# SITE PERCOLATION ON PLANAR GRAPHS

ZHONGYANG LI

Abstract. We prove that for a non-amenable, locally finite, connected, transitive, planar graph with one end, any automorphism invariant site percolation on the graph does not have exactly 1 infinite 1-cluster and exactly 1 infinite 0-cluster a.s. If we further assume that the site percolation is insertion-tolerant and a.s. there exists a unique infinite 0-cluster, then a.s. there are no infinite 1-clusters. The proof is based on the analysis of a class of delicately constructed interfaces between clusters and contours. Applied to the case of i.i.d. Bernoulli site percolation on infinite, connected, locally finite, transitive, planar graphs, these results solve two conjectures of Benjamini and Schramm (Conjectures 7 and 8 in [4]) in 1996.

## 1. Introduction

1.1. **Percolation on planar graphs and Benjamini-Schramm conjectures.** Let $G = (V(G), E(G))$ be an infinite, locally finite, connected graph. A manifold $M$ is plane if every self-avoiding cycle splits it into two parts. We say the graph $G$ is planar if it can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. We say that an embedded graph $G \subset M$ in $M$ is properly embedded if every compact subset of $M$ contains finitely many vertices of $G$ and intersects finitely many edges.

A site percolation configuration $\omega \in \{0,1\}^{V(G)}$ is a an assignment to each vertex in $G$ of either state 0 or state 1. A cluster in $\omega$ is a maximal connected set of vertices in which each vertex has the same state in $\omega$. A cluster may be a 0-cluster or a 1-cluster depending on the common state of vertices in the cluster. A cluster may be finite or infinite depending on the total number of vertices in the cluster. We say that percolation occurs in $\omega$ if there exists an infinite 1-cluster in $\omega$.

A bond percolation configuration $\phi \in \{0,1\}^{E(G)}$ is a an assignment to each edge in $G$ of either state 0 or state 1. A contour in $\phi$ is a maximal connected set of edges in which each edge has state 1. A contour may also be finite or infinite. In some cases, a bond percolation model is considered as a site percolation model on a different graph $\tilde{G}$, whose vertex set is the edge set of $G$, two vertices in $\tilde{G}$ are joined by an edge in $\tilde{G}$ if and only if their corresponding edges in $G$ share an endpoint. Therefore we can also use 1-clusters (resp. 0-clusters) to denote maximal connected sets of edges in which each edge has state-1 (resp. state-0) in a bond percolation configuration.

We say a vertex or an edge is open if it has state 1, and closed if it has state 0. The central question in the percolation theory is to study the conditions for the existence and the numbers of infinite clusters and contours. The percolation model is a natural mathematical model for structure of matter, magnetization, or spread of pandemic diseases.

Of particular interest is the i.i.d. Bernoulli site (resp. bond) percolation on a graph. In such a model, an independent Bernoulli random variable, which takes value 1 with probability $p \in [0,1]$, is associated to each vertex (resp. edge). For the i.i.d. Bernoulli percolation, define

$$\begin{aligned}
p_c^{site}(G) : &= \inf\{p \in [0,1] : \text{Bernoulli}(p) \text{ site percolation on } G \text{ has an infinite cluster a.s.}\}\\
p_c^{bond}(G) : &= \inf\{p \in [0,1] : \text{Bernoulli}(p) \text{ bond percolation on } G \text{ has an infinite cluster a.s.}\}\\
p_u^{site}(G) : &= \inf\{p \in [0,1] : \text{Bernoulli}(p) \text{ site percolation on } G \text{ has a unique infinite cluster a.s.}\}\\
p_u^{bond}(G) : &= \inf\{p \in [0,1] : \text{Bernoulli}(p) \text{ bond percolation on } G \text{ has a unique infinite cluster a.s.}\}
\end{aligned}$$

The main goal of this paper is to investigate the following two conjectures of Benjamini and Schramm about percolation on planar graphs.

**Conjecture 1.1.** *(Conjecture 7 of [4]) Suppose $G$ is planar, and the minimal degree in $G$ is at least 7. Then at every $p$ in the range $(p_c, 1-p_c)$, there are infinitely many infinite open clusters. Moreover, we conjecture that $p_c < \frac{1}{2}$, so the above interval is nonempty.*

**Conjecture 1.2.** *(Conjecture 8 of [4]) Let $G$ be a planar graph. Let $p = \frac{1}{2}$ be the probability that a vertex is open and assume that a.s. percolation occurs in the site percolation on $G$, then almost surely there are infinitely many infinite clusters.*

Conjecture 1.1 was proved in [4] when the graph $G$ is obtained by adding to the binary tree edges connecting all paris of nearest vertices of same level along a line. Conjecture 1.2 was proved in [4] when $G$ is a planar graph disjoint from the positive $x$-axis ($\{(x,0) : x > 0\}$), such that every bounded set in the plane meets finitely many vertices and edges of $G$. Conjectures 1.1 and 1.2 were also proved in [17] when $G$ is a regular triangular tiling of the hyperbolic plane $\mathbb{H}^2$ in which each vertex has the same degree $d \geq 7$. In this paper, we shall prove the above two conjectures for percolation on a large class of graphs, which include but are not restricted to the class of graphs discussed in [17]. A major advantage of the triangular tiling of the hyperbolic plane is that in its dual graph, each vertex has degree 3. However, this property does not hold for general planar graphs. We shall develop new technique, in particular, the analysis of a newly constructed "interface" between clusters and contours in order to prove Conjectures 1.1 and 1.2 for site percolation on general planar graphs.

## 1.2. **Percolation and graph structures.**

When study site percolation on infinite graphs, a natural assumption is that the graph should be "locally identical" at each vertex. Let $\text{Aut}(G)$ be the automorphism group of $G$. A graph $G = (V(G), E(G))$ is called vertex-transitive, or transitive, if there exists a subgroup $\Gamma \subseteq \text{Aut}(G)$, such that all the vertices are in the same orbit under the action of $\Gamma$ on $G$. The graph $G$ is called quasi-transitive if there exists a subgroup $\Gamma \subseteq \mathrm{Aut}(G)$, such that all the vertices are in finitely many different orbits under the action of $\Gamma$ on $G$.

The percolation properties of a graph are also closely related to the structure of the graph. Assume $\Gamma \subseteq \mathrm{Aut}(G)$ acts on $G$ quasi-transitively. We say the action of $\Gamma$ on $G$ is unimodular if for any $u, v \in V(G)$ in the same orbit of $\Gamma$,

$$|\mathrm{Stab}_u(v)| = |\mathrm{Stab}_v(u)|.$$

where $\mathrm{Stab}_u$ is the subgroup of $\Gamma$ defined by

$$\mathrm{Stab}_u := \{\gamma \in \Gamma : \gamma(u) = u\}.$$

The graph $G$ is called amenable if

$$\inf_{K \subseteq V(G), |K| < \infty} \frac{|\partial_E K|}{|K|} = 0, \tag{1.1}$$

where $\partial_E K$ consists of all the edges in $E(G)$ that have exactly one endpoint in $K$ and one endpoint not in $K$. If the left-hand side of (1.1) is strictly positive, then the graph $G$ is called non-amenable.

Let $G$ be an infinite, connected, planar, transitive graph, with finite vertex degree. Each such graph is quasi-isometric with one and only one of the following spaces: $\mathbb{Z}$, the 3-regular tree, the Euclidean plane $\mathbb{R}^2$, and the hyperbolic plane $\mathbb{H}^2$; see [1]. See [7] for background on hyperbolic geometry.

Since self-loops and multiple edges between the same pair of vertices have no effect on the existence of infinite clusters in the site percolation, without loss of generality, we assume that

- $G$ is simple.

Recall that the number of ends of a connected graph is the supremum over its finite subgraphs of the number of infinite components that remain after removing the subgraph. The number of ends of a graph is closely related to properties of statistical mechanical models on the graph; see [18], for example, about the effects of the number of ends of a graph on the speed of self-avoiding walks. The following proposition about the number of ends of a quasi-transitive graph was observed in [16] and proved in [1].

**Proposition 1.3.** *An infinite, connected, locally finite, quasi-transitive graph has either one or two or infinitely many ends. If the graph has two ends, then it is amenable. If it has infinitely many ends, then it is non-amenable.*

*Proof.* See Proposition 2.1 of [1]. □

The idea to solve Conjectures 1.1 and 1.2 is to classify all the infinite, connected, locally finite, transitive, planar graphs according to the number of ends, and then prove these conjectures for each subclass of graphs. More precisely, let $G$ be an infinite, connected, locally finite, transitive, planar graph, the following cases may occur

(1) $G$ is amenable and has one-end. The classical percolation model on the 2D square grid belongs to this case and has been studied extensively, see, for example, [10].
(2) $G$ is non-amenable and has one-end. It is proved that $p_c^{site} < p_u^{site}$, and $p_c^{bond} < p_u^{bond}$ for this case in [5].
(3) $G$ has two ends.
(4) $G$ has infinitely many ends.

We shall study percolation on each subclass of graphs listed above, and develop general techniques which can be applied beyond the i.i.d. Bernoulli percolation case. The major new contribution is an analysis of a type of "interface" separating contours and clusters, which can only be a union of disjoint self-avoiding contours or doubly infinite self-avoiding path. The planar duality will also play an important role in our analysis.

### 1.3. Main results.

Here are the main theorems proved in this paper.

**Theorem 1.4.** *Let $G = (V(G), E(G))$ be a non-amenable, locally finite, planar graph with one end. Assume one of the following conditions holds:*

*(1) $G$ is transitive; or*
*(2) $G$ is quasi-transitive; and $G$ can be properly embedded into the hyperbolic plane $\mathbb{H}^2$ in such a way that each vertex $v$ is incident to faces of degree $m_1, m_2, \ldots, m_{l_v}$ ($l_v \geq 2$) in the cyclic order, where $m_1, \ldots, m_{l_v} \geq 3$ are positive integers; and the automorphisms of $G$ extend to isometries of $\mathbb{H}^2$.*

*Consider an automorphism-invariant site percolation measure $\mu$ on $G$ with sample space $\{0,1\}^{V(G)}$, that is, there exists a subgroup $\Gamma \subseteq \mathrm{Aut}(G)$ acting quasi-transitively on $G$, such that for any event $A \subseteq \{0,1\}^{V(G)}$ and $\gamma \in \Gamma$, $\mu(A) = \mu(\gamma A)$. Let $s_0$ (resp. $s_1$) be the total number of infinite 0-clusters (resp. infinite 1-clusters) in $\omega$, then*

$$\mu((s_0, s_1) = (1,1)) = 0.$$

Before stating the next main theorem, we first recall the following definition.

**Definition 1.5.** *Let $G = (V, E)$ be a graph. Given a set $A \in 2^V$, and a vertex $v \in V$, denote $\Pi_v A = A \cup \{v\}$. For $\mathcal{A} \subset 2^V$, we write $\Pi_v \mathcal{A} = \{\Pi_v A : A \in \mathcal{A}\}$. A site percolation process $(\mathbf{P}, \omega)$ on $G$ is insertion-tolerant if $\mathbf{P}(\Pi_v \mathcal{A}) > 0$ for every $v \in V$ and every event $\mathcal{A} \subset 2^V$ satisfying $\mathbf{P}(\mathcal{A}) > 0$.*

*A site percolation is deletion tolerant if $\mathbf{P}[\Pi_{\neg v} \mathcal{A}] > 0$ whenever $v \in V$ and $\mathbf{P}(\mathcal{A}) > 0$, where $\Pi_{\neg v} A = A \setminus \{v\}$ for $A \in 2^V$, and $\Pi_{\neg v} \mathcal{A} = \{\Pi_{\neg v} A : A \in \mathcal{A}\}$.*

We can similarly define the insertion or deletion tolerance for bond percolation by replacing a vertex with an edge in the above definition. It is straightforward to check that the i.i.d. Bernoulli($p$) site or bond percolation for $p \in (0,1)$ is both deletion tolerant and insertion tolerant.

**Theorem 1.6.** *Let $G$ be a graph as described in Theorem 1.4. Let $\mu$ be an automorphism-invariant percolation measure on $\{0,1\}^{V(G)}$. Then*

*(1) If $\mu$ is insertion-tolerant and $\mu$-a.s. there is a unique infinite 0-cluster, then $\mu$-a.s. there are no infinite 1-clusters.*

*(2) If $\mu$ is deletion-tolerant, and $\mu$-a.s. there is a unique infinite 1-cluster, then $\mu$-a.s. there are no infinite 0-clusters.*

When the graph $G$ is a vertex-transitive, regular triangular tiling of $\mathbb{H}^2$ with vertex degree $d \geq 7$, the results of Theorems 1.4 and 1.6 were proved in [17]. In this case, the dual graph $G^+$ of $G$ has vertex degree 3, therefore each vertex in a connected component of edges of $G^+$ separating state 0 vertices and state 1 vertices of $G$ has degree 2 in the component. Hence each such component is either a doubly infinite self-avoiding path or a self-avoiding cycle. This important fact simplifies the analysis. In the general case, we need to carefully construct an "interface" between clusters and contours to prove the theorem.

In Theorems 1.4 and 1.6, we do not require that the percolation is an i.i.d. Bernoulli percolation. When theorems 1.4 and 1.6 are applied to i.i.d. Bernoulli percolation, we obtain following theorems.

**Theorem 1.7.** *Let $G$ be an infinite, connected, locally finite, transitive, planar graph in which each vertex has degree at least 7. Consider the i.i.d. Bernoulli(p) site percolation of $G$. Then*

*(A) $p_c^{site} < \frac{1}{2}$.*

*(B) For every $p$ in the range $(p_c^{site}, 1 - p_c^{site})$, there are infinitely many infinite open clusters and infinitely many infinite closed clusters a.s.*

*(C) For every $p$ in the range $[0, 1]$, a.s. there exists at least 1 infinite open or closed cluster.*

**Theorem 1.8.** *Let $G$ be an infinite, connected, locally finite, transitive, planar graph. Consider the i.i.d. Bernoulli(p) site percolation of $G$. Then*

*(A) $p_u^{site} + p_c^{site} \geq 1$.*

*(B) Assume each vertex is open independently with probability $\frac{1}{2}$. Assume that a.s. percolation occurs in the site percolation on $G$, then almost surely there are infinitely many infinite 1-clusters and infinitely many infinite 0-clusters.*

When considering site percolation on an infinite, connected, locally finite, transitive, planar graph, Theorem 1.7(A)(B) confirms Conjecture 1.1, and Theorem 1.8(B) confirms Conjecture 1.2.

The organization of the paper is as follows. In Section 2, we review some known results about planar graphs and percolation, which will be used to prove the main theorems. In Section 3, we prove Theorems 1.4 and 1.6. In Section 4, we prove two inequalities of $p_c^{site}$ and $p_u^{site}$, which will be used to prove Theorem 1.7. In Section 5, we study i.i.d. Bernoulli bond or site percolation on two-ended, locally finite, connected transitive graphs, and show that $p_c = 1$. In Section 6, we study i.i.d. Bernoulli bond or site percolation on connected, locally finite, transitive graphs with infinitely many ends, and show that $p_u = 1$. In

Section 7, we study i.i.d. Bernoulli percolation on one-ended, amenable, connected, locally finite, planar graphs and show that when $p = \frac{1}{2}$, percolation cannot occur. In Section 8, we prove Theorems 1.7 and 1.8. In Section 9, we prove combinatorial results about infinite clusters and contours with the help of planar duality, which have been used to prove the main theorems.

## 2. Backgrounds

In this section, we review some known results about planar graphs and percolation on amenable and non-amenable graphs, which will be used to prove the main results in this paper.

### 2.1. Planar graphs with one end.

We refer to the spheres, the Euclidean plane, the hyperbolic planes as natural geometries. The natural geometries are two-dimensional Riemannian manifolds that possess a group of isometries. Each natural geometry is characterized by its Gauss curvature. The curvature is positive for the spheres, zero for the Euclidean plane, and negative for the hyperbolic planes. An Archimedean tiling of a two-dimensional Riemannian manifold is a tiling by regular polygons such that the group of isometries of the tiling acts transitively on the vertices of the tiling. Then one-ended vertex-transitive planar graphs can be characterized as tilings of natural geometries.

**Lemma 2.1.** *Let $G$ be a locally finite, connected, vertex-transitive planar graph with at most one end. The $G$ has an embedding on a natural geometry as an Archimedean tiling; all automorphisms of $G$ extend to automorphisms of the tiling and are induced by isometries of the geometry.*

*Proof.* See Theorem 3.1 of [1]. □

For an vertex-transitive Archimedean tiling, there is an simple criterion to determine whether the graph is amenable or not (see [21]).

