if $L = \bigcap_P L_P$, where P runs over the height one primes of A. Thus two semi-divisorial ideals in a Krull domain are equal if they agree at the height one localizations. It is then clear that if B is a semi-divisorial overring of a Krull domain A, then B is a Krull domain. In particular, an ideal transform of a Krull domain is semi-divisorial and hence a Krull domain.

Observe also that (2.6) displays a semi-divisorial extension as a ring of quotients with respect to a topologizing set of ideals [1].

3. H-DOMAINS

The usefulness of the ideal $I.I^{-1}$ in the proof of (2.4) suggests that we look more closely at the ideals J such that $J^{-1} = A$. For Krull domains or Noetherian rings these are quite large and contain finitely generated ideals with the same property. We shall refer to such rings as H-domains. For a fuller discussion of these rings let us bring in the set $P(A)$ of primes associated to principal ideals-that is, the primes minimal over an ideal $(a):b$.

(3.1) PROPOSITION. *Let I be a finitely generated ideal of A. Then $I^{-1} = A$ if and only if I is not contained in any $P \in P(A)$.*

Proof. Immediate [7].

We consider now several statements on H-domains.

(3.2a) A is an H-domain if we restrict I in the definition to prime ideals.

Proof. Let $C = \{I \mid I^{-1} = A$ and fails to contain a finitely generated ideal J, $J^{-1} = A\}$. C is clearly inductive. Let I be a maximal element in C. We claim that I is prime. Suppose $x,y \notin I$ and $xy \in I$. Let $J = (I,x)$ and $L = (I,y)$. Then $J^{-1} = L^{-1} = A$; let $J_0 = (x,a_1,\ldots,a_m)$ and $L_0 = (y,b_1,\ldots,b_n)$ with $a_i,b_j \in I$ and $J_0^{-1} = L_0^{-1} = A$. But then $J_0.L_0 \subset I$ is a finitely generated ideal that contradicts the choice of I.

(3.2b) A is an H-domain if and only if $P(A)$ is compact as a subset of Spec(A) and $P^{-1} \neq A$ for each $P \in P(A)$.

(3.2c) $A[T]$ is an H-domain along with A.

Proof. Let P be a prime ideal of $A[T]$ such that $P^{-1} = A[T]$. Let I be the ideal of A generated by the coefficients of the polynomials in P. If $I^{-1} \neq A$ we would have $I^{-1}P \subset A[T]$, contradicting $P^{-1} = A[T]$. Thus $I^{-1} = A$ and we can find a polynomial f in P such that the ideal $J = c(f)$ = ideal of A generated by the coefficents of f is such that $J^{-1} = A$. If $0 \neq a \in P \cap A$, then a,f form a regular sequence of two elements in P and the ideal (a,f) works. If $P \cap A = 0$ then $P = Q \cap A[T]$, where Q is a prime ideal of $K[T]$. In $K[T]$ let $Q_1,\ldots,Q_n$ be the other prime ideals containing the element f. Let $f = up^e.p_1^{e_1}\ldots p_n^{e_n}$ be a primary decomposition of f, u = unit of $K[T]$. Write $Q_i \cap A[T] = P_i$; it is clear that the P's are the only prime ideals of $A[T]$ of height one that contain f. Let now $g \in P_1^{e_1}\ldots P_n^{e_n} \setminus P$. We claim that $P^e \subset (f):g$. Indeed, if $F \in P^e$ there is a nonzero constant r such that $rFg \in (f)$. Since also $fFg \in (f)$ we get $gF \in (f)$, since (f) is semi-divisorial! It is also clear that $P^{e-1} \not\subset (f):g$. Thus $gf^{-1}P^{e-1} \subset P^{-1}$ and $gf^{-1}P^{e-1} \not\subset A[T]$.

What are the H-domains? Clearly Noetherian rings, and as remarked earlier, Krull domains. We shall consider other examples soon, but before we point out

(3.2d) A completely integrally closed H-domain is a Krull domain.

Proof. We use the following characterization of Krull domains: (K_1) For $P \in \mathcal{P}(A)$, A_P is a discrete valuation domain, and (K_2) each nonzero element of A is contained in only finitely many elements of $\mathcal{P}(A)$.

(1) To prove (K_1), let $P \in \mathcal{P}(A)$ and let I be a nonzero ideal $I \subset P$. If $I.I^{-1} \subset P$ we would have $(I.I^{-1})^{-1} \supset P^{-1} \neq A$. This contradicts the complete closure hypothesis. Thus I is invertible at P.

