Instructions:

- All answers must be written clearly.
- You must show all work to receive credit.
- You may use a calculator, but you must show all your work in order to receive credit.
- Be sure to erase or cross out any work that you do not want graded.
- If you need extra space, you may use the back sides of the exam pages (if you do, please write me a note so that I know where to look).

<table>
<thead>
<tr>
<th>Question:</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>Total</th>
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</table>
1. Solve the following initial value problems.

(a) \( \frac{dy}{dt} - \frac{2y}{1+t^2} = 3, \quad y(0) = 1 \).

Integrating Factor

Multiply both sides by \( M(t) \)

\[ \frac{dy}{y} \left( \frac{1}{1+t^2} \right) = \frac{3}{1+t^2} \]

\[ \int \frac{1}{1+t^2} \, dt = \int 3 \, \frac{1}{1+t^2} \, dt + C \]

\[ \tan^{-1} t + C = 3 \tan^{-1} t + C \]

\[ y(t) = (1+t^3)\left(3\tan^{-1} t + C\right) \]

(b) \( \frac{dy}{dt} = t + \frac{2y}{1+t^2} \)

Integrating Factor

Write \( \frac{dy}{y} \left( \frac{1}{1+t^2} \right) = \frac{t}{t^2} \)

\[ \int \frac{2}{1+t^2} \, dt = \int \frac{1}{1+t^2} \, dt + C \]

\[ 2\ln(1+t) = \ln(1+t^2) + C \]

\[ y(t) = (1+t^2)^2 \left[ \ln(1+t) + C \right] \]

So 

\[ t = u - 1 \]

\[ u = 1 + t \]

\[ \int u - \frac{1}{u^2} \, du + C = \int (1+t^2) \left[ \ln(1+t) + C \right] \]
2. Consider the following initial value problem,

\[ \frac{dy}{dt} = y^2 - 2t + 1, \quad y(0) = 1. \]

By hand, use Euler's method with \( \Delta t = 1 \) (this is a horrible \( \Delta t \)) to approximate the value of \( y_2 \). Use the table to record the necessary values of \( k, t, y, \) and \( \frac{dy}{dt} = f(t, y) \) at each step. Show all work if you want to receive credit.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( t_k )</th>
<th>( y_k )</th>
<th>( f(t_k, y_k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
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<td>1</td>
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</tr>
<tr>
<td>2</td>
<td>2</td>
<td>11</td>
<td></td>
</tr>
</tbody>
</table>

Use the steps below to approximate the value of \( y_2 \).

\[ y_{n+1} = y_n + f(t_n, y_n) \Delta t \]

\[ f(t_0, y_0) = y_0^2 - 2t_0 + 1 \]
\[ = 1^2 - 2(0) + 1 \]
\[ = 2 \]

\[ f(t_1, y_1) = y_1^2 - 2t_1 + 1 \]
\[ = 3^2 - 2(1) + 1 \]
\[ = 9 - 2 + 1 \]
\[ = 8 \]

\[ y_2 = 11 \]

\[ y(2) \approx 11 \]
3. Consider the following differential equation \( \frac{dy}{dt} = \tan^{-1}y \)

(a) Explain why we know that a solution to any initial-value problem using this equation exists.

\[ f(t,y) = \tan^{-1} y \] is a continuous function everywhere. Thus by the Existence Theorem, there is a solution for every IVP.

(b) Explain why we know that a solution to any initial-value problem using this equation must be unique.

\[ \frac{df}{dy} = \frac{1}{1 + y^2} \] is a continuous function on any rectangle since the denominator can never be zero. Thus by the Uniqueness Theorem the solution is unique.

(c) Find the equilibrium solutions for this differential equation.

\[ \tan^{-1} y = 0 \] if and only if \( y = 0 \). Thus \( y = 0 \) is the only equilibrium solution.

(d) Draw a Phase Line for this differential equation. Identify the equilibrium points as sinks, sources or nodes.