**Lemma 2.2.** *Assume the graph $G$ can be realized as a vertex-transitive Archimedean tiling on a natural geometry. Assume that each vertex had degree $d \geq 3$, and is incident to $d$ faces of degree $m_1, m_2, \ldots, m_d$.*

*(1) If $\frac{1}{m_1} + \frac{1}{m_2} + \ldots + \frac{1}{m_d} = \frac{d-2}{2}$, then $G$ is infinite and amenable can be embedded into the Euclidean $\mathbb{R}^2$ such that all automorphisms of $G$ extend to automorphisms of the tiling and are induced by isometries of $\mathbb{R}^2$;*

*(2) If $\frac{1}{m_1} + \frac{1}{m_2} + \ldots + \frac{1}{m_d} > \frac{d-2}{2}$, then $G$ is finite and can be embedded into the sphere $\mathbb{S}^2$ such that all automorphisms of $G$ extend to automorphisms of the tiling and are induced by isometries of $\mathbb{S}^2$.;*

*(3) If $\frac{1}{m_1} + \frac{1}{m_2} + \ldots + \frac{1}{m_d} < \frac{d-2}{2}$, then $G$ is non-amemable can be embedded into the hyperbolic plane $\mathbb{H}^2$ such that all automorphisms of $G$ extend to automorphisms of the tiling and are induced by isometries of $\mathbb{H}^2$.*

The characterization of one-ended vertex-transitive planar graphs as tilings of natural geometries makes it possible to develop universal techniques to study statistical mechanical models on all these graphs; see [11], for example, about a universal lower bound of connective constants on all the infinite, connected, transitive, planar, cubic graphs.

2.2. **Percolation.** We always assume that the graph $G$ is infinite, connected, and locally finite in this subsection.

**Lemma 2.3.** *Let $G = (V(G), E(G))$ be a nonamenable graph with a transitive unimodular automorphism group, and consider the i.i.d. Bernoulli site or bond percolation on $G$. Then at the corresponding critical value $p = p_c$, almost surely there is no infinite cluster.*

*Proof.* See Theorem 1.3 of [3]; see also [2]. □

**Lemma 2.4.** *Suppose that $G$ is a quasi-transitive graph, and consider the i.i.d. Bernoulli bond or site percolation on $G$. Assume that $0 < p_1 < p_2 \leq 1$, and that at level $p_1$ there is a.s. a unique infinite cluster. Then also at level $p_2$ there is a.s. a unique infinite cluster.*

*Proof.* See Corollary 1.2 of [22]; see also [12]. □

**Lemma 2.5.** *Let $G$ be a transitive, non-amenable, planar graph with one end. Then Bernoulli($p_u$) percolation on $G$ has a unique infinite cluster a.s.*

*Proof.* See Theorem 1.2 of [5]. □

**Lemma 2.6.** *Let $G$ be a quasi-transitive nonamenable planar graph with one end, and let $\omega$ be an invariant percolation on $G$. Then a.s. the number of infinite components of $\omega$ is 0, 1 or $\infty$.*

*Proof.* See Lemma 3.5 of [5]. □

**Lemma 2.7.** *Let $G$ be a non-amenable, quasi-transitive, unimodular graph, and let $\omega$ be an invariant percolation on $G$ which has a single component a.s. Then $p_c(\omega) < 1$ a.s.*

*Proof.* See Theorem 3.4 of [5]. □

**Lemma 2.8.** *Consider the i.i.d. Bernoulli site percolation on the regular tiling $G$ of the hyperbolic plane with triangles, such that each vertex has the same degree $d \geq 7$. Assume that each vertex of $G$ takes value 1 with probability $\frac{1}{2}$. Let $s_0$ (resp. $s_1$) be the total number of infinite 0-clusters (resp. 1-clusters). Then a.s. $(s_0, s_1) = (\infty, \infty)$.*

*Proof.* See Example 2.3 of [17]. □

Lemma 2.6 requires non-amenability and planarity of the graph to obtain that the number of infinite open-clusters is 0,1, or $\infty$ a.s. If the graph is amenable or nonplanar, similar results can also be obtained under the additional assumption that the percolation is insertion-tolerant.

**Lemma 2.9.** *Let $G = (V(G), E(G))$ be a connected, locally finite, quasi-transitive graph. Consider insertion-tolerant, invariant percolation on $G$, then the number of infinite 1-clusters is a.s. $0, 1, \infty$.*

*Proof.* The proof is based on an adaptation of an argument of Newman and Schulman ([20]), where the results were proved for percolation on $\mathbb{Z}^d$.

Let $\mu$ be the corresponding percolation measure. Without loss of generality, assume that $\mu$ is ergodic. Let $s_1$ be the total number of infinite 1-clusters in a random percolation configuration on $G$. Then there exists an integer $k \in \{0, 1, \dots\} \cup \{\infty\}$, such that $\mu(s_1 = k) = 1$. If $k \notin \{0, 1, \infty\}$, let $\xi_1$ and $\xi_2$ be two distinct infinite clusters closest to a fixed vertex $v_0$. Then

$$\mu(\cup_{1\leq n<\infty}\{d_G(\xi_1, \xi_2) \leq n\}) = 1 = \lim_{n\to\infty} \mu(\{d_G(\xi_1, \xi_2) \leq n\})$$

Here $d_G(\xi_1, \xi_2)$ denotes the graph distance of $\xi_1$ and $\xi_2$ as two subsets of $V(G)$. Then there exists a positive integer $N$, such that

$$\mu(\{d_G(\xi_1, \xi_2) \leq N\}) \geq \frac{1}{2}.$$

Find a path of length at most $N$ in $G$ joining the $\xi_1$ and $\xi_2$, and make all the vertices (or edges, depending on whether it is a bond or a site percolation) on $G$ open. Then by the insertion-tolerance of $\mu$, with strictly positive probability, $s_1 \leq k - 1$. This contradicts the fact that $\mu(s_1 = k) = 1$. Then the proposition follows. □

Let $\omega_1, \omega_2 \in \{0, 1\}^{V(G)}$. Define $\omega_1 \vee \omega_2,\ \omega_1 \wedge \omega_2 \in \{0, 1\}^{V(G)}$ as follows

$$\omega_1 \vee \omega_2(v) = \max\{\omega_1(v), \omega_2(v)\}, \qquad v \in V(G);$$
$$\omega_1 \wedge \omega_2(v) = \min\{\omega_1(v), \omega_2(v)\}, \qquad v \in V(G).$$

An event $A \subseteq \{0, 1\}^{V(G)}$ is called increasing if for any $\omega_1 \in A$, $\omega_2 \geq \omega_1$, we have $\omega_2 \in A$. The following F.K.G. inequality is well known.

**Lemma 2.10.** ([8],[14])*Let $\mu$ be a strictly positive probability measure on $\{0, 1\}^{V(G)}$ satisfying the following F.K.G. lattice condition:*

$$\mu(\omega_1 \vee \omega_2)\mu(\omega_1 \wedge \omega_2) \geq \mu(\omega_1)\mu(\omega_2), \qquad \omega_1, \omega_2 \in \{0, 1\}^{V(G)}.$$

*Then for any increasing events $A, B \subseteq \{0, 1\}^{V(G)}$,*

$$\mu(A \cap B) \geq \mu(A)\mu(B). \tag{2.1}$$

It is straight forward to check that the F.K.G. inequality (2.1) holds for the i.i.d. Bernoulli($p$) percolation when $p \in (0, 1)$.

## 3. Proof of Theorems 1.4 and 1.6

In this section, we prove Theorem 1.4 and Theorem 1.6. The idea to prove Theorem 1.4 is that based on a site configuration $\omega \in \{0,1\}^{V(G)}$ with invariant distribution and satisfying $(s_0, s_1) = (1,1)$, we construct a new bond configuration $\psi_\omega$ on a new graph which also has invariant distribution, and moreover, $\psi_\omega$ consists of a single component which is a doubly-infinite self-avoiding path. This contradicts Lemma 2.7. Theorem 1.6 can be proved in the similar spirit. We shall start with the construction of the new graph on which $\psi_\omega$ is defined.

3.1. **Construction of bond configurations on the dual superposition graph.** Let $G$ be a connected, locally finite, quasi-transitive, non-amenable planar graph with one end satisfying the assumptions of Theorem 1.4. By Lemmas 2.1 and 2.2, we can identify the graph $G$ with its embedding in $\mathbb{H}^2$ in which the action of $\Gamma$ on $G$ extends to an isometric action on $\mathbb{H}^2$. Let $G^+$ be the planar dual graph of $G$, that is, $G^+$ is obtained by placing a vertex on each face of $G$; two vertices of $G^+$ are joined by an edge of $G^+$ if and only if the corresponding faces in $G$ share an edge of $G$.

We shall always use $*^+$ to denote the dual of $*$. If $*$ is an edge, then $*^+$ is its dual edge. If $*$ is a vertex, then $*^+$ is its dual face. If $*$ is a face, then $*^+$ is its dual vertex.

Let $\overline{G}$ be the superposition of $G$ and $G^+$; that is, each vertex of $\overline{G}$ is either a vertex of $G$, a vertex of $G^+$ or the midpoint of an edge of $G$. Two vertices $u, v$ of $\overline{G}$ are joined by an edge of $\overline{G}$ if and only if one of the following two conditions holds.

(1) $u$ is a vertex of $G$, and $v$ is the midpoint of an edge $e(v)$ of $G$, such that $e(v)$ is incident to $u$, or vice versa;
(2) $u$ is a vertex of $G^+$, and $v$ is the midpoint of an edge $e^+(v)$ of $G^+$, such that $e^+(v)$ is incident to $u$, or vice versa.

Let $\overline{G}^+$ be the dual graph of $\overline{G}$. Since in $\overline{G}$, each face has degree 4, in $\overline{G}^+$, each vertex has degree 4.

For each site configuration $\omega \in \{0,1\}^{V(G)}$, we define a bond configuration $\phi_\omega^+ \in \{0,1\}^{E(G^+)}$ such that for each dual edge $e^+ \in E(G^+)$, $\phi_\omega^+(e^+) = 1$ if and only if the edge $e \in E(G)$ (dual edge of $e^+$) joins two endpoints with different states in $\omega$. See Figure 3.1 for the graph $G$, site configuration $\omega \in \{0,1\}^{V(G)}$, the dual graph $G^+$, the induced bond configuration $\phi_\omega^+ \in \{0,1\}^{E(G^+)}$, and the graph $\overline{G}^+$.

It is straightforward to check that each vertex of $G^+$ has an even degree in the subgraph $\phi_\omega^+$. We define the interface $\eta_\omega$ for $\phi_\omega^+$ to be a bond configuration in $\{0,1\}^{E(\overline{G}^+)}$, where an edge $f \in E(\overline{G}^+)$ satisfies $\eta_\omega(f) = 1$ if and only if its dual edge $f^+ \in E(\overline{G})$ is an half edge of $e \in E(G) \cup E(G^+)$ such that one of the following two conditions holds:

(1) If $e \in E(G)$, then the dual edge $e^+ \in E(G^+)$ satisfies $\phi_\omega^+(e^+) = 1$. In this case we say that the edge $f$, or the contour $I_f$ in $\eta_\omega$ containing $f$, is incident to the contour $C_{e^+}$ in $\phi_\omega^+$ including $e^+$. In Figure 3.1, the state-1 edge $(B, C) \in E(\overline{G}^+)$ of $\eta_\omega$ is of

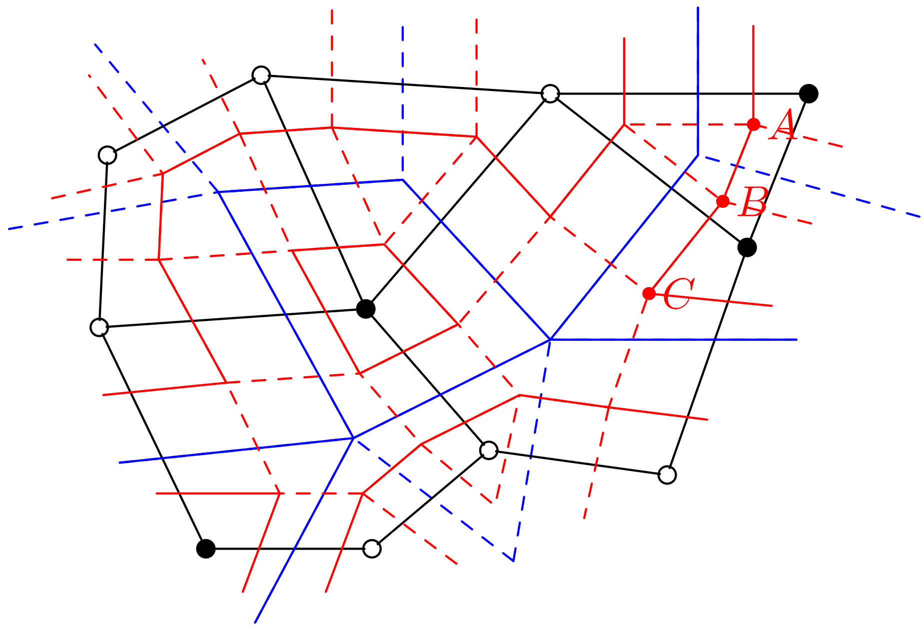


FIGURE 3.1. The graph $G$ where site percolation $\omega$ is defined is represented by black lines; state-1 (open) vertices in $\omega$ are represented by dots; state-0 (closed vertices) in $\omega$ are represented by circles. The dual graph $G^+$ is represented by blue lines; state-1 edges in $\phi_\omega^+$ are represented by solid blue lines; state-0 edges in $\phi_\omega^+$ are represented by dashed blue lines. The graph $\overline{G}^+$ is represented by red lines; state-1 edges in $\eta_\omega$ are represented by solid red lines; state-0 edges in $\eta_\omega$ are represented by dashed red lines.

this type. Let $u$ be the common vertex of $e$ and $f^+$, we also say that the edge $f$, or the contour $I_f$, is incident to the cluster $\xi_u$ in $\omega$ containing $u$.

(2) If $e \in E(G^+)$, then $e^+ \in E(G)$ (the dual edge of $e$) joins two vertices $u, v \in V(G)$ such that $\omega(u) = \omega(v)$, and the endpoint $w$ of $f^+$ in $V(G^+)$ is in a contour of $\phi_\omega^+$. In Figure 3.1, the state-1 edge $(A, B) \in E(\overline{G}^+)$ of $\eta_\omega$ is of this type. In this case we say that the edge $f$, or the contour $I_f$ in $\eta_\omega$ containing $f$, is incident to the contour $C_w$ in $\phi_\omega^+$ containing $w$ and the cluster $\xi_{u,v}$ in $\omega$ including both $u$ and $v$.

We say two infinite clusters $A$, $B$ in $\omega$ are adjacent if there exists a path $l_{ab}$, joining a vertex $a \in A$ and $b \in B$, and consisting of edges of $G$, such that $l_{ab}$ does not intersect any other infinite clusters in $\omega$. In particular, if there are exactly two infinite clusters in $\omega$, then the two infinite clusters must be adjacent. Let $L$ be an infinite contour in $\phi_\omega^+$. We say $A$ is incident to $L$ if there exists a vertex $z \in A$ and an edge $e^+$ in $L$ such that $z$ is an endpoint of the dual edge $e$ of $e^+$.

**Lemma 3.1.** *Each contour in $\eta_\omega$ is either a self-avoiding cycle or a doubly infinite self-avoiding path.*

*Proof.* We first show that each vertex along a contour in $\eta_\omega$ has two incident edges in the contour. By definition, each contour in $\eta_\omega$ is a connected subgraph in $\overline{G}^+$, and recall that each vertex in $\overline{G}^+$ has four incident edges. For a vertex $v \in V(\overline{G}^+)$, let $e_1, e_2, e_3, e_4 \in E(\overline{G}^+)$ be the four incident edges of $v$ in the cyclic order. Assume that $e_1^+$ and $e_2^+$ (the dual edges of $e_1$ and $e_2$) are half edges of $G$, while $e_3^+$ and $e_4^+$ are half edges of $G^+$. Then we claim that at most one of $e_1$ and $e_4$ is present in a contour of $\eta_\omega$, and at most one of $e_2$ and $e_3$ is present in a contour of $\eta_\omega$.

Assume that both $e_1$ and $e_4$ are present in a contour of $\eta_\omega$; we shall obtain a contradiction. Note that $e_1^+$ and $e_4^+$ are half edges of two dual edges $f = (u, v) \in E(G)$ and $f^+ = (z, w) \in E(G^+)$, respectively. If both $e_1$ is present in a contour of $\eta_\omega$, then $\phi_\omega^+(f^+) = 1$, which means that the dual edge $f$ of $f^+$ joins two vertices $u, v \in V(G)$ such that $\omega(u) \neq \omega(v)$. But $e_4$ is present in a contour of $\eta_\omega$ implies that $\omega(u) = \omega(v)$. The contradiction implies that at most one of $e_1$ and $e_4$ is present in a contour of $\eta_\omega$. Similarly at most one of $e_2$ and $e_3$ is present in a contour of $\eta_\omega$.