(2) First note that the directed union $I = \cup I_\alpha$ of divisorial ideals in A is a divisorial ideal: Indeed, if $x \in (I^{-1})^{-1}$, $xI^{-1} \subset A$ and thus $x(I.I^{-1}) \subset I$. Since $(I.I^{-1})^{-1} = A$, by the H-condition we have a finitely generated ideal $J \subset I.I^{-1}$, $J^{-1} = A$. Thus $xJ \subset I_\alpha$ for some α. Taking inverses twice we get $x \in I_\alpha \subset I$.

Before proving (K_2) we record the following useful observation:

(3.2e) If I is a semi-divisorial module and $\mathcal{P}(A)$ is compact, then $I = \bigcap_{P \in \mathcal{P}(A)} I_P$.

(3) Let us now show (K_2). Let a be an element of A and denote by $\{P_\alpha\}$ the subset of primes in $\mathcal{P}(A)$ that contain a. For each such P let $n = \text{ord}_P(a)$. Form the directed set of divisorial ideals

$$I_{\alpha_1,\ldots,\alpha_r} = a(P_{\alpha_1}^{n_1}\cdots P_{\alpha_r}^{n_r})^{-1}$$

and denote its union by I. Let $x \in I^{-1}$; for each P as above we have $xa(P^n)^{-1} \subset A$ and consequently $x \in A_P$. Thus, since $A = \cap A_P$, $P \in \mathcal{P}(A)$, we get $x \in A$. Since I is a divisorial ideal we have $I = A$. The desired conclusion now follows easily.

<u>EXAMPLE</u>. A locally Noetherian domain A with Noetherian prime spectrum Spec(A) is an H-domain.

<u>Proof</u>. According to (3.2b) it suffices to show that if $P \in \mathcal{P}(A)$ then $P^{-1} \neq A$. Suppose $P \in \mathcal{P}(A)$, $P^{-1} = A$ and is maximal with respect to these properties. Then P

is maximal in $\mathcal{P}(A)$. Let I be a finitely generated ideal with radical P; let J be a finitely generated ideal such that $J_P = P_P$. Put $L = I + J$. Since $L^{-1} \neq A$, by the maximality of P we can write

$$L \subset (a):b \subset P$$

We claim $(a):b = P$, that will contradict $P^{-1} = A$ since $b/a \in ((a):b)^{-1}$. If $x \in P$, there is $t \notin P$ so that $tx \in (a):b$. Thus $((a):b,t)x \subset (a):b$. But $((a):b,t)$ is not contained in any element of $\mathcal{P}(A)$ and thus get from the compactness of $\mathcal{P}(A)$ a finitely generated ideal N, $N \subset ((a):b,t)$, avoiding all elements of $\mathcal{P}(A)$. Thus $N^{-1} = A$ and since $(a):b$ is divisorial, $x \in (a):b$.

We now begin a discussion of finiteness of flat ideals in H-domains. To help in focusing we consider special cases of the following question.

CONJECTURE. A faithfully flat ideal in an H-domain is finitely generated.

Under this light a closer analysis of (2.4) is

(3.3) PROPOSITION. Let A be a domain with the property that each nonzero element is contained in only finitely many elements of $\mathcal{P}(A)$. Let I be a flat ideal that is finitely generated at each $P \in \mathcal{P}(A)$. Then I is finitely generated.

Proof. It is clear that $\mathcal{P}(A)$ is compact. It suffices then to show that $I.I^{-1}$ is not contained in any element of $\mathcal{P}(A)$. Let $P_1,\dots,P_m$ be the primes of $\mathcal{P}(A)$ that contain I, let a be an element of I generating I at these primes, and let $P_1,\dots,P_m,Q_1,\dots,Q_n$ be the elements of $\mathcal{P}(A)$ containing (a). Let $b \in I \setminus \cup Q_i$. Then

$I.(a,b)^{-1} \subset A$, since it is clear that the inclusion holds at each localization A_P, $P \in P(A)$. Thus $(a,b)^{-1} = I^{-1}$ and $I.I^{-1}$ is not contained in any such prime.