(e) Sketch a general graph of possible solutions to this differential equation. Include the equilibrium solutions in your graph.
4. A 1000-gallon tank contains 300 gallons of water and 50 pounds of salt. A solution with a concentration of 15 pounds of salt per gallon is added to the tank at a rate of 5 gallons per minute. Water is also allowed to leave the tank through a separate spout at a rate of 5 gallons per minute. We may assume the water in the tank is always well-mixed. How much salt is in the tank after 8 minutes?

\[
\frac{dy}{dt} = \text{Rate in} - \text{Rate out} \\
= \frac{15 \text{ lb}}{\text{gal}} \times 5 \text{ gal/min} - \frac{y}{300} \text{ lb/5 gal} \times 5 \text{ gal/min} \\
\text{Rearrange} \quad 10 \quad y' + \\
\begin{align*}
\frac{dy}{dt} &= \frac{y(500) - y}{60} \\
\int \frac{dy}{4500 - y} &= \int \frac{1}{60} dt \\
-\ln |4500 - y| &= \frac{t}{60} + c_1 \\
|4500 - y| &= e^{-t/60 + c_1} \\
4500 - y &= c_2 e^{-t/60} \\
4500 - y &= c_3 e^{-t/60} \\
4500 - y &= c_4 e^{-t/60} \\
y &= 4500 + c_5 e^{-t/60} \\
y(8) &\approx 605.48 \text{ lb}
\end{align*}
\]

\[y(0) = 50\]

\[\therefore y \approx 4450 \text{ lb}\]
5. For the following one-parameter family, locate the bifurcation values and then draw the phase lines for values of the parameter slightly smaller than, slightly larger than, and at the bifurcation values.

\[
\frac{dy}{dt} = y^2 - ay + 1
\]

Use the quadratic formula to solve for the equilibrium solutions:

\[
y = \frac{a \pm \sqrt{a^2 - 4y}}{2}
\]

The number of solutions occur at \( a = \pm 2 \) so these are the bifurcation values.

<table>
<thead>
<tr>
<th>( a )</th>
<th># of solutions</th>
</tr>
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<tr>
<td>1.2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>1.72</td>
<td>2</td>
</tr>
</tbody>
</table>

Graph of \( y^2 - ay + 1 \)
6. Consider the linear system given by

\[
\frac{d\mathbf{y}}{dt} = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{y} + \begin{pmatrix} -1 \\ -1 \end{pmatrix}
\]

(a) Find the general solution.

\[
\mathbf{v}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}
\]

\[
\mathbf{v}_1 = \begin{pmatrix} A - \lambda I \end{pmatrix} \mathbf{v}_0
\]

\[
= \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}
\]

\[
= \begin{pmatrix} x_0 + y_0 \\ -x_0 + y_0 \end{pmatrix}
\]

The general solution is

\[
\mathbf{y}(t) = e^{\lambda t} \mathbf{v}_0 + t e^{\lambda t} \mathbf{v}_1
\]

\[
= e^{-2t} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t e^{-2t} \begin{pmatrix} x_0 + y_0 \\ -x_0 + y_0 \end{pmatrix}
\]

(b) Find the particular solution for when \(\mathbf{y} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}\).

\[
U \sin y, \quad x_0 = -3, \quad y_0 = 1
\]

\[
\mathbf{y}(t) = e^{-2t} \begin{pmatrix} -3 \\ 1 \end{pmatrix} + t e^{-2t} \begin{pmatrix} 4 \\ y \end{pmatrix}
\]

(c) What type of equilibrium point does this system have?

A. spiral sink  B. spiral source  C. center  D. almost spiral  E. none of these
7. Find the general solution to the following differential equation

\[
\frac{d^2 y}{dt^2} + 3y = 2t + \cos 4t.
\]

You may use the Method of Undetermined Coefficients or Complexification.