Therefore each vertex along a contour in $\eta_\omega$ has one or two incident edges in the contour. Assume there exists $p \in V(\overline{G}^+)$ incident to exactly one edge $e \in E(\overline{G}^+)$ in a contour of $\eta_\omega$. The following cases might occur

(1) The dual edge $e^+ \in E(\overline{G})$ is a half edge of an edge $g \in E(G)$. Assume $g = (a, b)$, where $a, b \in V(G)$, and $a$ is also an endpoint of $e^+$. Since $\eta_\omega(e) = 1$, $\phi_\omega^+(g^+) = 1$. Since $a$ is incident to at least two edges in $G$, let $(a, c) \in E(G)$ be an edge incident to $a$ other than $(a, b)$, and assume that $a$ has no other incident edges in $E(G)$ between $(a, b)$ and $(a, c)$. More precisely, assume that $(a, b)$ and $(a, c)$ are on the boundary of a face $F$ of $G$, such that $F$ contains the vertex $p$ in $V(\overline{G}^+)$
   (a) If $\omega(c) \neq \omega(a)$, then $\phi_\omega^+((a, c)^+) = 1$. Recall that $\phi_\omega^+((a, b)^+) = 1$, $(a, b)^+$ and $(a, c)^+$ share a vertex in $G^+$, we obtain that $(a, c)^+$ and $(a, b)^+$ are in the same contour of $\phi_\omega^+$ containing $(a, b)^+$. Let $d$ be the midpoint of the edge $(a, c)$, then $\eta_\omega((a, d)^+) = 1$, and $p$ has incident edge $(a, d)^+$ other than $e$ in the contour of $\eta_\omega$ containing $e$.
   (b) If $\omega(c) = \omega(a)$, let $F^+ \in V(G^+)$ be the vertex dual to the face $F$ of $G$. Since $\phi_\omega^+(g^+) = 1$, $F^+$ is an endpoint of $g^+$, $F^+$ is in a contour of $\phi_\omega^+$. Then $\eta_\omega((d, F^+)^+) = 1$. Hence and $p$ has incident edge $(d, F^+)^+$ other than $e$ in the contour of $\eta_\omega$ containing $e$.

(2) The dual edge $e^+ \in E(\overline{G})$ is a half edge of an edge $h^+ \in E(G^+)$. Then the dual edge $h = (x, y) \in E(G)$ of $h^+$ joins two vertices $x, y$ with $\omega(x) = \omega(y)$. Moreover, assume that $h^+ = (s, t)$ such that $s$ is also an endpoint of $e^+$, then $s$ is in a contour of $\phi_\omega^+$. Assume that $x$ and $p$ are in the same face of $G^+$ with the edge $(s, t)$ on its boundary. Since each vertex in $V(G)$ has at least two incident edges in $E(G)$, there is an edge $(x, \eta) \in E(G)$ different from $(x, y)$, such that $(x, y)$ and $(x, \eta)$ are bounding a face of $G$ containing $p$. Assume the face of $G$ corresponds to the dual vertex $s \in V(G^+)$, denoted by $s^+$.

(a) If $\omega(\eta) \neq \omega(x)$, then $\phi_\omega^+((x,\eta)^+) = 1$. Assume $(x,\eta)^+ = (s,r)$, then $(s,r)$ is in a contour of $\phi_\omega^+$ containing $s$. Let $\theta$ be the midpoint of the edge $(x,\eta)$, then $\eta_\omega((x,\theta)^+) = 1$, and $(x,\theta)^+ \in E(\overline{G}^+)$ is incident to $p$. Hence $p$ is incident to at least two edges in a contour of $\eta_\omega$.

(b) If $\omega(\eta) = \omega(x)$, since $s$ is in a contour of $\eta_\omega$, $\theta$ is the midpoint of the edge $(x,\eta)$, we have $\eta_\omega((s,\theta)^+) = 1$. Hence $(s,\theta)^+$ is an edge incident to $p$ other than $e$ but is in a contour of $\eta_\omega$.

In all the cases above, an arbitrary vertex $p \in V(\overline{G}^+)$ in a contour of $\eta_\omega$ is incident to exactly two edges in $\eta_\omega$. Hence each contour in $\eta_\omega$ is either a cycle or a doubly-infinite self-avoiding path. $\square$

**Lemma 3.2.** *Let* $\omega \in \{0,1\}^{V(G)}$.

*(1) The contours in* $\eta_\omega$ *and* $\phi_\omega^+$ *never cross.*

*(2) Let* $f \in E(\overline{G}^+)$ *such that* $\eta_\omega(f) = 1$. *If* $f$ *crosses an edge* $(u,v) \in E(G)$, *then* $\omega(u) \neq \omega(v)$.

*Proof.* Let $f \in E(\overline{G}^+)$ such that $\eta_\omega(f) = 1$. From the definition of $\eta_\omega$, the following cases might occur.

(1) $f^+$ is a half edge of an edge $e = (u,v) \in E(G)$, then $\phi_\omega^+(e^+) = 1$, therefore $\omega(u) \neq \omega(v)$. In this case $f$ crosses a unique edge $(u,v) \in E(G)$ with $\omega(u) \neq \omega(v)$, but does not cross any edge in $E(G^+)$.

(2) $f^+$ is a half edge of an edge $e^+ \in (G^+)$, then the edge $e = (u,v)$ satisfies $\omega(u) = \omega(v)$, thus $\phi_\omega^+(e^+) = 0$. In this case $f$ crosses exactly one edge $e^+$ in $E(G^+)$ which is not present in any contour of $\phi_\omega^+$, and $f$ does not cross any edge in $E(G)$.

In either case, an arbitrary edge in a contour of $\eta_\omega$ never cross a contour in $\phi^+$. If the edge in a contour of $\eta_\omega$ cross an edge of $E(G)$ as in the first case, $\omega(u) \neq \omega(v)$. Then the lemma follows. $\square$

**Lemma 3.3.** *Let* $\omega \in \{0,1\}^{V(G)}$. *Each contour* $D$ *in* $\eta_\omega$ *is incident to exactly one cluster* $\xi$ *in* $\omega$ *and exactly one contour* $C$ *in* $\phi_\omega^+$. *Moreover,* $\xi$ *and* $C$ *are in two distinct components of* $\mathbb{H}^2 \setminus D$.

*Proof.* Since $D$ is a contour in $\eta_\omega$, by Lemma 3.1, $D$ is either a self-avoiding cycle or doubly infinite self-avoiding path. In either case $\mathbb{H}^2 \setminus D$ has two components.

From the construction of $\eta_\omega$, we see that $D$ is incident to a cluster $\xi$ in $\omega$ (resp. a contour $C$ in $\phi^+(\omega)$) if and only if there exists an edge $f \in D \cap E(\overline{G}^+)$ incident to $\xi$ (resp. $C$). Let $f \in E(\overline{G}^+)$ be an arbitrary edge in $D$. Then $f$ must be incident to exactly one a cluster $\xi$ in $\omega$ and exactly one contour $C$ in $\phi^+(\omega)$. If $D$ is a self-avoiding cycle, we can visit every edge of $D$ by traversing the cycle starting at $f$ in either direction. If $D$ is a doubly infinite self-avoiding path, we can still visit every edge of $D$ by traversing the path starting from $f$ in both directions. To show that $D$ is incident to exactly one cluster in

$\omega$ and one contour in $\phi_\omega^+$, it suffices to show that traversing along $D$, each edge in $D$ is always incident to the same cluster $\xi$ and the same contour $C$.

The following cases might happen

(1) $f^+$ is a half edge for an edge $e = (u, v) \in E(G)$, then $\phi_\omega^+(e^+) = 1$, therefore $\omega(u) \neq \omega(v)$. Assume $u$ is also a vertex of $f^+$. Let $\xi$ be the cluster of $\omega$ at $u$, and $C$ be the contour of $\phi_\omega^+$ including $e^+$. Let $e^+ = (x, y)$, then $y^+$ is a face of $G$. Let $f = (a, b)$, such that $b$ is in the face $y^+$. Then $b^+$ is a degree-4 face in $\overline{G}$, and $b$ has 4 incident edges $(b, a)$, $(b, r)$, $(b, s)$, $(b, t)$ in $E(\overline{G}^+)$ in cyclic order, such that $(b, a)$ and $(b, t)$ are dual edges of a half edge of $G$, and $(b, r)$ and $(b, s)$ are dual edges of a half edge in $G^+$. As in the proof of Lemma 3.1, since $\eta_\omega((b, a)) = 1$, $\eta_\omega((b, r)) = 0$. Since each vertex in $V(\overline{G}^+) \cap D$ has two incident edges in $D \cap E(\overline{G}^+)$, the other edge along $D$ incident to $b$ must be either $(b, s)$ or $(b, t)$. Let $(u, w)$ be an edge on the boundary of the face $y^+$, such that $w \neq v$
   (a) If $\eta_\omega((b, s)) = 1$ and $\eta_\omega((b, t)) = 0$, then $\omega(u) = \omega(w)$, and the edge $(b, s)$ is still incident to the cluster $\xi$ including $u$ and $w$ and the cluster $C$ containing $y$.
   (b) If $\eta_\omega((b, s)) = 0$ and $\eta_\omega((b, t)) = 1$, then $\omega(u) \neq \omega(w)$, and $\phi_\omega^+((u, w)^+) = 1$. Let $(u, w)^+ = (y, z)$ where $z \in V(G)$, $z \neq x$. In this case the edge $(b, t)$ is still incident to the cluster $\xi$ including $u$ and the contour $C$ containing $y$ and $z$.

(2) $f^+$ is a half edge of an edge $e^+ \in (G^+)$, then the edge $e = (u, v)$ satisfies $\omega(u) = \omega(v)$, thus $\phi_\omega^+(e^+) = 0$. Then $v^+$ is a face in $G^+$. Assume $f = (a, b)$, such that $b$ is in the face $v^+$. Let $(b, a)$, $(b, r)$, $(b, s)$, $(b, t)$ be the 4 incident edges of $b$ in $\overline{G}^+$ in the cyclic order such that $(b, r)$ is the dual edge of an half edge of $(u, v)$. As in the proof of Lemma 3.1, since $\eta_\omega((b, a)) = 1$, we have $\eta_\omega(b, r) = 0$. Since each vertex in $V(\overline{G}^+) \cap D$ has two incident edges in $D \cap E(\overline{G}^+)$, the other edge along $D$ incident to $b$ must be either $(b, s)$ or $(b, t)$. Let $w \in V(G^+)$ be the common endpoint of $(u, v)^+$ and $f^+$. Let $C$ be the contour in $\phi_\omega^+$ containing $w$, and $\xi$ be the cluster in $\omega$ containing $u$ and $v$. Let $(w, z) \in E(G^+)$ be on the boundary of the face $v^+$ such that $(w, z) \neq (u, v)^+$.
   (a) If $\eta_\omega((b, s)) = 1$ and $\eta_\omega((b, t)) = 0$. Note that $(b, s)^+$ is an half edge of $(w, z)^+$. Then $\eta_\omega((b, s)) = 1$ implies that $\phi_\omega^+((w, z)) = 1$. Then the edge $(b, s)$ is still incident to the contour $C$ in $\phi_\omega^+$ including $w$ and $z$, and the cluster $\xi$ in $\omega$ including $u$ and $v$.
   (b) If $\eta_\omega((b, s)) = 0$ and $\eta_\omega((b, t)) = 1$, let $(v, x) = (z, w)^+$, then $\omega(v) = \omega(x)$. Hence the edge $(b, t)$ is still incident to cluster $\xi$ in $\omega$ including $u, v$ and $x$, and the contour $C$ in $\phi_\omega^+$ containing $w$.

This completes the proof that $D$ is incident to a unique contour $C$ in $\phi_\omega^+$ and a unique cluster $\xi$ in $\omega$, since every time we move along the doubly-infinite self-avoiding path or self-avoiding cycle $D$ to the next edge along either direction, the next edge is always incident to the same cluster $\xi$ in $\omega$ and the same contour $C$ in $\phi_\omega^+$.

The cluster in $\xi$, or the contour $C$, is in a component of $\mathbb{H}^2 \setminus D$ because by Lemma 3.2, $C \cap D = \emptyset$, and if we let $G_\xi$ be the graph whose vertex set if $\xi$, such that two vertices in $\xi$ are joined by an edge in $G_\xi$ if and only if they are joined by an edge in $G$, then $G_\xi \cap D = \emptyset$.

To check that $\xi$ and $C$ are in two distinct components of $\mathbb{H}^2 \setminus D$, let $f \in D \cap E(\overline{G}^+)$.

(1) If $f^+ = (u, \theta)$, such that $u \in V(G)$ and $\theta$ is the midpoint of an edge $e^+ \in V(G^+)$, then $u \in \xi$ and $e^+ \in C$. Since $(u, \theta)$ crosses $D$ exactly once, $\xi$ and $C$ are in two distinct components of $\mathbb{H}^2 \setminus D$.

(2) If $f^+ = (w, \theta)$, such that $w \in V(G^+)$ and $\theta$ is the midpoint of an edge $e \in V(G) = (u, v)$, then $u, v \in \xi$ and $w \in C$. Since $(w, \theta)$ crosses $D$ exactly once, $\xi$ and $C$ are in two distinct components of $\mathbb{H}^2 \setminus D$.

□

**Lemma 3.4.** *Let $\xi$ be an infinite cluster in $\omega$ and let $C$ be an infinite contour in $\phi_\omega^+$ such that $\xi$ and $C$ are incident. Then there exists a unique contour $D$ in $\eta_\omega$ that is incident to both $\xi$ and $C$.*

*Proof.* The existence of a contour $D$ in $\eta_\omega$ that is incident to both $\xi$ and $C$ follows directly from the definitions. More precisely, since $\xi$ and $C$ are incident, there exists $e = (u, v) \in E(G)$, such that $\omega(u) \neq \omega(v)$, $\phi_\omega^+(e^+) = 1$, $u \in \xi$ and $e^+ \in C$. Let $\theta$ be the midpoint of $(u, v)$, then $\eta_\omega((u, \theta)^+) = 1$. Therefore $(u, \theta)^+$ is in a contour $D$ of $\eta_\omega$ that is incident to both $\xi$ and $C$.

By Lemma 3.1, $D$ is either a self-avoiding cycle or a doubly infinite self-avoiding path. To show that $D$ is infinite, it suffices to exclude the possibility that $D$ is a self-avoiding cycle.

Suppose that $D$ is a self-avoiding cycle, then $\mathbb{H}^2 \setminus D$ has exactly two components, one bounded and the other unbounded. By Lemma 3.3, $C$ and $\xi$ must be in two distinct components of $\mathbb{H}^2 \setminus D$. Then one of them must be in the bounded component of $\mathbb{H}^2 \setminus D$, but this is impossible since the embedding of $G$ into $\mathbb{H}^2$ is proper. Therefore we conclude the existence of an infinite contour $D$ in $\eta_\omega$ incident to both $\xi$ and $C$.

Recall that each infinite contour in $\eta_\omega$ must be a doubly infinite self-avoiding path. If there exist more than two infinite contours in $\eta_\omega$, the two doubly infinite paths divide $\mathbb{H}^2$ into 3 distinct infinite components. The infinite contour $C$ in $\phi_\omega^+$ and the infinite cluster $\xi$ in $\omega$ can not be in all of the 3 components, and therefore the two doubly infinite paths cannot both be incident to $\xi$ and $C$. □

3.2. **Proof of Theorem 1.4.** Without loss of generality, assume that $\mu$ is ergodic. Suppose that $\mu((s_0, s_1) = (1, 1)) = 1$; we shall obtain a contradiction.

Let $\omega \in \{0, 1\}^{V(G)}$ be such that $(s_0, s_1) = (1, 1)$. Let $\xi_0$ be the infinite 0-cluster and $\xi_1$ be the infinite 1-cluster in $\omega$. The by Lemma 9.11, there exists a unique infinite contour $C$ in $\phi_\omega^+$ that is incident to both $\xi_0$ and $\xi_1$. Since $\xi_1$ is incident to $C$, by Lemma 3.4, there exists a unique infinite contour $L$ in $\eta_\omega$ that is incident to both $\xi_1$ and $C$.

Since $\overline{G}^+$ is a quasi-transitive planar graph, it is unimodular; see [19]. Then $L$ forms an invariant bond percolation on the non-amenable, quasi-transitive, unimodular graph $\overline{G}^+$ which has a single component a.s., but the critical percolation probability on $L$ is 1 since $L$ is a doubly-infinite self-avoiding path by Lemma 3.1. This contradiction to Lemma 2.7 implies the conclusion of the theorem. □

3.3. **Proof of Theorem 1.6.** We only prove Part (1) here; Part (2) can be proved using the same technique. Without loss of generality, assume that $\mu$ is ergodic. Assume that $s_0 = 1$ a.s.