(3.4) PROPOSITION. *Let I be a faithfully flat prime ideal. Then I is finitely generated.*

Proof. Let $J = I.I^{-1}$; we must show that J is not contained in any $P \in P(A)$. Assume $I.I^{-1} \subset I$; then, since I is locally finitely generated (actually only need that it be finitely generated at the primes of $P(A)$), we conclude $I^{-1} = A$. But then $I = A$, against the hypothesis. Suppose J is contained in a larger prime Q of $P(A)$. Localize at Q; then Q_Q will contain a regular sequence $\{a,b\}$, where a is the generator of the prime ideal I_Q and b is an element of $Q \setminus I$. Since Q_Q lies in $P(A_Q)$, again we get a contradiction.

REMARK. To prove the conjecture sufficed to show that faithfully flat ideals of H-domains are divisorial: If $I.I^{-1} \subset P \in P(A)$ then $(I.I^{-1})^{-1} \supset P^{-1} \supsetneq A$. Thus for $x \in P^{-1} \setminus A$ we have $xI.I^{-1} \subset A$ and $xI^{-1} \subset I^{-1}$ or $x(I^{-1})^{-1} \subset (I^{-1})^{-1}$. Since $I = (I^{-1})^{-1}$, and I is faithfully flat, $x \in A$, a contradiction.

4. CONTENT

Let A be a ring and denote by B the polynomial ring in the set $T = \{T_1,\ldots,T_n\}$ of independent indeterminates over A. We write $B = A[T]$. Given an ideal I of B considerable information is contained in the ideal c(I) generated by the coefficients of the polynomials in I; c(I) will be called the A-*content* of I. Flat ideals of B are particularly amenable to this kind of examination. The analysis of the content ideal is also useful in several questions occurring in the theory of stably coherent domains (i.e. A[T] is coherent). For instance, if $J = (a_1,\ldots,a_n)$ is a finitely generated ideal of A, form the ring homomorphism

$$A[T_1,\ldots,T_n] \rightarrow A[a_1t,\ldots,a_nt] \hookrightarrow A[t]$$

that sends $T_i \longrightarrow a_it$. $A[a_1t,\ldots,a_nt]$ is the so-called Rees's ring of J and its finiteness properties are quite useful about obtaining information on integral closure properties of J. A key to this is the nature of the kernel of the homomorphism, especially whether it is finitely generated when A[T] is coherent. More generally, there is the question of extending to coherent schemes certain results on the cohomology groups of projective morphisms [2,Th 2.2.1]. Although we shall not elaborate upon here, there are general features shared by these kernels and flat B-ideals.

We begin by disposing of the simplest case of flat ideals.

<u>(4.1) PROPOSITION.</u> <u>Let</u> I <u>be a projective ideal of</u> B. <u>Then</u> I <u>is finitely generated if and only if</u> c(I) <u>is finitely generated</u>.

<u>Proof</u>. Here A does not have to be a domain. Assume I projective and let L be the trace ideal of I, that is, $L = \Sigma f(I)$, where f runs over $\mathrm{Hom}_B(I,B)$. Since L is a pure ideal of B [6], we have $L = L_0[T]$, where $L_0 = L \cap A$ is a pure ideal of A. Since $I = L.I = L_0.I$, we have $J = L_0.J$, where J is the content ideal of I. By the so-called determinantal trick, there is $e \in L_0$ so that $(1-e)J = 0$. But $(1-e)J = 0$ implies $(1-e)I = 0$ and thus

$(1-e)L = 0 = (1-e)L_0$ and $L_0 = (e)$. Thus L is finitely generated, and by [6], I is also such.

The similar statement for flat ideals of B is not always true as pointed out in [3]. It is known [5] that for the case of one indeterminate flat ideals of finite content are locally finitely generated and, if A is a domain, they are finitely generated. Since the hypothesis on the A-content of a flat B-ideal is not amenable to induction on the number of indeterminates we give an argument to extend [5] to any number of indeterminates.

(4.2) THEOREM. *Let I be a flat ideal of B. If c(I) is finitely generated, then I is locally finitely generated. If moreover A is a domain, then I is finitely generated.*

Proof. According to the appropriate version of Nakayama's lemma, it suffices to show that for a maximal ideal P of B with $I_P \neq 0$ $I \neq P.I$ holds. We may assume that A is a local ring of maximal ideal $\underline{m}$. Construct a finitely generated (free) extension C of A such that there is a prime ideal Q of C[T] lying above P, with

$$Q = (\underline{m}', T_1-c_1,\ldots,T_n-c_n)$$

where $\underline{m}'$ is a maximal ideal of C lying above $\underline{m}$. Push I over to C[T] and consider the equation CI = Q.I. Change the notation back to B and consider an A-automorphism of B defined by $T_i \rightarrow T_i+c_i$. We may assume that we have $I = P.I$, $P = (\underline{m},T_1,\ldots,T_n)$.