Find \( Y_h \):

\[
\begin{align*}
\frac{d^3 y}{dt^3} + 3y &= 0 \\
\frac{d^2 y}{dt^2} + 3y &= 0 \\
\frac{d}{dt} (3y) &= 0 \\
s^2 + 3 &= 0 \\
s &= \pm \sqrt{3}i.
\end{align*}
\]

Thus, the general solution is

\[
Y_h(t) = K_1 e^{\sqrt{3}t} \cos \sqrt{3}t + K_2 e^{\sqrt{3}t} \sin \sqrt{3}t.
\]

Find \( Y_p \):

Since \( \frac{dY}{dt} \) is not in the LHS, we do not expect to have a \( \sin \) term. Thus our guess is

\[
Y_p(t) = A \cos 4t + Bt + C
\]

\[
\begin{align*}
LHS &= \frac{d^3 y}{dt^3} + 3y_p \\
&= \frac{d^3}{dt^3} \left( A \cos 4t + Bt + C \right) + 3 \left( A \cos 4t + Bt + C \right) \\
&= (-16A \cos 4t) + 3A \cos 4t + 3B + 3C \\
&= -13A \cos 4t + 3B + 3C.
\end{align*}
\]

\[
\begin{align*}
\frac{d^2}{dt^2} \left( A \cos 4t + Bt + C \right) &= -16A \sin 4t + B \\
&= 2t + \cos 4t
\end{align*}
\]

So the General Solution is

\[
Y(t) = K_1 e^{\sqrt{3}t} \cos \sqrt{3}t + K_2 e^{\sqrt{3}t} \sin \sqrt{3}t + \frac{2}{3}t - \frac{1}{15} \cos 4t.
\]

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8. Find the general solution to the following differential equation

\[
\frac{d^2 y}{dt^2} + 9y = 2 \cos 3t.
\]

You may use the Method of Undetermined Coefficients or Complexification.

Method of Undetermined Coefficients

Find \( Y_h \):

Since \( Y_h \) contains a \( \cos 3t \) term than there is a repeat, thus we guess:

\[
Y_p(t) = a \cos 3t + b t \sin 3t
\]

\[
Y_p'(t) = -3a \sin 3t + 3b t \cos 3t
\]

\[
Y_p''(t) = -9a \cos 3t + 9b t \sin 3t
\]

LHS:

\[
J^3 Y_p + 9Y_p = -6a \sin 3t + 6b \cos 3t
\]

\[
\alpha = 0 \quad \Rightarrow \quad \text{the general solution is}
\]

\[
y(t) = K_1 \cos 3t + K_2 \sin 3t + \frac{1}{3} t \sin 3t.
\]
9. (a) Compute $\mathcal{L}^{-1} \left[ \frac{e^{-2s}}{s+1} \right] = \frac{e^{-2t}}{s+1} = u_a(t) e^{-(t-a)}$

\[ a = 2 \]
\[ f(t) = e^t \]
\[ f(t-a) = e^{-(t-2)} \]

(b) Compute $\mathcal{L} \left[ u_a(t)e^{3(t-2)} \right] = \frac{e^{-2s}}{s-3} \]

\[ f(t) = e^{3t} \]
\[ f(t-a) = e^{3(t-2)} \]
\[ F(s) = \frac{1}{s-3} \]

(c) Compute $\mathcal{L}^{-1} \left[ \frac{4e^{-2s}}{s(s+3)} \right] = \mathcal{L}^{-1} \left[ \frac{4}{s} - \frac{3}{s+3} \right]$

\[ s \to \mathcal{L}^{-1} \left[ \frac{1}{s} - \frac{3}{s+3} \right] = \frac{1}{3} u_a(t) - \frac{1}{3} e^{-3(t-2)} \]

(d) Compute $\mathcal{L}^{-1} \left[ \frac{s+1}{s^2 + 6s + 10} \right] = \mathcal{L}^{-1} \left[ \frac{1}{s+3} + \frac{1}{(s+3)^2} \right]$

\[ s \to \mathcal{L}^{-1} \left[ \frac{1}{s+3} + \frac{1}{(s+3)^2} \right] = -3e^{-3t} - 2e^{-3t} \sin t \]

(e) Compute $\mathcal{L}^{-1} \left[ \frac{2s+3}{s^2 - 2s + 4} \right] = \mathcal{L}^{-1} \left[ \frac{2(s-1)}{(s-1)^2 + 3} \right] + \frac{5}{\sqrt{3}} \mathcal{L}^{-1} \left[ \frac{\sqrt{3}}{(s-1)^2 + (\sqrt{3})^2} \right]$

\[ \text{Next a} \]
\[ 2(s-1) \]

\[ 2e^t (\sqrt{3})t + \frac{5}{\sqrt{3}} \sin \sqrt{3} t \]
10. Solve the initial value problem using Laplace transforms.

