Inverse Trigonometric Functions

1. Trig functions don’t really have inverses, because they’re not ________.

However, we can find partial inverses by restricting the domain.

2. Let’s say we want to find the inverse sine function.
   
   (a) Draw a graph of sin(x).

   (b) Pick an interval \([a, b]\) such that sin(x) is 1-1 on that interval. There are options – try to pick the most reasonable option.

   (c) What is the domain and range of sin(x) on this interval?

   (d) What should the domain and range of the inverse of sin(x) be? (We write \(\sin^{-1}(x)\) or \(\arcsin(x)\) for the inverse of sin(x). Those two notations are equivalent.)
(e) We’re going to try to graph \( \sin^{-1}(x) \). First, let’s figure out a few points. Fill in the blanks:

(a) \( \sin(0) = \), so \( \sin^{-1}(\ ) = \).

(b) \( \sin\left(\frac{\pi}{6}\right) = \), so \( \sin^{-1}(\ ) = \).

(c) \( \sin\left(\frac{\pi}{4}\right) = \), so \( \sin^{-1}(\ ) = \).

(d) \( \sin\left(\frac{\pi}{3}\right) = \), so \( \sin^{-1}(\ ) = \).

(e) \( \sin\left(-\frac{\pi}{6}\right) = \), so \( \sin^{-1}(\ ) = \).

(f) \( \sin\left(-\frac{\pi}{4}\right) = \), so \( \sin^{-1}(\ ) = \).

(g) \( \sin\left(-\frac{\pi}{3}\right) = \), so \( \sin^{-1}(\ ) = \).

(h) Sketch a graph of \( \arcsin(x) \).

3. Now let’s do the same for \( \cos(x) \) and \( \arccos(x) \).

(a) What would a good restricted domain be for \( \cos(x) \) to make it 1-1?

(b) What should the domain and range of \( \cos^{-1}(x) \) be?
(c) Find some points on the graph of \( \cos^{-1}(x) \).

(d) Sketch a graph of \( \cos^{-1}(x) \).

Before doing the same thing for \( \tan^{-1}(x) \), let’s talk a little more about computing inverse trig functions. Remember – for trig functions, you plug in an angle and get out a ratio/number, which means for inverse trig functions, you plug in a ratio/number and get out an angle. What makes them a little bit more difficult is the restrictions we have in place, because trig functions are not 1-1.

4. What is \( \arccos(1) \)?

5. What is \( \arccos(-1) \)?

6. What is \( \arcsin\left(\frac{1}{2}\right) \)?

7. What is \( \arcsin\left(-\frac{\sqrt{2}}{2}\right) \)?

8. What is \( \arcsin(-1) \)?

9. What is \( \arccos\left(-\frac{1}{2}\right) \)?

10. What is \( \arctan(1) \)?
11. Let’s get back to graphing, and find the graph of \( \arctan(x) \).

(a) Find a domain on which \( \tan(x) \) is 1-1.

(b) Sketch a graph of \( \arctan(x) \).

Remember how functions and their inverses should cancel each other out? That is, \((f \circ f^{-1})(x) = x\) and \((f^{-1} \circ f)(x) = x\). Well, when we deal with partial inverses and restricted domains, things can get a little screwy. Let’s just see what might happen.

12. Start with a non-trig example. Hopefully you remember that \( \sqrt{x} \) is the partial inverse of \( f(x) = x^2 \).

(a) We can compute \((f^{-1} \circ f)(x)\), which in this case is \( \sqrt{x^2} \). If \( x = 4 \), what is \((f^{-1} \circ f)(x)\)?

(b) What is \((f \circ f^{-1})(x)\), if \( x = 4 \)?

(c) Hopefully the first two parts worked out nicely. However, what if \( x = -4 \)? Then what is \((f^{-1} \circ f)(x)\)?

(d) What is \((f \circ f^{-1})(x)\), if \( x = -4 \)?

These weird things happen because \( \sqrt{x} \) is only a partial inverse of \( x^2 \). Similar weird things can happen when you compose trig functions and their inverses. Sometimes it works out nicely, but sometimes not...
13. What is $\sin(\arcsin(1))$?

14. What is $\sin(\arcsin(-\frac{1}{2}))$?

15. What is $\sin^{-1}(\sin(0))$?

16. What is $\arcsin(\sin(\frac{2\pi}{3}))$?

17. What is $\sin^{-1}(\sin(\frac{\pi}{3}))$?

18. What is $\arcsin(\sin(\frac{11\pi}{6}))$?

19. What is $\cos(\arccos(-\frac{1}{2}))$?

20. What is $\cos^{-1}(\cos(3\pi))$?

21. What is $\tan(\arctan(\sqrt{3}))$?

22. What is $\tan^{-1}(\tan(\pi))$?

23. What is $\tan^{-1}(\tan(-\frac{\pi}{4}))$?

One thing that comes up in calculus is simplifying expressions that involves trig and inverse trig functions. For example, you might want to simplify the expression $\sin(\arctan(\frac{x}{4}))$. This isn't too hard to do, with the aid of a triangle.
24. Consider \( \arctan\left(\frac{x}{3}\right) \). This quantity is an angle, because the arctan function spits out angles. Call this angle \( \theta \). Draw a right triangle, with \( \theta \) as one of the angles.

25. If \( \theta = \arctan\left(\frac{x}{3}\right) \), then \( \tan(\ ) = \).

26. Label the sides of the triangle you drew with expressions involving \( x \) and 3. The Pythagorean Theorem may come in handy.

27. What is \( \sin(\arctan\left(\frac{x}{3}\right)) \)?

28. What is \( \tan(\arccos\left(\frac{1}{x}\right)) \)?

29. What is \( \csc(\sin^{-1}(x)) \)? (This is an easy one...why?)

30. What is \( \sec(\sin^{-1}(x)) \)?