# MATCHINGS ON RANDOM REGULAR HYPERGRAPHS

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Abstract. We prove the convergence in probability of free energy for matchings on random regular, uniform hypergraphs and explicitly compute the limit. We also obtain a region on parameters where replica symmetry holds.

## 1. Introduction

Counting the number of matchings on a graph has been an interesting problem to scientists dating back to at least 1960s (see [9, 16, 17]). Unlike perfect matchings, the number of which on a graph can be computed explicitly by a determinant formula (see [19, 24, 21]), counting the number of matchings (or monomer dimers) on a graph is complicated in general (see [18]); and algorithms are designed for that purpose ([20, 10]).

In this paper, we study the free energy of matchings on random, uniform hypergraphs. Let $d, l$ be positive integers such that $l \geq 2$ and $d \geq 2$. A hypergraph $G = (V, E, H)$ consists of a collection $H$ of labeled half-edges, a set $E$ of hyperedges such that each hyperedge $e \in E$ consists of $l$ distinct half-edges, and a set $V$ of vertices in which each vertex is represented by $d$ distinct half-edges incident to it. Here each half edge is part of a unique hyperedge and is incident to a unique vertex. However, it is allowed that two distinct half edges are incident to the same vertex and part of the same hyperedge. We call such a hypergraph a $(d, l)$-regular hypergraph.

A subset $M \subset E$ is a matching of $G = (V, E, H)$ if for any two hyperedges $e_1, e_2 \in M$, let $f_1, \ldots, f_l$ be half edges in $e_1$ and $g_1, \ldots, g_l$ be half edges in $e_2$, then no two half edges in $\{f_1, \ldots, f_l, g_1, \ldots, g_l\}$ share a vertex.

Let $m$ be a positive integer, and assume that $lm$ is an integer multiple of $d$. We may consider a $(d, l)$-regular hypergraph with hyperedge set $[m] = \{1, 2, \ldots, m\}$ as a division on the set $[lm]$ of labeled half-edges into disjoint size-$d$ subsets, where half-edge $i$ is part of the edge $\lceil \frac{i}{l} \rceil$, and each size-$d$ subset of half-edges represent a vertex. Then the total number of $(d, l)$-regular hypergraphs with hyperedge set $[m]$ is

$$\prod_{j=0}^{\frac{lm}{d}-1} \prod_{k=1}^{d-1} (k + jd) = \frac{(lm)!}{d^{\frac{lm}{d}} \left(\frac{lm}{d}\right)!} \tag{1.1}$$

When $l, d$ are fixed and let $m \to \infty$, by Stirling's formula we obtain

$$\frac{(lm)!}{d^{\frac{lm}{d}} \left(\frac{lm}{d}\right)!} = e^{O\left(\frac{1}{m}\right)} \sqrt{d} \left(\frac{lm}{e}\right)^{\frac{lm(d-1)}{d}}$$

Let $\Omega_{m,d,l}$ consists of all the $(d,l)$-regular hypergraphs with $m$ hyperedges. A random $(d,l)$-regular graph $\mathcal{G}_{m,d,l} \in \Omega_{m,d,l}$ is a random graph uniformly distributed in $\Omega_{m,d,l}$.

We shall apply the first and second moments method; and find a region for parameters when the exponential growth rate of the second moment is twice of that of the first moment. Similar approach was used to determine the asymptotic independence number for the dense Erdös-Rényi random graph ([14]); and to show the existence and non-existence of perfect matchings on hypergraphs ([5]). The major difference between a matching problem and a perfect matching problem lies in the fact that in the matching problem, we need to find the global maximizer for a function of two real variables, in contrast to the case that in a perfect matching problem, the global maximizer of a single variable function is studied. When the second moment is approximately the square of the first moment, we use a generalized version of the subgraph conditioning technique to show the convergence in probability of the free energy. The subgraph conditioning method was first used to study Hamiltonian cycles on a random graph (see [23]); the major difference here is that we prove that the asymptotic Poisson distribution of the number of cycles on a random graph conditional on the existence of a specific matching.