\[
\frac{d^2 y}{dt^2} + 9y = e^{4t}, \quad y(0) = 1, \quad y'(0) = 0
\]

**Step 1: Take Laplace Transform of both sides.**

\[
\mathcal{L}\{y''(t) + 9y(t)\} = \mathcal{L}\{e^{4t}\}
\]

\[
(s^2 \mathcal{L}\{y\}) - sy(0) - y'(0) + 9(s \mathcal{L}\{y\}) = \frac{1}{s-4}
\]

\[
s^2 \mathcal{L}\{y\} - s - 0 + 9s \mathcal{L}\{y\} = \frac{1}{s-4}
\]

\[
s \mathcal{L}\{y\} = \frac{1}{s-4} + \frac{s}{s^2 + 9}
\]

**Step 2: Solve for \(\mathcal{L}\{y\}\).**

\[
\mathcal{L}\{y\} = \frac{1}{s-4} + \frac{s}{s^2 + 9}
\]

**Step 3: Partial Fractions.**

\[
\frac{1}{(s-4)(s^2 + 9)} = \frac{A}{s-4} + \frac{Bs + C}{s^2 + 9}
\]

Set:

\[
A + B = 0
\]

\[
+4B + C = 0
\]

\[
9A - 4C = 1
\]

Solve:

\[
A = \frac{1}{25}, \quad B = -\frac{1}{25}, \quad C = \frac{1}{25}
\]

\[
\mathcal{L}\{y\} = \frac{1}{s-4} + \frac{-\frac{1}{25}}{s^2 + 9} + \frac{\frac{1}{25}}{s^2 + 9}
\]

\[
= \frac{1}{s-4} + \frac{3}{5(s^2 + 9)}
\]

**Step 4: Take inverse Laplace transform.**

\[
y = \mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{5(s^2 + 3^2)}\right\}
\]

\[
y = \frac{1}{5} e^t + \frac{2}{25} \left(\cos 3t - \frac{4}{25} \sin 3t\right)
\]
Laplace Transforms Table:

<table>
<thead>
<tr>
<th>( y(t) )</th>
<th>( Y(s) )</th>
<th>( \mathcal{L}^{-1} { Y(s) } )</th>
<th>( \mathcal{L} { y(t) } )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y(t) = e^{at} )</td>
<td>( Y(s) = \frac{1}{s-a} ) ((s &gt; a))</td>
<td>( y(t) = e^{at} )</td>
<td>( y(t) = \frac{n!}{s^n} ) ((s &gt; 0))</td>
</tr>
<tr>
<td>( y(t) = \sin \omega t )</td>
<td>( Y(s) = \frac{\omega}{s^2 + \omega^2} )</td>
<td>( y(t) = \sin \omega t )</td>
<td>( y(t) = \frac{s}{s^2 + \omega^2} )</td>
</tr>
<tr>
<td>( y(t) = e^{at} \sin \omega t )</td>
<td>( Y(s) = \frac{a}{(s-a)^2 + \omega^2} )</td>
<td>( y(t) = e^{at} \sin \omega t )</td>
<td>( y(t) = \frac{s-a}{(s-a)^2 + \omega^2} )</td>
</tr>
<tr>
<td>( y(t) = t \sin \omega t )</td>
<td>( Y(s) = \frac{2\omega s}{(s^2 + \omega^2)^2} )</td>
<td>( y(t) = t \cos \omega t )</td>
<td>( y(t) = \frac{s^2}{(s^2 + \omega^2)^2} )</td>
</tr>
<tr>
<td>( y(t) = u_a(t) )</td>
<td>( Y(s) = \frac{a}{s} ) ((s &gt; 0))</td>
<td>( y(t) = \delta_a(t) )</td>
<td>( y(t) = e^{as} )</td>
</tr>
</tbody>
</table>