Let $\omega \in \{0,1\}^{V(G)}$ be such that $s_0 = 1$. Let $\xi_0$ be the unique infinite 0-cluster. Let $G \setminus \xi_0$ be the subgraph of $G$ obtained by removing all the vertices in $\xi_0$ and edges incident to at least one vertex in $\xi_0$. Then the following cases might occur

(1) $G \setminus \xi_0$ has no infinite component $\mu$-a.s.
(2) $G \setminus \xi_0$ has exactly one infinite component $\mu$-a.s.
(3) $G \setminus \xi_0$ has at least two infinite components $\mu$-a.s.

If case (2) occurs, then we can construct a new configuration $\tilde{\omega} \in \{0,1\}^{V(G)}$ as follows:

$$\tilde{\omega}(v) = \begin{cases} 0, & \text{if } v \in \xi_0; \\ 1, & \text{otherwise.} \end{cases} \tag{3.1}$$

Then $\tilde{\omega}$ is an invariant site percolation on $G$ with a unique infinite 0-cluster and a unique infinite 1-cluster a.s. But this contradicts Theorem 1.4.

If case (3) occurs, again we construct the new configuration $\tilde{\omega}$ as in (3.1). Then $\tilde{\omega}$ has at least two distinct infinite 1-clusters $\zeta_1$ and $\zeta_2$, such that both $\zeta_1$ and $\zeta_2$ are adjacent to $\xi_0$. By Lemma 9.11, there exists an infinite contour $C_1$ in $\phi^+_{\tilde{\omega}}$ that is incident to both $\zeta_1$ and $\xi_0$, and an infinite contour $C_2$ in $\phi^+_{\tilde{\omega}}$ that is incident to both $\zeta_2$ and $\xi_0$.

We claim that $C_1 \neq C_2$. Indeed, if $C_1 = C_2$, then there exists a vertex $u \in V(G^+)$, such that

- $u$ is incident to two edges $e_1, e_2 \in E(G^+) \cap \phi^+_{\tilde{\omega}}$; and
- $e_1^+ = (x, y)$ with $x \in \xi_0$, and $y \in \zeta_1$; and
- $e_2^+ = (z, w)$, with $z \in \xi_0$, and $w \in \zeta_3$; and
- $\zeta_1$ and $\zeta_3$ are two distinct 1-clusters in $\tilde{\omega}$.

Since there is exactly one 0-cluster $\xi_0$ in $\tilde{\omega}$, moving along the boundary of the face $u^+$ in $G$ from $y$ to $w$ in each one of the two possible directions, we must both pass through a vertex in $\xi_0$, given that $y$ and $w$ are in two distinct 1-clusters $\zeta_1$ and $\zeta_3$ of $\tilde{\omega}$. If $\zeta_1$ and $\zeta_3$ satisfy the above conditions, we say that $\zeta_1$ and $\zeta_3$ are $*$-connected. See Figure 3.2.

Therefore $\zeta_3$ must be a finite 1-cluster. Moreover, we cannot find another vertex $v \in V(G^+)$, such that there are two edges $e_3$ and $e_4$ incident to $v$ in $G^+$, satisfying

- $e_3^+ = (a, b)$ with $a \in \xi_0$, and $b \in \zeta_3$; and
- $e_2^+ = (s, t)$, with $s \in \xi_0$, and $t \in \zeta_4$; and
- $\zeta_4$ is an infinite 1-cluster in $\tilde{\omega}$;

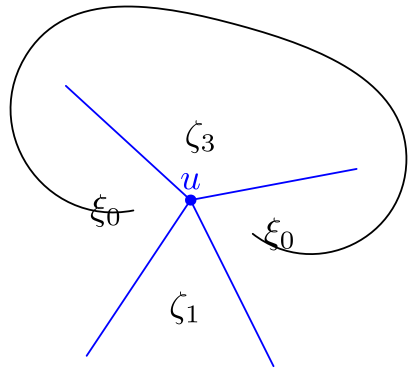


FIGURE 3.2. Incident to a vertex $u \in V(G^+)$, there are two distinct 1-clusters $\zeta_1$ and $\zeta_3$ in $\tilde{\omega}$, and a unique 0-cluster $\xi_0$. Contours in $\phi^+_{\tilde{\omega}}$ are represented by blue lines. The 0-cluster $\xi_0$ is represented by black lines.

because $\zeta_3$ is completely surrounded by $\xi_0$ except at $u$. Hence $\zeta_1$ can only be $*$-connected to finite 1-clusters, and any finite clusters that are $*$-connected to $\zeta_1$ cannot be $*$-connected to other infinite 1-clusters, in particular, they cannot be $*$-connected to $\zeta_2$. Therefore $C_1 \cap C_2 = \emptyset$, that is, $C_1$ and $C_2$ are distinct contours in $\phi^+_{\tilde{\omega}}$.

Then we can find a doubly-infinite self-avoiding path $L_1$ in $\overline{G}^+$ incident to both $\xi_0$ and $C_1$ and a doubly-infinite self-avoiding path $L_2$ in $\overline{G}^+$ incident to both $\xi_0$ and $C_2$, such that $L_1$ and $L_2$ are contours in $\eta_{\tilde{\omega}}$, $L_1 \cap L_2 = \emptyset$, by Lemmas 3.3 and 3.4.

We can also find two vertices $x, y \in V(G)$ such that $x \in \zeta_1$ and $y \in \zeta_2$, two vertices $u, v \in \xi_0$, such that $(x, u) \in E(G)$ and $(y, v) \in E(G)$. Since $\xi_0$ is connected, we can find a path $l_{uv}$ consisting of edges of $G$ and joining $u$ and $v$ such that all the vertices along the path are in $\xi_0$. Since

$$1 = \mu\{\cup_{n\geq 1}|l_{uv}| \leq n\} = \lim_{n\to\infty} \mu\{|l_{uv}| \leq n\},$$

there exists a sufficient large $N > 0$, such that

$$\mu\{|l_{uv}| \leq N\} \geq \frac{1}{2}.$$

Moreover, since $\mu$ is insertion-tolerant, we can make all the vertices on the path $l_{uv}$ to have state 1 and obtain a new configuration $\omega'$ such that

$$\mu\{\omega' \in \{0,1\}^{V(G)} : s_0(\omega) = 1\} > 0.$$

We claim that in $\omega'$, there are at least two distinct infinite 0-clusters. To see why that is true, give $l_{uv}$ an orientation from $u$ to $v$. Then $(x, u)$ crosses $L_1$ exactly once and splits $L_1$ into two parts: the left part $L_1^-$ and the right part $L_1^+$. Similarly $(y, v)$ splits $L_2$ into two parts: the left part $L_2^-$ and the right part $L_2^+$. Let $F^-$ (resp. $F^+$) be all the faces of $G$ intersecting $L_1^-$ (resp. $L_1^+$), and let $\eta^+ = [F^+ \cap \xi_0] \setminus \{u\}$, $\eta^- = [F^- \cap \xi_0] \setminus \{u\}$, then $\eta^+$ and $\eta^-$ are in two distinct infinite 0-clusters in $\omega'$. See Figure 3.3.

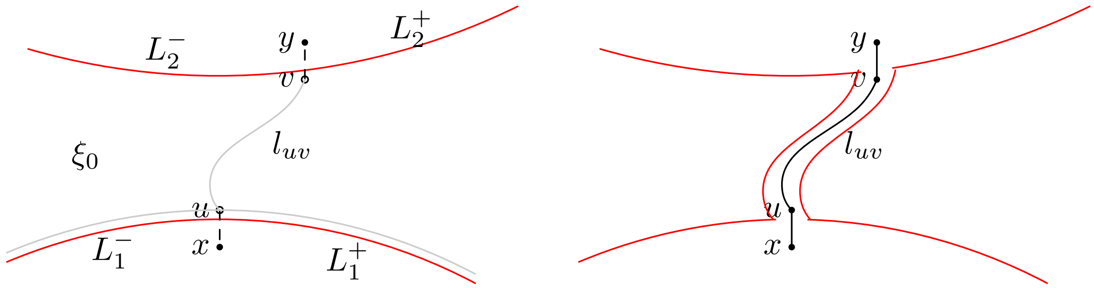


FIGURE 3.3. Site configurations $\tilde{\omega}$ and $\omega'$. The left graph represents $\tilde{\omega}$, and the right graph represents $\omega'$. Contours in $\eta_{\tilde{\omega}}$ and $\eta_{\omega'}$ are represented by red lines. State 1 vertices are represented by black dots, and state 0 vertices are represented by circles.

Then with positive probability there exist at least two infinite 0-clusters. This contradicts the assumption that $\mu$-a.s. $s_0 = 1$. Then cases (2) and (3) cannot occur. The theorem follows from case (1). $\square$

## 4. INEQUALITIES OF $p_c^{site}$ AND $p_u^{site}$

In this section, we prove two inequalities of $p_c^{site}$ and $p_u^{site}$, which are of independent interest and will also be used to prove Theorem 1.7. The idea to prove the first inequality, $p_c^{site}+p_u^{site} \geq 1$, is that by Lemma 2.9, we can enumerate all the 9 possible values of $(s_0, s_1)$, each one of which implies an inequality of $p_c^{site}$ and $p_u^{site}$; and the inequality $p_c^{site}+p_u^{site} \geq 1$ follows from Theorem 1.6. The idea to prove the second inequality $p_c^{site} < \frac{1}{2}$ when the vertex degree of the graph is at least 7, is to construct a coupling between the site configuration on $G$ and the site configuration on a regular triangular tiling of the hyperbolic plane in which each vertex has degree $d \geq 7$. The critical site percolation probability on the latter is known to be less than $\frac{1}{2}$; see [17].

**Lemma 4.1.** *Let $G$ be an infinite, connected, locally finite, quasi-transitive, non-amemable graph. Consider the i.i.d. Bernoulli site percolation on $G$. Let $\omega \in \{0,1\}^{V(G)}$, let $s_0$ (resp. $s_1$) be the total number of infinite 0-clusters (resp. infinite 1-clusters) in $\omega$. If for any $p \in [0,1]$, almost surely*

$$(s_0, s_1) \notin \{(1,1),(1,\infty),(\infty,1)\}, \tag{4.1}$$

*Then*

$$p_c^{site} + p_u^{site} \geq 1. \tag{4.2}$$

*Conversely if (4.2) holds and $p_c^{site} < p_u^{site}$ then (4.1) holds.*

*Proof.* Throughout the proof, we use $p_c$ (resp. $p_u$) to denote $p_c^{site}$ (resp. $p_u^{site}$). Let $\mu$ be the percolation probability measure on $\{0,1\}^{V(G)}$. Let $p$ be the probability that $\omega(v) = 1$

for each vertex $v \in V(G)$. Then for all $p \in [0,1]$, we have $\mu$-a.s.

$$(s_0, s_1) \in \{(0,0),(0,1),(0,\infty),(1,0),(1,1),(1,\infty),(\infty,0),(\infty,1),(\infty,\infty)\}. \tag{4.3}$$

More precisely, when $p = 0$, $(s_0, s_1) = (1, 0)$; when $p = 1$, $(s_0, s_1) = (0, 1)$; when $p \in (0, 1)$, i.i.d. Bernoulli percolation is insertion-tolerant, and (4.3) follows from Lemma 2.9.

We shall analyze each case. By symmetry it suffices to consider whether or not there exists $p \in [0,1]$, s.t.

$$(s_0, s_1) \in \{(0,0),(0,1),(0,\infty),(1,1),(1,\infty),(\infty,\infty)\}.$$

The following cases might occur

1. If there exists $p \in [0,1]$, such that $\mu$-a.s., $(s_0, s_1) = (0, 0)$, then $p \leq p_c \leq p_u$, $1 - p \leq p_c \leq p_u$, and therefore $p_c + p_u \geq 1$ and $p_c \geq \frac{1}{2}$.
2. If there exists $p \in [0,1]$, such that $\mu$-a.s., $(s_0, s_1) = (0, 1)$, then $1 - p \leq p_c \leq p_u$, $p_c \leq p_u \leq p$,
3. If there exists $p \in [0,1]$, such that $\mu$-a.s., $(s_0, s_1) = (0, \infty)$, then $1 - p \leq p_c \leq p_u$, $p_c < p \leq p_u$, and therefore $p_c + p_u \geq 1$.
4. If there exists $p \in [0,1]$, such that $\mu$-a.s., $(s_0, s_1) = (1, 1)$, then $p_u \leq p$, $p_u \leq 1 - p$, and therefore $p_c \leq p_u \leq \frac{1}{2}$.
5. If there exists $p \in [0,1]$, such that $\mu$-a.s., $(s_0, s_1) = (1, \infty)$, then $p_u \leq 1 - p$, $p_c < p \leq p_u$, and therefore $p_c + p_u < 1$.
6. If there exists $p \in [0,1]$, such that $\mu$-a.s., $(s_0, s_1) = (\infty, \infty)$, then $p_c < p \leq p_u$, $p_c < 1 - p \leq p_u$, and therefore $p_c < \frac{1}{2} \leq p_u$.

Assume that $p_c + p_u < 1$; we shall obtain a contradiction. Let $\epsilon := 1 - p_c - p_u > 0$, and $p = 1 - p_u - \frac{\epsilon}{2}$. Then we have

$$1 - p > p_u; \qquad \text{and } p = p_c + \frac{\epsilon}{2} > p_c;$$

there for $\mu$-a.s. $(s_0, s_1) \in \{(1, \infty), (1, 1)\}$. But this is a contradiction to the assumption. Thus we must have $p_c + p_u \geq 1$. ☐

Recall that in [5], it is proved that for the i.i.d. Bernoulli bond percolation on a transitive, non-amenable, planar graph $G$ with one end $p_c^{bond}(G^+) + p_u^{bond}(G) = 1$. Here by planar duality, we prove an analogous result for the i.i.d. Bernoulli site percolation.

**Proposition 4.2.** *Let $G$ be a connected, locally finite, vertex-transitive, non-amemable, planar graph with one end. Consider the i.i.d. Bernoulli site percolation on $G$. Then*

$$p_c^{site} + p_u^{site} \geq 1$$

*Proof.* By Lemma 4.1, it suffices to show that for any percolation probability $p \in [0,1]$, (4.1) holds. When $p = 0$, a.s. $(s_0, s_1) = (1, 0)$. When $p = 1$, a.s $(s_0, s_1) = (0, 1)$. When $p \in (0, 1)$, the i.i.d. Bernoulli site percolation is both insertion tolerant and deletion tolerant. Hence the proposition follows from Theorem 1.6. ☐

**Lemma 4.3.** *Let $G$ be a simple, transitive graph that can be properly embedded into the hyperbolic plane $\mathbb{H}^2$. Assume that the vertex degree in $G$ is at least 7, then $p_c^{site}(G) < \frac{1}{2}$.*

*Proof.* Let $G_T$ be a vertex-transitive, regular tiling of the hyperbolic plane by triangles in which each vertex has degree has at least 7, then the i.i.d. Bernoulli($\frac{1}{2}$) site percolation has infinitely many infinite 1-clusters and infinitely many infinite 0-clusters by Lemma 2.8. Hence in this case $p_c^{site}(G_T) \leq \frac{1}{2}$. It is also known that for a non-amenable graph with a transitive, unimodular automorphism group, at $p_c^{site}$ there are no infinite clusters in the i.i.d. Bernoulli site percolation a.s. by Lemma 2.3. Then we obtain in this case $p_c^{site}(G_T) < \frac{1}{2}$.