Note that when $l = 2$, the model is the same as monomer dimers on a random graph, whose free energy was computed in [1] and the convergence of the density of maximal matching was proved in [8, 4] using the technique of weak convergence (see [2, 6]). When $d = 2$, the model is the same as the independent set model on a random graph, of which the fluctuations of the maximum were studied in [7] when $l$ is sufficiently large. Free energies of vertex models on a graph with more general constraints were computed in [22, 13, 11, 12].

Here are the main results of the paper.

**Theorem 1.1.** *(1) Let $\beta_*$ be the unique root of*

$$(1 - d\beta)^l = \beta(1 - \beta)^{l-1} \tag{1.2}$$

*in $\left(0, \frac{1}{d}\right)$. Let $Z$ be the total number of matchings on a random $d,l$-regular graph $\mathcal{G}_{m,d,l}$. Then when $m$ is sufficiently large*

$$\lim_{m\to\infty} e^{-m\Phi_d(\beta_*)}\mathbb{E}Z = \frac{\sqrt{1-\beta_*}}{\sqrt{1 + (ld - d - l)\beta_*}}$$

*(2) Let*

$$L_1 : = \min\left\{\frac{1}{dl - d - l + 2}, \frac{1}{\sqrt{(d-1)(l-1)} + 1}\right\} \tag{1.3}$$

$$L_2 : = \min\left\{\frac{1}{d}\left(1 - \sqrt{\frac{d-1}{d^{\frac{l}{l-1}} - 1}}\right), \frac{dl + l^2 - 2l - d + 1}{2dl^2 - dl}, \frac{1}{\sqrt{(d-1)(l-1)} + 1}\right\} \tag{1.4}$$

*and*

$$\Phi_{d,l}(\beta) = -\beta \ln \beta + (l-1)(1-\beta)\ln(1-\beta) - \frac{l}{d}(1 - d\beta)\ln(1 - d\beta) \tag{1.5}$$

*Assume $l \geq 2$, $d \geq 2$ satisfy one of the following conditions*
*(a) $l = 2$ and $\beta \in \left(0, \frac{1}{d}\right)$; or*

*(b)* $d \geq 3$, $l \geq 2$ *such that* $\Phi'_{d,l}(L_1) \leq 0$.

*Then as* $m \to \infty$,

$$\frac{1}{m} \log Z \longrightarrow \Phi_{d,l}(\beta_*)$$

*in probability.*

(3) *Let*

$$Z(x) := \sum_{k=0}^{\frac{m}{d}} Z_k x^k, \tag{1.6}$$

*where* $Z_k$ *is the total number of matchings on* $\mathcal{G}_{m,d,l}$ *with exactly* $k$ *present hyperedges. Assume* $l \geq 2$, $d \geq 2$ *satisfy one of the following conditions:*

*(a)* $l = 2$, *and* $\beta \in \left(0, \frac{1}{d}\right)$; *or*

*(b)* $d \geq 3$, $l \geq 2$, *such that* $\Phi'_{d,l}(L_1) + \ln(x) \leq 0$; *or*

*(c)* $d \geq 3$, $l \geq 2$, *such that* $\Phi'_{d,l}(L_2) + \ln(x) \leq 0$.

*Then as* $m \to \infty$

$$\frac{1}{m} \log Z(x) \longrightarrow \Phi_{d,l}(\beta_*(x)) + \beta_*(x) \ln(x).$$

*in probability.*

Theorem 1.1(2) has the following applications.

**Example 1.2.** *It is straight forward to check that when*

$$(d, l) \in \{(2, 3), (2, 4), (2, 5), (3, 3), (4, 3)\},$$

*when have* $\Phi'_{d,l}(L_1) < 0$. *Then the free energy of matchings on* $\mathcal{G}_{m,d,l}$ *converges in probability to* $\Phi_{d,l}(\beta_*)$ *as* $m \to \infty$.

The organization of the paper is as follows. In Section 2, we compute the first moment of the total number of matchings of a given density, and the first moment of the totoal number of all matchings on $\mathcal{G}_{m,d,l}$. In section 3, we compute the second moment of matchings with a given density and find explicit conditions under which the second moment and the square of the first moment have the same exponential growth rate. In section 4, we show that conditional on a matching , the distributions of the number of cycles of different lengths on $\mathcal{G}_{m,d,l}$ converge to independent Poisson random variables as $m \to \infty$. In Section 5, we show the convergence in probability of the free energy to an explicit limit. In Section 6, we show the convergence in probability of the weighted energy to an explicit limit when each hyperedge in the matching is given a sufficiently small weight $x > 0$ and prove Theorem 1.1(3b). In Seciton 7, we prove Theorem 1.1(3c). In Section 8, we discuss the implications of our results on maximal matchings.

