Instructions:

- All answers must be written clearly.
- You must show all work to receive credit.
- You may use a calculator, but you must show all your work in order to receive credit.
- Be sure to erase or cross out any work that you do not want graded.
- If you need extra space, you may use the back sides of the exam pages (if you do, please write me a note so that I know where to look).
1. Solve the following initial value problems.

(a) \( \frac{dy}{dt} - \frac{2t}{1+t^2} y = 3, \ y(0) = 1. \)

(b) \( \frac{dy}{dt} = t + \frac{2y}{1+t} \)
2. Consider the following initial value problem,

\[
\frac{dy}{dt} = y^2 - 2t + 1, \quad y(0) = 1.
\]

By hand, use Euler’s method with \( \Delta t = 1 \) (this is a horrible \( \Delta t \)) to approximate the value of \( y_2 \). Use the table to record the necessary values of \( k, t, y, \) and \( \frac{dy}{dt} = f(t, y) \) at each step. Show all work if you want to receive credit.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( t_k )</th>
<th>( y_k )</th>
<th>( f(t_k, y_k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
3. Consider the following differential equation \( \frac{dy}{dt} = \tan^{-1} y \)

(a) Explain why we know that a solution to any initial-value problem using this equation exists.

(b) Explain why we know that a solution to any initial-value problem using this equation must be unique.

(c) Find the equilibrium solutions for this differential equation.

(d) Draw a Phase Line for this differential equation. Identify the equilibrium points as sinks, sources or nodes.

(e) Sketch a general graph of possible solutions to this differential equation. Include the equilibrium solutions in your graph.
4. A 1000-gallon tank contains 300 gallons of water and 50 pounds of salt. A solution with a concentration of 15 pound of salt per gallon is added to the tank at a rate of 5 gallons per minute. Water is also allowed to leave the tank through a separate spout at a rate of 5 gallons per minute. We may assume the water in the tank is always well-mixed. How much salt is in the tank after 8 minutes?
5. For the following one-parameter family, locate the bifurcation values and then draw the phase lines for values of the parameter slightly smaller than, slightly larger than, and at the bifurcation values.

\[
\frac{dy}{dt} = y^2 - ay + 1
\]
6. Consider the linear system given by

\[
\frac{d\vec{Y}}{dt} = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix} \vec{Y}.
\]

(a) Find the general solution.

(b) Find the particular solution for when \(\vec{Y} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}\).

(c) What type of equilibrium point does this system have?

   A. spiral sink   B. spiral source   C. center   D. almost spiral   E. none of these
7. Find the general solution to the following differential equation

\[ \frac{d^2y}{dt^2} + 3y = 2t + \cos 4t. \]

You may use the Method of Undetermined Coefficients or Complexification.
8. Find the general solution to the following differential equation

\[ \frac{d^2y}{dt^2} + 9y = 2 \cos 3t. \]

You may use the Method of Undetermined Coefficients or Complexification.
9. (a) Compute \( \mathcal{L}^{-1} \left[ \frac{e^{-2s}}{s + 1} \right] \).

(b) Compute \( \mathcal{L} \left[ u_2(t)e^{3(t-2)} \right] \).

(c) Compute \( \mathcal{L}^{-1} \left[ \frac{4e^{-2s}}{s(s + 3)} \right] \).

(d) Compute \( \mathcal{L}^{-1} \left[ \frac{s + 1}{s^2 + 6s + 10} \right] \).

(e) Compute \( \mathcal{L}^{-1} \left[ \frac{2s + 3}{s^2 - 2s + 4} \right] \).
10. Solve the initial value problem using Laplace transforms.

\[ \frac{d^2 y}{dt^2} + 9y = e^{4t}, \quad y(0) = 1, \ y'(0) = 0 \]
### Laplace Transforms Table:

<table>
<thead>
<tr>
<th>$y(t)$</th>
<th>$Y(s) = \mathcal{L}[y]$</th>
<th>$Y(t) = \mathcal{L}^{-1}[Y]$</th>
<th>$Y(s) = \mathcal{L}[y]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^{at}$</td>
<td>$\frac{1}{s - a}$ ($s &gt; a$)</td>
<td>$t^n$</td>
<td>$\frac{n!}{s^{n+1}}$ ($s &gt; 0$)</td>
</tr>
<tr>
<td>$\sin \omega t$</td>
<td>$\frac{\omega}{s^2 + \omega^2}$</td>
<td>$\cos \omega t$</td>
<td>$\frac{s}{s^2 + \omega^2}$</td>
</tr>
<tr>
<td>$e^{at} \sin \omega t$</td>
<td>$\frac{\omega}{(s - a)^2 + \omega^2}$</td>
<td>$e^{at} \cos \omega t$</td>
<td>$\frac{s - a}{(s - a)^2 + \omega^2}$</td>
</tr>
<tr>
<td>$t \sin \omega t$</td>
<td>$\frac{2\omega}{s^2 + \omega^2}$</td>
<td>$t \cos \omega t$</td>
<td>$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$</td>
</tr>
<tr>
<td>$u_d(t)$</td>
<td>$\frac{e^{-as}}{s}$ ($s &gt; 0$)</td>
<td>$\delta_d(t)$</td>
<td>$e^{-as}$</td>
</tr>
</tbody>
</table>

### Table 6.2

 Rules for Laplace Transforms:

Given functions $y(t)$ and $w(t)$ with $\mathcal{L}[y] = Y(s)$ and $\mathcal{L}[w] = W(s)$ and constants $\alpha$ and $\alpha$.

<table>
<thead>
<tr>
<th>Rule for Laplace Transform</th>
<th>Rule for Inverse Laplace Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{L} \left[ \frac{dy}{dt} \right] = sY(s) - y(0)$</td>
<td>$Y^{-1}(s) = \frac{Y(s)}{s}$</td>
</tr>
<tr>
<td>$\mathcal{L}[y + w] = \mathcal{L}[y] + \mathcal{L}[w] = Y(s) + W(s)$</td>
<td>$\mathcal{L}^{-1}[Y + W] = \mathcal{L}^{-1}[Y] + \mathcal{L}^{-1}[W] = y(t) + w(t)$</td>
</tr>
<tr>
<td>$\mathcal{L}[\alpha y] = \alpha \mathcal{L}[y] = \alpha Y(s)$</td>
<td>$\mathcal{L}^{-1}[\alpha Y] = \alpha \mathcal{L}^{-1}[Y] = \alpha y(t)$</td>
</tr>
<tr>
<td>$\mathcal{L}[u_d(t) y(t - a)] = e^{-as} \mathcal{L}[y] = e^{-as} Y(s)$</td>
<td>$\mathcal{L}^{-1}[e^{-as} Y] = u_d(t) y(t - a)$</td>
</tr>
<tr>
<td>$\mathcal{L}[e^{at} y(t)] = Y(s - a)$</td>
<td>$\mathcal{L}^{-1}[Y(s - a)] = e^{at} \mathcal{L}^{-1}[Y] = e^{at} y(t)$</td>
</tr>
</tbody>
</table>