Assume $G$ and $G_T$ have the same vertex degree and percolation occurs for a site configuration $\omega \in \{0,1\}^{V(G_T)}$. We shall construct a coupling between $G$ and $G_T$ as follows. Let $v_0 \in V(G_T)$ be a vertex in an infinite 1-cluster of $\omega$. We shall construct a configuration $\eta \in \{0,1\}^{V(G)}$ by induction. Let $w_0 \in V(G)$ and make $\eta(w_0) = 1$. Let $v_1, v_2, \ldots, v_d$ be all the neighboring vertices of $v_0$ in counterclockwise order and let $w_1, \ldots, w_d$ be all the neighboring vertices of $w_0$ in counterclockwise order. Make $\eta(w_i) = \omega(v_i)$. Let

$$\begin{aligned} A_1 &= \{v_0\} \cup \{v_i : i \in [d], \omega(v_i) = 1\} \\ B_1 &= \{w_0\} \cup \{w_i : i \in [d], \eta(v_i) = 1\} \\ C_1 &= \{v_i : i \in [d], \omega(v_i) = 0\} \\ D_1 &= \{w_i : i \in [d], \eta(v_i) = 0\} \end{aligned}$$

where

$$[d] = \{1, 2, \ldots, d\}$$

Let $\alpha(v_i) = w_i$, for $i \in \{0\} \cup [d]$. Then $\alpha$, restricted to $A_1$, is a 1-to-1 correspondence between $A_1$ and $B_1$; restricted to $C_1$, is a 1-to-1 correspondence between $C_1$ and $D_1$. Let $\mathcal{N}(\cdot)$ denote the set of all the neighboring vertices of a vertex. For each $v \in A_1$, the vertex set $\mathcal{N}(v) \cap [V(G_T) \setminus [A_1 \cup C_1]]$ has an injection into the vertex set $\mathcal{N}(\alpha(v)) \cap [V(G) \setminus [B_1 \cup D_1]]$. More precisely, let $u$ be a neighboring vertex of $v_i$ for some $i \in [d]$, such that $u \notin [A_1 \cup C_1]$, then the corresponding vertex of $u$ in $\mathcal{N}(\alpha(v_i)) \cap [V(G) \setminus [B_1 \cup D_1]]$ can be obtained as follows. Let $[e_1, \ldots, e_d]$ be all the incident edges of $v_i$ in counterclockwise order such that $e_1 = (v_0, v_i)$ and $e_j = (u, v_i)$. Let $[f_1, f_2, \ldots, f_d]$ be all the incident edges of $w_i$ in counterclockwise order such that $f_1 = (w_0, w_i)$. Then the corresponding vertex $x$ of $u$ is the endpoint of $f_j$ other than $w_i$. Obviously $x \in \mathcal{N}(\alpha(v_i))$. To see why $x \notin [B_1 \cup D_1]$, first of all $x \neq w_0$, since the graph $G$ is simple and therefore has no length-2 loop, and obviously $x \neq w_i$, since the graph $G$ has no self-loop. If $x = w_k$ for some $k \in [d] \setminus \{i\}$, then $w_k (= x), w_0, w_i$ form a degree-3 face in $G$, then $k \in \{i-1, i+1\}$ (here we take the convention that $d + 1 = 1$). More precisely,

(1) If $k = i+1$, $(x, w_i) = f_d$, then $(v_i, u) = e_d$. Hence $e_d = (v_i, u)$ and $e_0 = (v_i, v_0)$ are the boundary of a degree-3 face in $G_T$, that means $u$ is adjacent to $v_0$, hence $u \in A_1 \cup B_1$, which is a contradiction.

(2) If $k = i-1$, $(x, w_i) = f_2$, then $(v_i, u) = e_2$. Hence $e_2 = (v_i, u)$ and $e_0 = (v_i, v_0)$ are the boundary of a degree-3 face in $G_T$, that means $u$ is adjacent to $v_0$, hence $u \in A_1 \cup B_1$, which is a contradiction.

Therefore we have $x \in \mathcal{N}(\alpha(v_i)) \cap [V(G) \setminus [B_1 \cup D_1]]$. It is straightforward to check that that for two distinct vertices $u_1, u_2 \in \mathcal{N}(v_i) \cap [V(G_T) \setminus [A_1 \cup C_1]]$, they do not correspond to the same vertex in $\mathcal{N}(\alpha(v_i)) \cap [V(G) \setminus [B_1 \cup D_1]]$ by the construction above.

Now let $n \geq 1$ be a positive integer. Assume that we have determined

$$A_1 \subset A_2 \subset \cdots \subset A_n \subset V(G_T),$$
$$B_1 \subset B_2 \subset \cdots \subset B_n \subset V(G),$$
$$C_1 \subset C_2 \subset \cdots \subset C_n \subset V(G_T),$$
$$D_1 \subset D_2 \subset \cdots \subset D_n \subset V(G)$$

such that

(1) there is a 1-to-1 correspondence $\alpha : A_n \cup C_n \to B_n \cup D_n$, such that $\alpha(A_i) = B_i$, $\alpha(C_i) = D_i$ for all $i \in [n]$; and
(2) for each $z \in A_n \cup C_n$, there is a self-avoiding path

$$z_{s_0}(:= v_0), z_{s_1}, \ldots, z_{s_m}(:= z),$$

such that
   (a) for all $i \in [m]$, $z_{s_{i-1}}$ and $z_{s_i}$ are adjacent vertices in $G_T$;
   (b) for all $i \in [m]$, $z_{s_i} \in [A_{s_i} \cup C_{s_i}] \setminus [A_{s_i-1} \cup C_{s_i-1}]$
   (c) for all $i \in [m]$, $n \geq s_i > s_{i-1} \geq 0$
   (d) $z_{s_i}$ is obtained as a neighboring vertex of $z_{s_{i-1}}$ in the $s_i$th step of the induction.
(3) for each $z \in A_n$, such that $\mathcal{N}(z) \cap [V(G_T) \setminus [A_n \cup B_n]] \neq \emptyset$, there is an injection from $\mathcal{N}(z) \cap [V(G_T) \setminus [A_n \cup C_n]]$ to $\mathcal{N}(\alpha(z)) \cap [V(G) \setminus [B_n \cup D_n]]$ as follows. Let $w \in \mathcal{N}(z) \cap [V(G_T) \setminus [A_n \cup C_n]]$. Let $e_1, \ldots, e_d$ be all the incident edges of $z$ in $G_T$ in counterclockwise order, such that $e_1 = (z_{s_{m-1}}, z)$, and $[z, w] = e_j$. Let $f_1, \ldots, f_d$ be all the incident edges of $\alpha(z)$ in $G$ in counterclockwise order, such that $f_1 = (\alpha(z_{s_{m-1}}), \alpha(z))$, then the corresponding vertex for $w$ in $\mathcal{N}(\alpha(z)) \cap [V(G) \setminus [B_n \cup D_n]]$ is the endpoint of $f_j$ other than $\alpha(z)$.

Now let $z$ be an arbitrary vertex in $A_n$ such that $\mathcal{N}(z) \cap [V(G_T) \setminus [A_n \cup C_n]] \neq \emptyset$, and let $w \in \mathcal{N}(z) \cap [V(G_T) \setminus [A_n \cup C_n]]$. Find the corresponding vertex $y$ for $w$ in $\mathcal{N}(\alpha(z)) \cap [V(G) \setminus [B_n \cup D_n]]$ as described above, and make $\eta(y) = \omega(w)$. The following cases might occur:

(1) If $\eta(y) = \omega(w) = 1$, let $A_{n+1} = A_n \cup \{w\}$, $B_{n+1} = B_n \cup \{y\}$, $C_{n+1} = C_n$, $D_{n+1} = D_n$.
(2) If $\eta(y) = \omega(w) = 0$, let $A_{n+1} = A_n$, $B_{n+1} = B_n$, $C_{n+1} = C_n \cup \{w\}$, $D_{n+1} = D_n \cup \{y\}$.

Let $\alpha(w) = y$. Then we can check that

(1) there is a 1-to-1 correspondence $\alpha : A_{n+1} \cup C_{n+1} \to B_{n+1} \cup D_{n+1}$, such that $\alpha(A_i) = B_i$, $\alpha(C_i) = D_i$ for all $i \in [n+1]$; and

(2) for each $z \in A_{n+1} \cup C_{n+1}$, there is a self-avoiding path

$$z_{s_0}(:= v_0), z_{s_1}, \ldots, z_{s_m}(:= z),$$

such that

(a) for all $i \in [m]$, $z_{s_{i-1}}$ and $z_{s_i}$ are adjacent vertices in $G_T$;
(b) for all $i \in [m]$, $z_{s_i} \in [A_{s_i} \cup C_{s_i}] \setminus [A_{s_i-1} \cup C_{s_i-1}]$
(c) for all $i \in [m]$, $n+1 \geq s_i > s_{i-1} \geq 0$
(d) $z_{s_i}$ is obtained as a neighboring vertex of $z_{s_{i-1}}$ in the $s_i$th step of the induction.

(3) for each $z \in A_{n+1}$, such that $\mathcal{N}(z) \cap [V(G_T) \setminus [A_{n+1} \cup C_{n+1}]] \neq \emptyset$, there is an injection from $\mathcal{N}(z) \cap [V(G_T) \setminus [A_{n+1} \cup C_{n+1}]]$ to $\mathcal{N}(\alpha(z)) \cap [V(G) \setminus [B_{n+1} \cup D_{n+1}]]$ as follows. Let $w \in \mathcal{N}(z) \cap [V(G_T) \setminus [A_{n+1} \cup C_{n+1}]]$. Let $e_1, \ldots, e_d$ be all the incident edges of $z$ in $G_T$ in counterclockwise order, such that $e_1 = (z_{s_{m-1}}, z)$, and $[z, w] = e_j$. Let $f_1, \ldots, f_d$ be all the incident edges of $\alpha(z)$ in $G$ in clockwise order, such that $f_1 = (\alpha(z_{s_{m-1}}), \alpha(z))$, then the corresponding vertex $x$ for $w$ in $\mathcal{N}(\alpha(z)) \cap [V(G) \setminus [B_{n+1} \cup D_{n+1}]]$ is the endpoint of $f_j$ other than $\alpha(z)$.

To check that the vertex $x \notin B_{n+1} \cup D_{n+1}$, assume that $x \in B_{n+1} \cup D_{n+1}$; we shall find a contradiction. Since $x \in B_{n+1} \cup D_{n+1}$, we have $a := \alpha^{-1}(x) \in A_{n+1} \cup C_{n+1}$. Therefore there exists a self-avoiding path

$$a_{t_0}(:= v_0), a_{t_1}, \ldots, a_{t_k}(:= a),$$

such that

(a) for all $i \in [m]$, $a_{t_{i-1}}$ and $a_{t_i}$ are adjacent vertices in $G_T$;
(b) for all $i \in [m]$, $a_{t_i} \in [A_{t_i} \cup C_{t_i}] \setminus [A_{t_i-1} \cup C_{t_i-1}]$
(c) for all $i \in [m]$, $n+1 \geq t_i > t_{i-1} \geq 0$
(d) $a_{t_i}$ is obtained as a neighboring vertex of $a_{t_{i-1}}$ in the $t_i$th step of the induction.

Let

$$h := \max_{0 \leq i \leq k} \{i : a_{t_i} = z_{s_j}, \text{ for some } j \in [m]\}$$

Assume $a_{t_h} = z_{s_g}$. Then the path

$$a_k, a_{k-1}, \ldots, a_{t_h}, z_{s_{g+1}}, \ldots, z_{s_m}, w \tag{4.4}$$

in $G_T$ is self-avoiding. However, the path

$$\alpha(a_k), \alpha(a_{k-1}), \ldots, \alpha(a_{t_h}), \alpha(z_{s_{g+1}}), \ldots, \alpha(z_{s_m}), x \tag{4.5}$$

form a cycle in $G$ since $\alpha(a_k) = x$. Moreover, the cycle (4.5) is self-avoiding because $\alpha$ is 1-to-1. But if (4.5) is a self-avoiding cycle, then it is the outer boundary of a simply-connected region formed by the union of finitely many faces in $G$, since $G$ can be properly embedded into $\mathbb{H}^2$. Recall also that we obtain (4.5) from (4.4) by preserving the relative positions of incident edges at each vertex, and each vertex in $G$ and $G_T$ have the same degree, but each face in $G$ has degree at least 3 - the degree of each face in $G_T$. Therefore if (4.5) is a cycle, (4.4) cannot be self-avoiding. The contradiction implies that $x \notin B_{n+1} \cup D_{n+1}$.

The coupling process above shows that if in $G_T$, there is a strictly positive probability that $v_0$ is in an infinite 1-cluster, then with at least the same probability in $G$, $w_0$ is in an infinite 1-cluster, if each vertex in $G_T$ and $G$ has the same probability $p$ to be open. This implies

$$p_c^{site}(G) \leq p_c^{site}(G_T).$$

Combining with the result that $p_c^{site}(G_T) < \frac{1}{2}$, the lemma follows. □

## 5. Two-ended Graphs

The aim of this section is to prove the following results about i.i.d. Bernoulli percolation on a class of two-ended graphs.

**Theorem 5.1.** *Let $G = (V(G), E(G))$ be a connected, locally finite, quasi-transitive graph with two ends. Consider the i.i.d. Bernoulli percolation on $G$. Then the critical percolation probabilities satisify*

$$p_c^{bond}(G) = p_c^{site}(G) = p_u^{bond}(G) = p_u^{site}(G) = 1.$$

In particular, Theorem 5.1 implies that for the the i.i.d. Bernoulli percolation on a connected, locally finite, quasi-transitive, planar graph $G$ with two ends, if each vertex is open with probability $\frac{1}{2}$, then almost surely there are no infinite 1-clusters, that is, percolation does not occur.

We first recall the following proposition is proved in [1].

**Proposition 5.2.** *If an infinite, locally finite, quasi-transitive graph has two ends then it has linear growth rate.*

For $v \in V(G)$ and a positive integer $r$, let $B_G(v, r)$ consists of all the vertices in $G$ whose graph distance to the vertex $v$ is at most $r$. If a quasi-transitive graph $G$ has linear growth rate, then there exist constants $C_1, C_2 > 0, D_1, D_2 \geq 0$ such that for all $v \in V(G)$,

$$C_1 r + D_1 \leq |B_G(v, r)| \leq C_2 r + D_2 \tag{5.1}$$

We may write

$$B_G(v, r) = S_0 \cup S_1 \cup S_2 \cup \cdots S_r$$

where for $1 \leq i \leq r$, $S_i$ consists of all the vertices whose graph distance to the vertex $v$ is exactly $i$. To prove Theorem 5.1, it suffices to show that for i.i.d. Bernoulli percolation on a locally finite, quasi-transitive, connected graph $G$ with two ends, if each vertex has probability $p \in [0, 1)$ to be open, then a.s. percolation does not occur.

Let

$$\begin{aligned} I_{1,r} &= \{0 \leq i \leq r : |S_i| \geq 2(C_2 + D_2)\} \\ I_{2,r} &= \{0 \leq i \leq r : |S_i| < 2(C_2 + D_2)\} \end{aligned}$$

By (5.1),

$$|I_1| \leq \frac{r+1}{2}, \qquad |I_2| \geq \frac{r+1}{2}$$

For $0 \leq i$, let $E_i$ be the event that all the vertices in $S_i$ have state 0. Note that $\{E_i\}_{i=1}^{\infty}$ are mutually independent. If percolation occurs a.s., the probability that $E_i$ occurs infinitely many times is 0. Note that

$$\mathbb{P}(E_i) = (1-p)^{|S_i|}$$

Therefore

$$\sum_{i=1}^{\infty} \mathbb{P}(E_i) \geq \lim_{r\to\infty} \sum_{i\in I_{2,r}} \mathbb{P}(E_i) \geq \lim_{r\to\infty} \sum_{i\in I_{2,r}} (1-p)^{2(C_2+D_2)} \geq (1-p)^{2(C_2+D_2)} \lim_{r\to\infty} \frac{r+1}{2} = \infty$$

for all $p<1$. By Borel-Contelli lemma, if $p<1$, then a.s. $E_i$ occurs infinitely many times, then percolation does not occur. Then Theorem 5.1 follows.

## 6. Graphs with Infinitely Many Ends

The goal of this section is to prove the following theorem about i.i.d. Bernoulli percolation on a class of graphs with infinitely many ends.

**Theorem 6.1.** *Let $G = (V(G), E(G))$ be a connected, locally finite, quasi-transitive graph with infinitely many ends. Consider the i.i.d. Bernoulli percolation on $G$, then $p_u^{site}(G) = p_u^{bond}(G) = 1$.*

The case for an independent bond percolation on a transitive graph with infinitely many ends was proved in [23]; see also [13]. In the same spirit, we prove the case for an independent bond or site percolation on a quasi-transitive graph.

Let $p \in (0,1)$. Again let $\mu$ be the probability measure for the Bernoulli($p$) percolation on $G$. Since $\mu$ is ergodic, it suffices to show that $\mu$-a.s. the number of infinite 1-clusters is not 1.

Let $\Lambda$ be a finite connected subgraph of $G$ such that $G\setminus\Lambda$ has $k\geq 3$ infinite components, $Y_1,\ldots,Y_k$. Let $d$ be the diameter of $\Lambda$ and $s$ the greatest distance between orbits of $V(G)$ under the action of $\Gamma$. Then there is a copy $\Lambda_i$ of $\Lambda$ (under some automorphism) inside each $Y_i$, at distance $\leq d+s+1$ from $\Lambda$. Now $\Lambda_i$ splits $G$ into $k$ infinite components, one of which contains $\Lambda$ and all the $Y_j$ except $Y_i$. Then $G\setminus[\Lambda\cup_{i=1}^{k}\Lambda_i]$ has at least $k(k-1)$ infinite components.

This way we can construct a $k$-regular tree by contracting each one of $\Lambda$, $\Lambda_1,\ldots,\Lambda_k$ into a vertex, and joining an edge between $\Lambda$ and $\Lambda_i$, for $1\leq i\leq k$. In each one of the $k-1$ infinite components of $\Lambda_i$ which do not contain $\Lambda$, we can find a copy $\Lambda_{i,j}$ $(1\leq j\leq k-1)$ of $\Lambda$ with distance $\leq d+s+1$ from $\Lambda_i$, such that $\Lambda_{i,j}$ splits $G$ into $k$ infinite components, one of which contains $\Lambda_1,\ldots,\Lambda_k$ and $\Lambda$. Then we contract each one of $\Lambda_{i,j}$ into a vertex, and join an edge between $\Lambda_i$ and $\Lambda_{i,j}$ for $1\leq i\leq k, 1\leq j\leq k-1$. Continuing this process we obtain an infinite $k$-regular tree $T$.