Let now $J = c(I)$ and let h be a polynomial in I such that $c(h) = J$. Reduce the coefficients of all polynomials modulo $\underline{m}J$. If degree(h) = s and $I = P.I = P^2.I = \ldots = (\underline{m} + (T_1,\ldots,T_n))^r.I$ we would have-where '*' denotes reduction modulo $\underline{m}J$- $h^* \in (T_1,\ldots,T_n)^r.I^*$ and would get a contradiction when $r > s$.

The remainder of the proof proceeds as in [5].

The following is an application to chain conditions on A. It answers, in the case of a domain, a question of C. Faith.

(4.3) COROLLARY. *Let A be a domain with the property that projective ideals satisfy the ascending chain condition. Then A[T] inherits this property.*

Proof. Here T may stand for a single indeterminate. Let $I_1 \subset I_2 \subset I_3 \subset \ldots$ be an ascending chain of invertible ideals of A[T]. We may assume that the I's are not contained in (T). Then the ideal $I_i^{(o)}$ generated by the constant terms of the polynomials in I_i is a projective ideal of A, being isomorphic to I_i/TI_i. Since the chain $I_1^{(o)} \subset I_2^{(o)} \subset I_3^{(o)} \subset \ldots$ is stationary $\cup I_i^{(o)}$ is a finitely generated ideal. But if $f_1,\ldots,f_m$ are polynomials in $I = \cup I_i$ such that their constant terms generate $\cup I_i^{(o)}$ it is clear that the contents of these polynomials generate the content of the flat ideal I. By the preceding I is finitely generated and the statement follows.

(4.4) PROPOSITION. *Let I be a flat ideal of A[T] that contains an element f such that $c(f)^{-1} = A$. Then I is finitely generated.*

Proof. It suffices to show that the ideal $I.I^{-1}$ contains a nonzero constant r: then {r,f} is a regular sequence and (2.1) applies.

Let $I_K = IK = Q_1^{(e_1)} \cap \ldots \cap Q_n^{(e_n)}$ be the primary representation of IK in K[T] and put $P_i^{(e_i)} = A[T] \cap Q_i^{(e_i)}$. It follows from the presence of f in I that the primary representation of I is given by $I = P_1^{(e_1)} \cap \ldots \cap P_n^{(e_n)}$.

Put $P = P_1$, $e = e_1$, and pick $g \in P^{(e)} \setminus \bigcup_{i=2}^{n} P_i \cup P^{(e+1)}$. Then $G = g + T^r f^2$, $r > \deg(g)$, is an element such as g with $c(G)^{-1} = A$ besides. As for I, we conclude that (G) has a primary decomposition $(G) = P^{(e)} \cap M_1^{(d_1)} \cap \ldots \cap M_s^{(d_s)}$, where none of the P_i's is among the M_j's. Now pick $F \in \cap M_j^{(d_j)} \setminus P$; then $FG^{-1}I \subset A[T]$, since given any $h \in I$, G and a nonzero constant c_h send $FG^{-1}h$ into A[T]--a divisorial ideal. Thus $FG^{-1} \in I^{-1}$ and it is clear that $I.I^{-1} \not\subset P$.

Consider $(I^{-1})_K = (I^{-1})K$; since $I^{-1} \supset A[T]$, $(I^{-1})_K = (t_1^{c_1} \ldots t_m^{c_m})$, where the t_i's are prime elements of K[T] and $c_i \leq 0$. The argument above shows that if t_1 corresponds to P, then $e + c_1 = 0$. It follows now that $(I.I^{-1})K = K[T]$, and $I.I^{-1}$ contains the nonzero constant as desired.

This result will be useful in the study of the 'new' flat ideals of A[T]. By this we mean a flat ideal such that A[T]/I is A-torsionfree. These are the ideals less likely to be isomorphic to extended ideals from A. Let I be one such finitely generated ideal. Then I is invertible and we have an equation $I.L = (f)$, where f is a nonzero element of A[T]. Let J be c(I); then, since $J.J^{-1}.I \subset I$ and $J^{-1}.I \subset A[T]$, we get $J^{-1}.I = I$. In the equation above we then have $(f) = J^{-1}(f)$ and thus $J^{-1} = A$. We use this remark in the sequel.