## 2. First Moment

Let $\beta \in (0, 1)$. Let $Z_{m\beta}$ be the total number of matchings consisting of $m\beta$ hyperedges on a random $(d, l)$-regular graph $\mathcal{G}_{m,d,l}$. In this section, we compute the first moment of $\mathbb{E}Z_{m\beta}$ and the first moment of $Z$, the totoal number of matchings on $\mathcal{G}_{m,d,l}$.

Note that if $\beta > \frac{1}{d}$ then $Z_{m\beta} = 0$. Now assume

$$\beta \in \left(0, \frac{1}{d}\right). \tag{2.1}$$

Then

$$\begin{aligned} \mathbb{E}Z_{m\beta} &= \binom{m}{m\beta} \prod_{j=0}^{d-2} \prod_{i=0}^{lm\beta-1} \frac{lm(1-\beta)-(d-1)i-j}{lm-1-di-j} \\ &= \frac{m!}{(m\beta)!(m(1-\beta))!} \frac{(lm(1-\beta))!}{(lm(1-d\beta))!} \frac{(lm(1-d\beta))!}{(lm)!} \frac{d^{lm\beta}\left(\frac{lm}{d}\right)!}{\left(\frac{lm}{d}(1-d\beta)\right)!} \end{aligned} \tag{2.2}$$

Fix $d, \beta, l$, and let $m \to \infty$, by Stirling's formula we obtain

$$\mathbb{E}Z_{m\beta} \asymp m^{-\frac{1}{2}} e^{m\Phi_{d,l}(\beta)} \tag{2.3}$$

where $\Phi_{d,l}$ is defined by (1.5).

Hence

$$\Phi'_{d,l}(\beta) = -\ln\beta - (l-1)\ln(1-\beta) + l\ln(1-d\beta) \tag{2.4}$$

$$\Phi''_{d,l}(\beta) = -\frac{1}{\beta} - \frac{ld - d\beta - (l-1)}{(1-\beta)(1-d\beta)} < 0; \qquad \forall \beta \in \left(0, \frac{1}{d}\right). \tag{2.5}$$

Then we have

**Lemma 2.1.** *(1)* $\max_{\beta\in\left(0,\frac{1}{d}\right)} \Phi_{d,l}(\beta) = \Phi_{d,l}(\beta_*)$*, where* $\beta_*$ *is given by (1.2).*

*(2) If*

$$\frac{(l-1)(d-1)}{d} \ln\left(\frac{d-1}{d}\right) - \frac{1}{d}\ln\left(\frac{1}{d}\right) := f_l\left(\frac{1}{d}\right) \geq 0$$

$\Phi_{d,l}(\beta) > 0$ *for all* $\beta \in \left(0, \frac{1}{d}\right)$.

*(3) If* $f\left(\frac{1}{d}\right) < 0$*, let* $\beta_0$ *be the unique number in* $\left(\beta_*, \frac{1}{d}\right)$ *such that*

$$\Phi_{d,l}(\beta_0) = 0.$$

*Then*

*(a) If* $\beta \in (0, \beta_0)$*, then* $\Phi_{d,l}(\beta) > 0$.

*(b) If* $\beta \in \left(\beta_0, \frac{1}{d}\right)$*, then* $\Phi_{d,l}(\beta) < 0$.

*Proof.* From (2.4) we obtain

$$\lim_{\beta\to 0+} \Phi'_{d,l}(\beta) = +\infty; \qquad \lim_{\beta\to\frac{1}{d}-} \Phi'_{d,l}(\beta) = -\infty.$$

By (2.5), the equation $\Phi'_{d,l}(\beta) = 0$ has exactly one root in $\beta_* \in \left(0, \frac{1}{d}\right)$ satisfying(1.2).

From (2.5) we see $\Phi'_{d,l}$ is monotone decreasing in $\left(0, \frac{1}{d}\right)$, therefore

- when $\beta \in (0, \beta_*)$, $\Phi'_{d,l} > 0$, and $\Phi_{d,l}(\beta)$ is monotone increasing;
- when $\beta \in \left(\beta_*, \frac{1}{d}\right)$, $\Phi'_{d,l} < 0$, and $\Phi_{d,l}(\beta)$ is monotone decreasing.