We say $\Lambda_{i_1,\ldots,i_t}$ for $t \geq 0$ is open if there is at least one vertex (or edge for bond percolation) in $\Lambda_{i_1,\ldots,i_t}$ open. Then

$$\mu(\Lambda_{i_1,\ldots,i_t} \text{ is open}) = 1-(1-p)^{|\Lambda|} \in (0,1) \tag{6.1}$$

when $p \in (0,1)$.

Let

$$\overline{\Lambda} := \{\Lambda\} \cup_{t=1}^{\infty} \cup_{i_1\in[k],i_2,\ldots,i_t\in[k-1]}\{\Lambda_{i_1,\ldots,i_t}\};$$

in other words, $\overline{\Lambda}$ consists of all the vertices of the tree $T$.

For $u, v \in V(G)$, let $u \leftrightarrow v$ be the event that $u$ and $v$ are joined by a path in $G$, such that each vertex, or edge along the path has state 1. Define

$$\bar{p}_{conn}(G) = \sup\left\{p : \inf_{u,v\in V(G)}\mu(u \leftrightarrow v) = 0\right\}.$$

Let $x, y \in V(G)$ be two vertices in two copies $\Lambda_x, \Lambda_y$ of $\Lambda$ in $\overline{\Lambda}$. There is a unique path $l_{xy}$ in $T$ joining $\Lambda_x$ and $\Lambda_y$. If $x \leftrightarrow y$ in $G$, then every copy of $\Lambda$ in $\overline{\Lambda}$ along $l_{xy}$ must be open. Therefore for all $p \in (0,1)$,

$$\mu(x \leftrightarrow y) \leq \left(1-(1-p)^{|\Lambda|}\right)^{|l_{x,y}|} \to 0$$

as $|l_{xy}| \to 0$ by (6.1). Then we have for all $p \in (0,1)$,

$$\bar{p}_{conn}(G) = 1.$$

**Lemma 6.2.** *Let $G$ be a quasi-transitive graph with countably many vertices. Consider the i.i.d. Bernoulli(p) (bond or site) percolation on $G$. Assume $p > p_c$. Let $\mu$ be the corresponding probability measure. Then*

$$\inf_{x\in V(G)} \mu\{x \leftrightarrow \infty\} > 0.$$

*Proof.* Since $G$ is quasi-transitive, the action of $\mathrm{Aut}(G)$ on $V(G)$ has finitely many orbits. Let $W \subset V(G)$ be a finite set of vertices consisting of exactly one representative in each orbit. Let $d_W$ be the diameter of $W$.

When $p > p_c$, a.s. the i.i.d. Bernoulli($p$) percolation on $G$ has an infinite 1-cluster. More precisely,

$$\begin{aligned} 1 &= \mu(\text{there is an infinite } 1-\text{cluster}) \\ &= \mu(\cup_{x\in V(G)}\{x \leftrightarrow \infty\}) \\ &\leq \sum_{x\in V(G)} \mu(x \leftrightarrow \infty) \end{aligned}$$

Since $G$ has countably many vertices and is quasi-transitive,

$$\max_{x\in W} \mu(x \leftrightarrow \infty) = \max_{x\in V(G)} \mu(x \leftrightarrow \infty) > 0$$

Let $y \in V(G)$. By quasi-transitivity, there exists $z \in V(G)$ whose graph distance to $y$ is at most $d_W$, such that

$$\mu(z \leftrightarrow \infty) > 0.$$

There exists a path in $G$ joining $y$ and $z$ with distance at most $d_W$. Then by the F.K.G. inequality (2.1)

$$\mu(y \leftrightarrow \infty) \geq \mu(y \leftrightarrow z, z \leftrightarrow \infty) \geq \mu(y \leftrightarrow z)\mu(z \leftrightarrow \infty) \geq p^{d_W}\mu(z \leftrightarrow \infty).$$

Hence we have

$$\min_{y \in V(G)} \mu(y \leftrightarrow \infty) \geq p^{d_W} \max_{z \in V(G)} \mu(z \leftrightarrow \infty) > 0.$$

Then the lemma follows. □

If $p > p_u$, by Lemma 2.4, there exists a unique infinite 1-cluster. In this case if both $x$ and $y$ are in the unique infinite 1-cluster, then $x \leftrightarrow y$. Therefore

$$\mu(x \leftrightarrow y) \geq \mu(x \leftrightarrow \infty, y \leftrightarrow \infty).$$

By F.K.G inequality (2.1), we have

$$\mu(x \leftrightarrow \infty, y \leftrightarrow \infty) \geq \mu(x \leftrightarrow \infty)\mu(y \leftrightarrow \infty) \geq \left[\inf_{z \in V(G)} \mu(z \leftrightarrow \infty)\right]^2$$

By Lemma 6.2, $\left[\inf_{z \in V(G)} \mu(z \leftrightarrow \infty)\right]^2$ is a positive constant independent of $x$ and $y$. Hence whenever $p > p_u$, $\inf_{x,y \in V(G)} \mu(x \leftrightarrow \infty, y \leftrightarrow \infty) > 0$, and therefore

$$p_u \geq \bar{p}_{conn}(G) = 1$$

Here $p_u$ represents either $p_u^{site}$ or $p_u^{bond}$. This completes the proof of Theorem 6.1.

## 7. Amenable Planar Graphs with One End

In this section, we prove that for the i.i.d. Bernoulli$\left(\frac{1}{2}\right)$ site percolation on any connected, locally finite, transitive, planar, amenable graph with one end, a.s. percolation does not occur.

By Lemmas 2.1 and 2.2, we can enumerate all the locally finite, connected, vertex-transitive, planar, amenable graph with one end by enumerating the vector $[m_1, \ldots, m_d]$ representing the degrees of all the faces each vertex is incident to in a cyclic order. It is know that there are exactly 11 Archimedean tilings of the Euclidean plane $\mathbb{R}^2$:

1. $d = 3$: [6,6,6], [3,12,12],[4,6,12],[4,8,8];
2. $d = 4$: [4,4,4,4],[3,6,3,6],[3,4,6,4];
3. $d = 5$: [3,3,3,3,6],[3,3,4,3,4],[3,3,4,3,4];
4. $d = 6$: [3,3,3,3,3,3]

Indeed, numerical experiments show that each one of these 11 Archimedean tilings of the Euclidean plane has $p_c^{site} \geq \frac{1}{2}$, and the equality holds only for the $[3, 3, 3, 3, 3, 3]$ lattice. Below we shall prove this fact by universal arguments that work for all the 11 tilings.

**Lemma 7.1.** *Let $G$ be one of the 11 Archimedean tilings of the Euclidean plane, as listed above. Consider the i.i.d. Bernoulli $\left(\frac{1}{2}\right)$ site percolation on $G$. Then a.s. there are no infinite 1-clusters.*

*Proof.* The proof makes use of symmetry and a "crossing" argument; see proof of Theorem (11.12) in [10], in which the case for bond percolation on the square grid is proved. Suppose that the i.i.d. Bernoulli $\left(\frac{1}{2}\right)$ site percolation on $G$ has infinite-1 clusters, then a.s. there is a unique infinite cluster since the graph is amenable; see [6, 9].

Note that the graph $G$ is invariant under translations of $\mathbb{Z}^2$. A fundamental domain is a subgraph of $G$, which is the quotient under the action of $\mathbb{Z}^2$ on $G$. Let $n$ be a positive integer. Let $T(n)$ be a box consists of $n \times n$ fundamental domains, such that the boundary $\partial T(n)$ of $T(n)$ does not pass through any vertex of $G$. Assume that $\partial T(n)$ can be divided into four congruent parts $(l, t, r, b)$ in cyclic order, such that

- $\partial T_n = l \cup t \cup r \cup b$; and
- any two parts in $\{l, t, r, b\}$ are non-overlapping except at endpoints; and
- there exists an automorphism of $G$ which can be extended to an isometry of $\mathbb{R}^2$ and maps $(l, t, r, b)$ to $(t, l, b, r)$; and
- Let $E_l$ (resp. $E_t$, $E_r$, $E_b$) be the set of all the edges in $E(G)$ crossing $l$ (resp. $t$, $r$, $b$), then $E_l$, $E_t$, $E_r$, $E_b$ are pairwise disjoint.

This can be done for each one of the 11 Archimedean tilings of the Euclidean plane.

Let $A^l(n)$ (resp. $A^r(n)$, $A^t(n)$, $A^b(n)$) be the event that both endpoints of some edge in $E_l$ (resp. $E_r$, $E_t$, $E_b$) boundary of $T(n)$ lies in an open infinite path of $G$ which uses no other vertex in $T(n)$. Note that $A^l(n)$, $A^r(n)$, $A^t(n)$ and $A^b(n)$ are increasing events having equal probability and whose union is the event that some vertex in $T(n)$ lies in an infinite open cluster. Since a.s. there exists an infinite open cluster, we obtain

$$\lim_{n\to\infty} \mu(A^l(n) \cup A^r(n) \cup A^t(n) \cup A^b(n)) = 1$$

Therefore by the F.K.G. inequality (2.1),

$$\begin{aligned} 0 &= \lim_{n\to\infty} \mu([A^l(n)]^c \cap [A^r(n)]^c \cap [A^t(n)]^c \cap [A^b(n)]^c) \\ &\geq \lim_{n\to\infty} \{\mu([A^u(n)]^c)\}^4 \\ &= \lim_{n\to\infty} \{1 - \mu([A^u(n)])\}^4 \end{aligned}$$

for $u \in \{l, r, t, b\}$, where $[\cdot]^c$ denotes the complement of an event. Hence we have

$$\lim_{n\to\infty} \mu(A^u(n)) = 1.$$

Then there exists a sufficiently large $N$, such that for each $u \in \{l, r, t, b\}$

$$\mu(A^u(N)) > \frac{7}{8}.$$

Similarly, let $B^l(n)$ (resp. $B^r(n)$, $B^t(n)$, $B^b(n)$) be the event that both endpoints of some edge in $E_l$ (resp. $E_r$, $E_t$, $E_b$) boundary of $T(n)$ lies in a closed infinite path of $G$

which uses no other vertex in $T(n)$. Then by symmetry we have for each $u \in \{l, r, t, b\}$,

$$\mu(B^u(N)) = \mu(A^u(N)) > \frac{7}{8}.$$

Now consider the event

$$F := A^l(N) \cap A^r(N) \cap B^t(N) \cap B^b(N),$$

i.e.,

- there exists an infinite open path starting from an edge crossing the $l$ boundary of $T(n)$ using no other vertices in $T(n)$; and
- there exists an infinite open path starting from an edge crossing the $r$ boundary of $T(n)$ using no other vertices in $T(n)$; and
- there exists an infinite closed paths starting from an edge crossing the $t$ boundary of $G_n$ using no other vertices in $T(n)$; and
- there exists an infinite closed paths starting from an edge crossing the $b$ boundary of $G_n$ using no other vertices in $T(n)$.

The probability that $F$ does not occur satisfies

$$\begin{aligned}\mu(F^c) &= \mu([A^l(N)]^c \cup [A^r(N)]^c \cup [B^t(N)]^c \cup [B^b(N)]^c) \\ &\leq \mu([A^l(N)]^c) + \mu([A^r(N)]^c) + \mu([B^t(N)]^c) + \mu([B^b(N)]^c) \\ &< \frac{1}{2}.\end{aligned}$$

Therefore $P(F) > \frac{1}{2}$. If $F$ occurs then $G \setminus T(N)$ contains two disjoint infinite open clusters, since the two open clusters are separated physically by infinite closed paths of the dual. Similarly $G \setminus T(N)$ contains two disjoint infinite closed clusters. Since $G$ contains a.s. a unique infinite open cluster and a unique infinite closed cluster, conditional on $F$, there exists an open connection of $G$ between the two fore-mentioned infinite open clusters. This connection forms a barrier to possible closed connections of $G$ joining the two infinite closed clusters. Hence, conditional on $A$, almost surely there exists two or more closed clusters in $G$. This contradiction to the uniqueness of infinite closed clusters (see [6]) implies the almost sure non-existence of infinite open clusters in $G$; then the lemma follows. □

## 8. Proof of Theorems 1.7 and 1.8

In this section, we prove Theorems 1.7 and 1.8. The idea is to classify the all the planar graphs by the number of ends as well as amenability, and apply results proved in Sections 3-7. We first prove a lemma.

**Lemma 8.1.** *Let $G$ be a quasi-transitive, connected, locally finite graph. Consider the i.i.d. Bernoulli(p) site percolation on $G$. Let $s_0$ (resp. $s_1$) be the total number of infinite 0-clusters (resp. infinite 1-clusters) in a random site configuration. If $p_c^{site} < \frac{1}{2}$. Then for any $p \in [0, 1]$ a.s.*

$$(s_0, s_1) \neq (0, 0).$$

*Proof.* Let $\mu$ be the corresponding probability measure, then $\mu$ is ergodic. Hence either $\mu((s_0, s_1) = (0,0)) = 0$ or $\mu((s_0, s_1) = (0,0)) = 1$.

Assume that there exists $p \in [0,1]$, such that $\mu((s_0, s_1) = (0,0)) = 1$, then $1-p \leq p_c^{site}$, and $p \leq p_c^{site}$, therefore we have $2p_c^{site} \geq 1$, which contradicts the assumption that $p_c^{site} < 1$. Then the lemma follows. □

8.1. **Proof of Theorem 1.7.** Let $G$ be the graph satisfying the assumptions of the Theorem. If each vertex has degree $d \geq 7$, since the graph is simple and planar, each face has degree at least 3. Assume the degree of faces around each vertex is given by $[m_1, \ldots, m_d]$ in cyclic order, we have

$$\frac{1}{m_1} + \frac{1}{m_2} + \ldots + \frac{1}{m_d} \leq \frac{d}{3} < \frac{d-2}{2}$$

Hence the graph $G$ is always non-amenable. By Proposition 1.3, the following cases might occur:

(1) $G$ is non-amenable and has one end.
(2) $G$ has infinitely many ends.

In either case by Lemma 4.3, we have $p_c^{site} < \frac{1}{2}$. The a.s. existence of at least 1 infinite open or closed clusters follows from Lemma 8.1. In case (1), $1 - p_c^{site} \leq p_u^{site}$ follows from Proposition 4.2. In case (2), $1 - p_c^{site} \leq p_u^{site}$ because $p_u^{site} = 1$ by Theorem 6.1. Therefore when $p \in (p_c^{site}, 1-p_c^{site}) \subseteq (p_c^{site}, p_u^{site})$, a.s. there are infinitely many infinite open clusters. There are also infinitely many infinite closed clusters because when $p \in (p_c^{site}, 1 - p_c^{site})$, $1 - p \in (p_c^{site}, 1 - p_c^{site})$. □

8.2. **Proof of Theorem 1.8.** Let $\mu$ be the percolation measure. Then $\mu$ is ergodic. By Lemma 2.6 and symmetry, $\mu$-a.s.$(s_0, s_1) \in \{(0,0), (1,1), (\infty, \infty)\}$. Under the assumption that a.s. percolation occurs. $\mu$-a.s. $(s_0, s_1) \in \{(1,1), (\infty, \infty)\}$

By Proposition 1.3, the following 4 cases might occur:

(a) $G$ is amenable and has one-end. By Lemma 7.1, when $p = \frac{1}{2}$, $\mu$-a.s. percolation does not occur. Hence in this case the assumption of the Part (B) does not hold. Still by Lemma 7.1, $p_u^{site} = p_c^{site} \geq \frac{1}{2}$, therefore $p_u^{site} + p_c^{site} \geq 1$.
(b) $G$ is non-amenable and has one-end. The conclusion follows from Theorem 1.4. In this case $p_u^{site} + p_c^{site} \geq 1$ follows from Proposition 4.2.
(c) $G$ has two ends. By Theorem 5.1, $p_c^{site} = 1$. Hence when $p = \frac{1}{2}$, $\mu$-a.s. percolation does not occur. Again in this case the assumption of Part (B) does not hold. Moreover $p_u^{site} + p_c^{site} \geq 2p_c^{site} = 2$.
(d) $G$ has infinitely many ends. By Theorem 5.1, $p_u^{site} = 1$. When $p = \frac{1}{2}$ and percolation occurs, $p_c^{site} < p < p_u^{site}$, then $\mu$-a.s. $(s_0, s_1) = (\infty, \infty)$. Moreover, $p_u^{site} + p_c^{site} \geq 1 + p_c^{site} \geq 1$.