<u>(4.5) PROPOSITION.</u> <u>Let</u> I <u>be</u> <u>a</u> <u>faithfully</u> <u>flat</u> <u>ideal</u> <u>of</u> A[T]. <u>If</u> A <u>is</u> <u>a</u> <u>local</u> <u>ring</u> <u>then</u> I <u>is</u> <u>finitely</u> <u>generated</u>.

<u>Proof</u>. We may assume that I is not contained in (T). Note that I/TI is faithfully flat over A. This implies that the ideal generated by the constant terms of poly-

nomials in I is principal, generated by the constant term of, say, f. But then it is clear that $c(f) = c(I)$ and (4.2) applies.

(4.6) THEOREM. *Let I be a faithfully flat ideal of A[T]. Assume that A[T]/I is A-torsionfree. Then I is finitely generated in the following cases: i) A is an H-domain; ii) A is a coherent ring.*

Proof. When (i) applies, the remark above says that c(I) is not contained in any $P \in \mathcal{P}(A)$. There is then an element f of content c(f) such that $c(f)^{-1} = A$.

In case of (ii), for each prime ideal $\underline{m}$ of A there is a finite set of elements $a_1,\ldots,a_n$ in c(I) such that $((a_1,\ldots,a_n)^{-1})_{\underline{m}} = A_{\underline{m}}$. There is then a neighborhood of $\underline{m}$ in Spec(A) where this equation holds. From the compactness of Spec(A) we get then a finite set of elements in c(I) so that the ideal J they generate is such that $J^{-1}=A$.

In both cases one finishes the proof with (4.4).

(4.7) COROLLARY. *Let A be an integrally closed coherent domain and let I be a flat ideal of A[T] such that A[T]/I is torsionfree. Then I is finitely generated.*

Proof. Let g be a nonzero element of I of least degree. Note $c(g)^{-1}g \subset I$. But $(c(g).c(g)^{-1})^{-1} = A$ and thus the finitely generated ideal $c(c(g)^{-1}g)$ is not contained in any $P \in \mathcal{P}(A)$. By (4.4) I is finitely generated.

5. COMPLETENESS

Let A be an integrally closed domain and let I be an ideal of A. The completion of I is the ideal, $I' = \bigcap IV_\alpha$, where the intersection is taken over all valuation rings between A and K [8].

CONJECTURE. Every flat ideal of an integrally closed domain is complete.

An affirmative answer would provide a valuative test of divisibility. Unfortunately, other than the few cases of [5] not much seems known. The following result allows concentrating on a rather large rings--e.g., the integral closure B of A in the algebraic closure of K.

(5.1) PROPOSITION. *Let* A *be an integrally closed domain, let* I *be a flat ideal of* A, *and let* B *be a torsionfree integral extension of* A. *Then* $I = IB \cap A$.

Proof. Consider the sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0$$

Note that B/A is a torsionfree A-module. Tensor this sequence by A/I to get the exact sequence

$$\mathrm{Tor}_1^A(A/I,B/A) \longrightarrow A/I \longrightarrow B/IB$$

Since B/A embeds in a vector space V over K, we get $\mathrm{Tor}_1^A(A/I,B/A) = \mathrm{Tor}_2^A(A/I,C) = 0$, where C is the cokernel of the embedding.

For a simple application let I be an idempotent flat ideal of the integrally closed domain A. It would be a consequence of the completeness that such ideals are radical ideals, since idempotent ideals in valuation domains are prime. This fact can however be resolved when A is of characteristic 2. It was shown in [5] that I^2 is generated by the squares of the elements in I (the similar fact for higher powers is still unasnwered). Let then x be an element of A with $x^2 \in I = I^2$. We then have an equation $x^2 = \Sigma r_i a_i^2$, $r_i \in A$, $a_i \in I$. Pass now to B (=integral closure of A in the algebraic closure of K) where we may write $r_i = s_i^2$. Using that A has characteristic 2, we conclude $x \in IB$. Now use (5.1).

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S. Glaz and W. V. Vasconcelos

Department of Mathematics

Rutgers University

New Brunswick, New Jersey 08903 USA

(Received April 10, 1977)