## 9. Combinatorial Results about Contours and Clusters

Throughout this section, we assume that $G$ is a graph satisfying the assumption of Theorem 1.4. The goal of this section is to prove Lemma 9.11, which states that if in a site configuration $\omega \in \{0,1\}^{V(G)}$ there is an infinite 0-cluster $\xi_0$ and an infinite 1-cluster $\xi_1$ such that $\xi_0$ and $\xi_1$ are adjacent, then there exists a unique infinite contour in $\phi_\omega^+$ incident to both $\xi_0$ and $\xi_1$. Similar result has been proved for the constrained percolation model on the amenable or nonamenable planar $[m,4,n,4]$ lattice; see [15, 17]. Here the result is generalized to the unconstrained percolation model on an arbitrary one-ended graph $G$ that can be quasi-transitively and isometrically embedded to the hyperbolic plane $\mathbb{H}^2$.

**Lemma 9.1.** *Let $\omega \in \{0,1\}^{V(G)}$. When interpreted as subsets of $\mathbb{H}^2$, $\eta_\omega \cap \phi_\omega^+ = \emptyset$.*

*Proof.* This follows directly from the definition of $\eta_\omega$. □

**Lemma 9.2.** *Let $\omega \in \{0,1\}^{V(G)}$. For any contour $I$ of $\eta_\omega$, let $E_I$ be the set consisting of all the edges $(u,v) \in E(G)$ satisfying the following condition*

- *there is an half edge $g$ of the dual edge $(u,v)^+$ satisfying $g^+ \in I$.*

*Let $F_I \subset V(G)$ be the vertex set of $E_I$. Then all the vertices in $F_I$ lie in a single cluster. Moreover,*

*(1) $F_I$ is the vertex set of a doubly-infinite self-avoiding path if $I$ is a doubly-infinite self-avoiding path.*

*(2) If $I$ is a self-avoiding cycle, then $F_I$ is the vertex set of either a finite self-avoiding path or a self-avoiding cycle.*

*Proof.* First of all, note that $F_I$ is a connected set of vertices in $G$. Now, if not all the vertices in $F_I$ are in the same cluster, then there exist a pair of adjacent vertices $x, y \in F_I$, such that the edge $(x,y)$ of $G$ crosses a contour. Then the contour crossing $(x,y)$ must cross the interface $I$ as well, but this is a contradiction to Lemma 9.1. □

**Lemma 9.3.** *Let $\omega \in \{0,1\}^{V(G)}$. If there exist at least two infinite contours in $\phi_\omega^+$, then there exists an infinite 0-cluster or an infinite 1-cluster in $\omega$. Moreover, if $C_1$ and $C_2$ are two infinite contours in $\phi_\omega^+$, then there exists an infinite cluster in $\omega$ incident to $C_1$.*

*Proof.* If there exist at least two infinite contours in $\phi_\omega^+$, then we can find two distinct infinite contours $C_1$ and $C_2$ in $\phi_\omega^+$, two points $x \in C_1$ and $y \in C_2$ and a self-avoiding path $p_{xy}$, consisting of edges of $G$ and two half-edges, one starting at $x$ and the other ending at $y$, and connecting $x$ and $y$, such that $p_{xy}$ does not intersect any infinite contours in $\phi_\omega^+$ except at $x$ and at $y$. Indeed, we may take any path intersecting two distinct contours in $\phi_\omega^+$, and then take a minimal subpath with this property.

Let $v \in V$ be the first vertex of $G$ along $p_{xy}$ starting from $x$. Let $u$ be the point along the line segment $[v,x]$ lying on $\eta_\omega$. Let $l_u$ be the contour of $\eta_\omega$ containing $u$. Then $l_u$ is either a doubly-infinite self-avoiding path or a self-avoiding cycle by Lemma 3.1.

We consider these two cases separately. Firstly, if $l_u$ is a doubly-infinite self-avoiding path, then we claim that $v$ is in an infinite (0 or 1-)cluster of $\omega$. Indeed, this follows from Lemma 9.2.

Secondly, if $l_u$ is a self-avoiding cycle, then $\mathbb{H}^2 \setminus l_u$ has two components, $Q_v$ and $Q'_v$, where $Q_v$ is the component including $v$. Since $l_u$ is a cycle, exactly one of $Q_v$ and $Q'_v$ is bounded, the other is unbounded. Since $C_1 \subseteq Q'_v$, and $C_1$ is an infinite contour, we deduce that $Q'_v$ is unbounded, and $Q_v$ is bounded. Since $y \notin l_u$, either $y \in Q_v$, or $y \in Q'_v$. If $y \in Q'_v$, then any path, consisting of edges of $G$ and one half-edge incident to $y$, connecting $v$ and $y$ must cross $C_1$. In particular, $p_{xy}$ crosses $C_1$ not only at $x$, but also at some point other than $x$. This contradicts the definition of $p_{xy}$. Hence $y \in Q_v$. Since $C_1 \cap C_2 = \emptyset$, this implies $C_2 \subseteq Q_v$; because if $C_2 \cap Q'_v \neq \emptyset$, then $C_2 \cap C_1 \neq \emptyset$. But $C_2 \subseteq Q_v$ is impossible since $C_2$ is infinite and $Q_v$ is bounded. Hence this second case is impossible.

Therefore we conclude that if there exist at least two infinite contours in $\phi^+_\omega$, then there exists an infinite (0 or 1)-cluster in $\omega$. □

**Lemma 9.4.** *Let $\omega \in \{0,1\}^{V(G)}$. Let $x \in V(G)$ be in an infinite 0-cluster of $\omega$, let $y \in V(G)$ be in an infinite 1-cluster of $\omega$, and let $l_{xy}$ be a path, consisting of edges of $G$ and connecting $x$ and $y$. Then $l_{xy}$ has an odd number of crossings with infinite contours in $\phi^+_\omega$ in total.*

*In particular, if there exist both an infinite 0-cluster and an infinite 1-cluster in $\omega$, then there exists an infinite contour in $\phi^+_\omega$.*

*Proof.* Throughout the proof, we use "contours" to denote contours in $\phi^+_\omega$.

Moving along $l_{xy}$, any two neighboring vertices $u, v \in V(G)$ have different states if and only if the edge $(u, v)$ crosses a contour. Since the states of $x$ and $y$ are different, moving along $l_{xy}$, the states of vertices must change an odd number of times. Therefore $l_{xy}$ crosses contours an odd number of times.

It remains to show that the total number of crossings of $l_{xy}$ with finite contours is even. Since $l_{xy}$ crosses finitely many finite contours in total, let $C_1, \ldots, C_m$ be all the finite contours intersecting $l_{xy}$, where $m$ is a nonnegative integer.

Let $G \setminus \cup_{i=1}^m C_i$ be the subgraph obtained from $G$ by removing all the edges of $G$ crossed by the $C_i$'s. Since all the $C_i$'s are finite and $G$ has one end, $G \setminus \cup_{i=1}^m C_i$ has exactly one infinite component. We claim that both $x$ and $y$ lie in the infinite connected component of $G \setminus \cup_{i=1}^m C_i$. Indeed, if $x$ is in a finite component of $G \setminus \cup_{i=1}^m C_i$, then it is a contradiction to the fact that $x$ is in an infinite 0-cluster, because the infinite 0-cluster including $x$ cannot be a subset of a finite component of $G \setminus \cup_{i=1}^m C_i$. Similarly $y$ is also in an infinite component of $G \setminus \cup_{i=1}^m C_i$. Since $G \setminus \cup_{i=1}^m C_i$ has a unique infinite component, we infer that both $x$ and $y$ are in the same infinite component of $G \setminus \cup_{i=1}^m C_i$.

Since both $x$ and $y$ lie in the infinite connected component of $G \setminus \cup_{i=1}^m C_i$, we can find a path $l'_{xy}$ connecting $x$ and $y$, using edges of $G$, such that the path does not intersect $\cup_{i=1}^m C_i$ at all. Moreover, each vertex of $G^+$ has an even number of incident edges in $\cup_{i=1}^m C_i$. We can transform $l_{xy}$ to $l'_{xy}$ using a finite sequence of moves; in each move, the path only changes along the boundary of a single face of $G$. Since the face contains a single vertex of even degree in $\cup_i C_i$ (the degree of the vertex in $\cup_i C_i$ can be 0), it is easy to verify that the parity of the total number of crossings is preserved. This implies that $l_{xy}$ must cross infinite contours an odd number of times, because $l_{xy}$ crosses (infinite and finite) contours an odd number of times in total, and $l_{xy}$ crosses finite contours an even number of times. □

**Lemma 9.5.** *Let $\omega \in \{0,1\}^{V(G)}$. Let $C_\infty$ be an infinite contour in $\phi_\omega^+$. Then each infinite component of $G \setminus C_\infty$ contains an infinite cluster in $\omega$ that is incident to $C_\infty$.*

*Proof.* Let $S$ be an infinite component of $G \setminus C_\infty$. Let $e^+ \in C_\infty$ be an edge of $G^+$ with midpoint $x$, and let $y \in S$ be a vertex of $G$, such that $y$ is an endpoint of $e \in V(G)$ (the dual edge of $e^+$). Let $v$ be the midpoint of the line segment $[x, y]$. Then $v$ lies on a contour of $\eta_\omega$. Let $l_v$ be the contour of $\eta_\omega$ containing $v$.

We claim that $l_v$ is infinite. Suppose that $l_v$ is finite. Then by Lemma 3.1, $l_v$ is a self-avoiding cycle. Let $Q_x$ (resp. $Q_y$) be the component of $\mathbb{H}^2 \setminus l_v$ containing $x$ (resp. $y$). Then exactly one of $Q_x$ and $Q_y$ is bounded, and the other is unbounded. Note that $C_\infty \subset Q_x$ by Lemma 9.1.

We claim that $S \subset Q_y$. To see why that is true, note that since $S$ is connected and $y \in S \cap Q_y$, if $S$ is not a subset of $Q_y$, there exist a pair of adjacent vertices $p, q \in S$, such that $p \in Q_y$ and $q \notin Q_y$. Then the edge $(p, q)$ of $G$ crosses $l_v \subseteq \eta_\omega$. From the definition of $\eta_\omega$, we obtain that in this case $\omega(p) \neq \omega(q)$, and therefore the edge $(p, q)$ crosses the contour $C_\infty$ as well. But this is impossible since $S$ is an infinite component of $G \setminus C_\infty$.

Since it is impossible that $C_\infty \subset Q_x$ and $S \subset Q_y$ both $C_\infty$ and $S$ are infinite, we infer that $l_v$ is infinite.

According to Lemma 9.2, all the vertices in $F_{l_v}$ lie in an infinite cluster incident to $C_\infty$. □

**Lemma 9.6.** *Let $\omega \in \{0,1\}^{V(G)}$. Let $\xi_0$ be an infinite-0 cluster in $\omega$ and $\xi_1$ be an infinite 1-cluster in $\omega$. Assume that there exist a vertex $x$ in the infinite 0-cluster, a vertex $y$ in the infinite 1-cluster, and a path $l_{xy}$, consisting of edges of $G$ and joining $x$ and $y$, such that $l_{xy}$ crosses exactly one infinite contour in $\phi^+(\omega)$, $C_\infty$. Then $C_\infty$ is incident to $\xi_0$ and $\xi_1$.*

*Proof.* Let $a, b$ be two midpoint of edges $e_1^+$, $e_2^+$ such that $a$ is the first point in $C_\infty$ visited by $l_{xy}$, and $b$ is the last point in $C_\infty$ visited by $l_{xy}$ when traveling from $x$ to $y$ along $l_{xy}$. Let $l_{xa}$ (resp. $l_{by}$) be the portion of $l_{xy}$ between $x$ and $a$ (resp. between $b$ and $y$). Let $p \in V(G)$ (resp. $q \in V(G)$) be the last (resp. first) vertex of $V(G)$ visited by $l_{xa}$ (resp. $l_{by}$) when traveling from $x$ to $a$ (resp. $b$ to $y$). Let $l_{xp}$ (resp. $l_{qy}$) be the portion of $l_{xy}$ between $x$ and $p$ (resp. between $q$ and $y$). Then $l_{xp}$ and $l_{qy}$ cross only finite contours in $\phi_\omega^+$.

As in the proof of Lemma 9.4, we may change paths around faces of $G$ and obtain new paths $l'_{xp}$, $l'_{qy}$ consisting of edges of $G$, such that $l'_{xp}$ and $l'_{qy}$ do not cross contours in $\phi_\omega^+$ at all. Then $p$ and $x$ are in the same cluster of $\omega$, hence $p \in \xi_0$; $q$ and $y$ are in the same cluster of $\omega$, hence $q \in \xi_1$. □

**Lemma 9.7.** *Let $\omega \in \{0,1\}^{V(G)}$. Assume that $\xi$ is an infinite cluster of $\omega$, and $C$ is an infinite contour of $\phi_\omega^+$. Assume that $x$ is a vertex of $G$ in $\xi$, and let $y \in C$ be the midpoint of an edge of $G$. Assume that there exists a path $p_{xy}$ connecting $x$ and $y$, consisting of edges of $G$ and a half-edge incident to $y$, such that $p_{xy}$ crosses no infinite contours in $\phi_\omega^+$ except at $y$. Let $z$ be the first vertex of $G$ along $p_{xy}$ starting from $y$. Then $z \in \xi$.*

*Proof.* Since $p_{xy}$ crosses no infinite contours in $\phi_\omega^+$ except at $y$, let $C_1, \ldots, C_m$ be all the finite contours crossing $p_{xy}$. We claim that $\mathbb{H}^2 \setminus \cup_{i=1}^m C_i$ has a unique unbounded component, which contains both $x$ and $y$. Indeed, since $x \in \xi$ and $y \in C$; neither the infinite cluster $\xi$ nor the infinite contour $C$ can lie in a bounded component of $\mathbb{H}^2 \setminus \cup_{i=1}^m C_i$.

Let $I$ be the intersection of the union of the contours in $\eta_\omega$ incident to $C_1, \ldots, C_m$ with the unique unbounded component of $\mathbb{H}^2 \setminus \cup_{i=1}^m C_i$. Since each $C_i$, $1 \leq i \leq m$, is a finite contour, each component of $I$ is finite. More precisely, by Lemma 3.1 $I$ consists of finitely many disjoint self-avoiding cycles, denoted by $D_1, \ldots, D_t$. For $1 \leq i \leq t$, $\mathbb{H}^2 \setminus D_i$ has exactly one unbounded component, and one bounded component. Moreover, for $i \neq j$, $D_i$ and $D_j$ are incident to distinct contours in $C_1, \ldots, C_m$.

Let $B_i$ be the bounded component of $\mathbb{H}^2 \setminus D_i$. We claim that each $B_i$ is simply-connected, and $B_i \cap B_j = \emptyset$, for $i \neq j$. Indeed, $B_i$ is simply connected, since the boundary of $B_i$, $D_i$ is a self-avoiding cycle, whose embedding in $\mathbb{H}^2$ is a simple closed curve, for $1 \leq i \leq t$. Let $1 \leq i < j \leq t$. Since $D_i$ and $D_j$ are disjoint, either $B_i \cap B_j = \emptyset$, or one of $B_i$ and $B_j$ is a proper subset of the other. Without loss of generality, assume $B_i$ is a proper subset of $B_j$. Then $D_i$ is a proper subset of $B_j$. Hence $D_i$ is in a bounded component of $\mathbb{H}^2 \setminus \cup_{k=1}^m C_k$, which contradicts the definition of $D_i$.

Let $R_i$ be the set of faces $F$ of $G$, for which $B_i \cap F \neq \emptyset$. Let $\widetilde{B}_i = \cup_{F \in R_i} F$. Note that for $1 \leq i \leq t$, each $\widetilde{B}_i$ is a simply-connected, closed set. Let $B_i'$ be the interior of $\widetilde{B}_i$. Then each $B_i'$ is a simply-connected, open set; moreover, $B_i' \cap B_j' = \emptyset$, if $i \neq j$. This follows from the fact that for $i \neq j$, $D_i$ and $D_j$ come from interfaces of distinct contours, and the fact that $B_i \cap B_j = \emptyset$, for $i \neq j$.

Let $B' = \cup_{i=1}^t B_i'$. Then $B'$ is open, and $x, y, z \in \mathbb{H}^2 \setminus B'$, although $x$ and $z$ may be on the boundary of $B'$.

There is a path $p_{xy}' \subseteq [p_{xy} \cap (\mathbb{H}^2 \setminus B')] \cup \partial B'$, connecting $x$ and $y$, where $\partial B'$ is the boundary of $B'$. More precisely, $p_{xy}$ is divided by $\partial B'$ into segments; on each segment of $p_{xy}$ in $P \setminus B'$, $p_{xy}'$ follows the path of $p_{xy}$; for each segment of $p_{xy}$ in $B'$, $p_{xy}'$ follows the boundary of $B'$ to connect the two endpoints of the segment. This is possible since $B'$ consists of bounded, disjoint, simply-connected, open sets $B_i'$, for $1 \leq i \leq t$, and both $x$ and $v$ are in the complement of $B'$ in $\mathbb{H}^2$.

All the vertices along $p_{xy}'$ are in the same cluster. In particular, this implies that $x$ and $z$ are in the same infinite cluster $\xi$. □

**Lemma 9.8.** *Let $\omega \in \{0,1\}^{V(G)}$. Let $\xi_0$ be an infinite 0-cluster in $\omega$ and $\xi_1$ be an infinite 1-cluster in $\omega$ such that $\xi_0$ and $\xi_1$ are adjacent. Then there exists an infinite contour in $\phi_\omega^+$ that is incident to both $\xi_0$ and $\xi_1$.*

*Proof.* Let $x$ be a vertex in $\xi_0$, and let $y$ be a vertex in $\xi_1$. Let $l_{xy}$ be a path joining $x$ and $y$ and consisting of edges of $G$. Since $\xi_0$ and $\xi_1$ are adjacent, we assume that $l_{xy}$ does not intersect any infinite clusters other than $\xi_0$ and $\xi_1$.

By Lemma 9.4, $l_{xy}$ must cross infinite contours in $\phi_\omega^+$ an odd number of times. By Lemma 9.6, if $l_{xy}$ crosses exactly one infinite contour in $\phi_\omega^+$, $C_\infty$, then $C_\infty$ is incident to both the infinite 0-cluster and the infinite 1-cluster, and so the lemma is proved in this case.

Suppose that there exist more than one infinite contour in $\phi_\omega^+$ crossing $l_{xy}$. Let $C_1$ and $C_2$ be two distinct infinite contours in $\phi_\omega^+$ crossing $l_{xy}$.

Let $u \in C_1 \cap l_{xy}$ and $v \in C_2 \cap l_{xy}$ (Here we interpret the contours and the paths as their embeddings to $\mathbb{H}^2$, so that $u, v$ are points in $\mathbb{H}^2$), such that the portion of $l_{xy}$ between $u$ and $v$, $p_{uv}$, does not cross any infinite contours in $\phi_\omega^+$ except at $u$ and at $v$. Let $u_1$ be the first vertex of $G$ along $p_{uv}$, starting from $u$; and let $v_1$ be the first vertex of $G$ along $p_{uv}$ starting from $v$. Let $u_2$ (resp. $v_2$) be the point along the line segment $[u, u_1]$ (resp. $[v, v_1]$) lying on $\eta_\omega$. Following the procedure in the proof of Lemma 9.3, we can find an infinite cluster $\theta$, such that $u_1 \in \theta$. The following cases might happen:

(1) $x \notin \theta$ and $y \notin \theta$;
(2) $x \notin \theta$ and $y \in \theta$;
(3) $x \in \theta$ and $y \notin \theta$;
(4) $x \in \theta$ and $y \in \theta$.

First of all, case (4) is impossible because we assume $x$ and $y$ are in two distinct infinite clusters. Secondly, if case (1) is true, then $l_{xy}$ intersects at least three infinite clusters, which is a contradiction to our assumption.

Case (2) and case (3) can be handled using similar arguments, and we write down the proof of case (2) here.

If case (2) is true, first note that $y \in \theta = \xi_1$ implies that $C_1$ is incident to the infinite 1-cluster $\xi_1$ since $C_1$ is incident to $u_1 \in \theta$. Let $z$ be the first point in $C_1 \cap l_{xy}$ (again interpret edges as line segments), when traveling along $l_{xy}$ starting from $x$. Let $p_{xz}$ be the portion of $l_{xy}$ between $x$ and $z$.

Next, we will prove the following claim by induction on the number of complete edges of $G$ along $p_{xz}$ (in contrast to the half edge along $p_{xz}$ with an endpoint $z$).

**Claim 9.9.** *Under case (2), there is an infinite contour incident to both $\xi_0$ and $\xi_1$.*

Assume that the number of complete edges of $G$ along $p_{xz}$ is $n$, where $n = 0, 1, 2, \ldots$.

First of all, consider the case when $n = 0$. This implies that $C_1$ is incident to the infinite 0-cluster $\xi_0$ at $x$. Recall that $C_1$ is also incident to the infinite 1-cluster $\xi_1$ at $y$, and so Claim 9.9 is proved.

We make the following induction hypothesis:

- Claim 9.9 holds for $n \leq k$, where $k \geq 0$.

Now we consider the case when $n = k + 1$. The interior points of $p_{xz}$ are all points along $p_{xz}$ except $x$ and $z$. We consider two cases:

(a) at interior points, $p_{xz}$ crosses only finite contours but not infinite contours in $\phi_\omega^+$;
(b) at interior points, $p_{xz}$ crosses infinite contours in $\phi_\omega^+$.

We claim that if case (a) occurs, then $C_1$ is incident to both $\xi_0$ and $\xi_1$. It suffices to show that $C_1$ is incident to $\xi_0$.

Let $z_1$ be the first vertex in $V$ along $p_{xz}$ starting from $z$. According to Lemma 9.7, both $x$ and $z_1$ are in $\xi_0$. We infer that $C_1$ is incident to $\xi_0$, if $p_{xz}$ intersects only finite contours at interior points.

Now we consider case (b). Let $C_3$ be an infinite contour in $\phi_\omega^+$ crossing $p_{xz}$ at interior points. Obviously, $C_3$ and $C_1$ are distinct, because $C_1$ crosses $p_{xz}$ only at $z$. Let $w$ be the last point in $C_3 \cap p_{xz}$, when traveling along $p_{xz}$, starting from $x$, and let $p_{wz}$ be the portion of $p_{xz}$ between $w$ and $z$. Assume $p_{wz}$ does not cross infinite contours at interior points.

Let $w_1$ be the first vertex of $G$ along $p_{wz}$, starting from $w$, and let $w_2$ be the midpoint of $w$ and $w_1$. According to the proof of Lemma 9.3, we can find an infinite cluster $\xi_3$ including $w_1$. The following cases might happen:

i $x \notin \xi_3$, and $y \notin \xi_3$;
ii $x \in \xi_3$, and $y \notin \xi_3$;
iii $x \notin \xi_3$, and $y \in \xi_3$;
iv $x \in \xi_3$, and $y \in \xi_3$.

First of all, Case iv is impossible because we assume $x$ and $y$ are in two distinct infinite clusters. Secondly, if Case i is true, then $l_{xy}$ intersects at least 3 infinite clusters, which contradicts to our assumption on $l_{xy}$.

If Case ii is true, then $C_3$ is incident to $\xi_0$. Since $w_1 \in \xi_3 = \xi_0$, and $p_{wz}$ does not cross infinite contours except at $w$ and $z$, by Lemma 9.7, we infer that $z \in \xi_3$, and $\xi_3$ is exactly the infinite 0-cluster $\xi_0$ including $x$. We conclude that $C_1$ is incident to $\xi_0$ as well, and Claim 9.9 is proved.

If Case iii is true, then $C_3$ is incident to $\xi_1 = \xi_3$. Let $t$ be the first vertex in $p_{xz} \cap C_3$, when traveling from $p_{xz}$, starting at $x$, and let $p_{xt}$ be the portion of $p_{xz}$ between $x$ and $t$. We explore the path $p_{xt}$ as we have done for $p_{xz}$. Since the length of $p_{xz}$ is finite, and the number of full edges of $G$ along $p_{xt}$ is less than that of $p_{xz}$ by at least 1, we apply the induction hypothesis with $C_1$ replaced by $C_3$, $C_2$ replaced by $C_1$, $\xi_1$ replaced by $\xi_3$, $p_{xz}$ replaced by $p_{xt}$, and we conclude that there exists an infinite contour adjacent to both $\xi_0$ and $\xi_1$. □

**Lemma 9.10.** *Let $\omega \in \{0, 1\}^{V(G)}$. Let $\xi_0$, $\xi_1$ be two distinct infinite clusters in $\omega$. Let $C_1$, $C_2$ be two distinct infinite contours in $\phi_\omega^+$. Then it is not possible that the following two conditions happen simultaneously.*

*(a) The infinite contour $C_1$ is incident to both $\xi_0$ and $\xi_1$.*
*(b) The infinite contour $C_2$ is incident to both $\xi_0$ and $\xi_1$.*

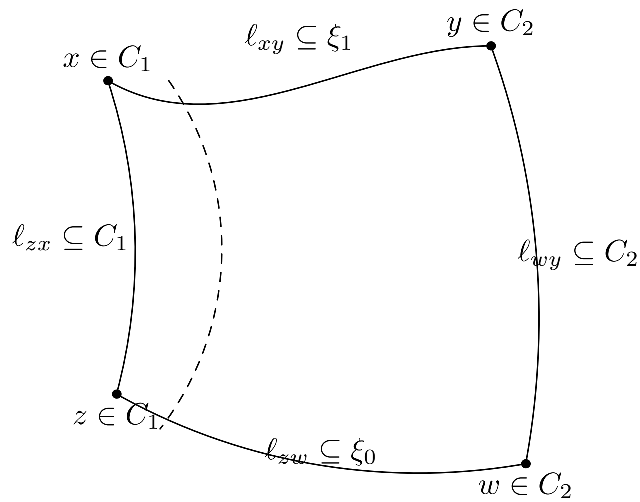


FIGURE 9.1. Infinite clusters and incident contours

*Proof.* We will prove the lemma by contradiction.

Assume that both (a) and (b) occur. We can find points $x \in C_1$ and $y \in C_2$, such that $x$ and $y$ are connected by a path $l_{xy}$, consisting of edges of $G$ and two half-edges, (one starting at $x$ and one ending at $y$), such that every vertex of $G$ along $l_{xy}$ is in $\xi_1$. Similarly, we can find a point $z \in C_1$ and $w \in C_2$, such that $z$ and $w$ are connected by a path $l_{zw}$, consisting of edges of $G$ and two half-edges, (one starting at $z$ and one ends at $w$), such that every vertex of $G$ along $l_{zw}$ is in $\xi_0$. Moreover, we can find a path $l_{zx} \subseteq C_1$ connecting $z$ and $x$ and $l_{wy} \subseteq C_2$ connecting $w$ and $y$. Viewed as subsets of $\mathbb{H}^2$, the four paths $l_{xy}$, $l_{wy}$, $l_{zw}$ and $l_{zx}$ are disjoint except for the endpoints. Therefore their union is a simple closed curve in $\mathbb{H}^2$. Let $R \subseteq \mathbb{H}^2$ be the bounded region enclosed by the curve; see Figure 9.1.

Let $x_1$ be the first vertex of $G$ along $l_{xy}$ starting from $x$; and let $z_1$ be the first vertex of $G$ along $l_{zw}$ starting from $z$. Let $x_2$ (resp. $z_2$) be the midpoint of the line segment $[x, x_1]$ (resp. $[z, z_1]$). Since $x \in C_1$ and $x_1 \in \xi_1$, a contour of $\eta_\omega$ incident to $C_1$ contains $x_2$. Similarly a contour of $\eta_\omega$ incident to $C_1$ contains $z_2$ as well.

We claim that $x_2$ and $z_2$ are in the same contour of $\eta_\omega$ incident to $C_1$. To see why this is true, consider the contour $\gamma$ of $\eta_\omega$ incident to $C_1$ containing $x_2$; Lemma 3.1 implies that $\gamma$ is either a self-avoiding cycle or a doubly-infinite self-avoiding path. Therefore $\gamma$ crosses $\partial R = l_{xy} \cup l_{zw} \cup l_{zx} \cup l_{wy}$ an even number of times. But the only other possible crossing of $\gamma$ with $\partial R$ is $z_2$, therefore $z_2 \in \gamma$. Indeed, $\gamma$ cannot cross $C_1$ or $C_2$ because contours in $\eta_\omega$ and contours in $\phi_\omega^+$ cannot cross by Lemma 4.2(1); moreover, $\gamma$ cannot cross $l_{xy}$ at a point other than $x_2$ because if that occurs, we have

- either an edge of $l_{xy}$ joins two vertices of different states in $\omega$ by Lemma 4.2(2), which is impossible;
- or $\gamma$ is incident to $C_2$; but this is impossible by Lemma 3.3.

Similarly, $\gamma$ cannot cross $l_{zw}$ at a point other than $z_2$. By similar reasoning any other contour of $\eta_\omega$ incident to $C_1$ does not cross $\partial R$.

By Lemma 9.2, all those vertices in $F_\gamma$ lie in the same cluster. Then $x_1$ and $z_1$ are in the same cluster of $\omega$. However $x_1 \in \xi_1$, $z_1 \in \xi_0$, and $\xi_1$ and $\xi_0$ are distinct clusters in $\omega$. The contradiction implies the conclusion of the lemma. □

**Lemma 9.11.** *Let $\omega \in \{0,1\}^{V(G)}$. Let $\xi_0$ be an infinite 0-cluster in $\omega$ and $\xi_1$ be an infinite 1-cluster in $\omega$ such that $\xi_0$ and $\xi_1$ are adjacent. Then there exists exactly one infinite contour in $\phi_\omega^+$ that is incident to both $\xi_0$ and $\xi_1$.*

*Proof.* The lemma follows from Lemmas 9.8 and 9.10. □

**Acknowledgements.** Z.L.'s research is supported by National Science Foundation grant 1608896 and Simons Collaboration Grant 638143.

## References

[1] L. Babai, *The growth rate of vertex-transitive planar graphs.*, Proceedings of the Eighth Annual ACM-SIAM Symposium on Discrete Algorithms (New Orleans, LA, 1997), New York, 1997, pp. 564–573.

[2] I. Benjamini, R. Lyons, Y. Peres, and O. Schramm, *Critical percolation on any nonamenable group has no infinite clusters*, Ann. Probab. **27** (1999), 1347–1356.

[3] ———, *Group-invariant percolation on graphs*, Geom. Funct. Anal. **9** (1999), 29–66.

[4] I. Benjamini and O. Schramm, *Percolation beyond $\mathbb{Z}^d$, many questions and a few answers*, Electronic Communications in Probability **1** (1996), 71–82.

[5] ———, *Percolation in the hyperbolic plane*, Journal of the American Mathematical Society **14** (2000), 487–507.

[6] R.M. Burton and M. Keane, *Density and uniqueness in percolation*, Communications in Mathematical Physics **121** (1989), 501–505.

[7] J. W. Cannon, W. J. Floyd, R. Kenyon, and W. R. Parry, *Hyperbolic geometry*, Flavors of Geometry, Cambridge Univ. Press, Cambridge, 1997, pp. 59–115.

[8] C. M. Fortuin, P. W. Kasteleyn, and J. Ginibre, *Correlation inequalities on some partially ordered sets*, Commun. Math. Phys. **22** (1971), 89–103.

[9] A. Gandolfi, M. Keane, and C. Newman, *Uniqueness of the infinite component in a random graph with appli-cations to percolation and spin glasses.*, Probability Theory and Related Fields **4** (1992), 511–527.

[10] G. Grimmett, *Percolation*, Springer, 1999.

[11] G. R. Grimmett and Z. Li, *Cubic graphs and the golden mean*, Discrete Mathematics **343** (2020), 11638.

[12] O. Häggström and Y. Peres, *Monotonicity of uniqueness for percolation on cayley graphs: All infinite clusters are born simultaneously*, Probability Theory and Related Fields **113** (1999), 273–285.

[13] O. Häggström, Y. Peres, and R.H. Schonmann, *Percolation on transitive graphs as a coalescent process: Relentless merging followed by simultaneous uniqueness*, Perplexing Problems in Probability. Festschrift in honor of Harry Kesten, Birkhäuser, Basel–Boston, 1999, p. 69–90.

[14] R. Holley, *Remarks on the fkg inequalities*, Commun. Math. Phys. **36** (1974), 227–231.

[15] A. Holroyd and Z. Li, *Constrained percolation in two dimensions*, Annales de L'institut Henri Poincaré D (2020).

[16] H. Hopf, *Enden offener räume und unendliche diskontinuierliche gruppen,*, Comment. Math. Helv. **16** (1944), 81–100.

[17] Z. Li, *Constrained percolation, Ising model and XOR Ising model on planar lattices*, Random Structures and Algorithms (2020).

[18] ———, *Positive speed self-avoiding walks on graphs with more than one end*, Journal of Combinatorial Theory, Series A. **175** (2020), 105257.

[19] R. Lyons and Y. Peres, *Probability on trees and networks*, Cambridge University Press, 2016.

[20] C.M. Newman and L.S. Shulman, *Infinite clusters in percolation models*, J. of Statis. Phys. **26** (1981), 613–628.

[21] D. Renault, *The vertex-transitive TLF-planar graphs*, Discrete Mathematics **309** (2009), 2815–2833.

[22] R. H. Schonmann, *Stability of infinite clusters in supercritical percolation*, Probability Theory and Related Fields **113** (1999), 287–300.

[23] ———, *Multiplicity of phase transitions and mean-field criticality on highly non-amenable graphs*, Commun. Math. Phys. **219** (2001), 271–322.

Department of Mathematics, University of Connecticut, Storrs, Connecticut 06269-3009, USA

*Email address*: zhongyang.li@uconn.edu

*URL*: https://mathzhongyangli.wordpress.com