Then Part (1) of the lemma follows. Moreover,

$$\inf_{\beta\in\left(0,\frac{1}{d}\right)}\Phi_{d,l}(\beta)=\min\left\{\lim_{\beta\rightarrow 0+}\Phi_{d,l}(\beta),\ \lim_{\beta\rightarrow\frac{1}{d}-}\Phi_{d,l}(\beta)\right\}$$

By (1.5) we obtain

$$\lim_{\beta\rightarrow 0+}\Phi_{d,l}(\beta)=0;$$

and

$$\lim_{\beta\rightarrow\frac{1}{d}-}\Phi_{d,l}(\beta)=\frac{(l-1)(d-1)}{d}\ln\left(\frac{d-1}{d}\right)-\frac{1}{d}\ln\left(\frac{1}{d}\right):=f\left(\frac{1}{d}\right)$$

Then it is straightforward to check Part (2) and (3) of the lemma. □

**Lemma 2.2.** *Let* $n\in\mathbb{N}$. *Then*

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}}\leq n!\leq\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}} \tag{2.6}$$

*Moreover,*

$$\sqrt{2\pi}\ n^{n+\frac{1}{2}}e^{-n}\leq n!\leq e\ n^{n+\frac{1}{2}}e^{-n}$$

*Proof.* See Exercise 3.1.9 of [15]. □

**Lemma 2.3.** *Let* $\beta\in\left(0,\frac{1}{d}\right)$. *Then*

$$\lim_{m\rightarrow\infty}\sqrt{m}e^{-m\Phi_{d,l}(\beta)}Z_{m\beta}=\frac{1}{\sqrt{2\pi\beta(1-d\beta)}}$$

*The convergence is uniform in any compact subset of* $\left(0,\frac{1}{d}\right)$.

*Proof.* Let $h\in[1,\frac{m}{d}-1]$ be a positive integer. By (2.2) and (2.6), we obtain

$$\frac{B(m,h)e^{m\Phi_{d,l}\left(\frac{h}{m}\right)}}{\sqrt{2m\pi\frac{h}{m}\left(1-\frac{dh}{m}\right)}}\leq\mathbb{E}Z_h\leq\frac{A(m,h)e^{m\Phi_{d,l}\left(\frac{h}{m}\right)}}{\sqrt{2m\pi\frac{h}{m}\left(1-\frac{dh}{m}\right)}}; \tag{2.7}$$

where

$$\begin{aligned}A(m,h)&=e^{\frac{1}{12m}+\frac{1}{12l(m-h)}-\frac{1}{12h+1}-\frac{1}{12(m-h)+1}-\frac{1}{12ml+1}+\frac{d}{12ml}-\frac{d}{12l(m-hd)+d}}\\B(m,h)&=e^{\frac{1}{12m+1}+\frac{1}{12l(m-h)+1}-\frac{1}{12h}-\frac{1}{12(m-h)}-\frac{1}{12lm}+\frac{d}{12lm+d}-\frac{d}{12l(m-hd)}}\end{aligned}$$

Then the lemma follows. □

**Proof of Theorem 1.1(1).** Note that

$$Z_0=1$$

$$Z_{\frac{m}{d}}=\frac{m!}{\left(\frac{m}{d}\right)!\left(\frac{m(d-1)}{d}\right)!}\frac{\left(\frac{lm(d-1)}{d}\right)!}{(lm)!}d^{2m\beta}\left(\frac{lm}{d}\right)!$$

We have

$$B\left(m,\frac{m}{d}\right)\sqrt{l}e^{m\Phi_{d,l}\left(\frac{1}{d}\right)}\leq \mathbb{E}Z_{\frac{m}{d}}\leq A\left(m,\frac{m}{d}\right)\sqrt{l}e^{m\Phi_{d,l}\left(\frac{1}{d}\right)};$$

where

$$\begin{aligned}A\left(m,\frac{m}{d}\right) &= e^{\frac{1}{12m}+\frac{d}{12lm(d-1)}-\frac{d}{12m+d}-\frac{d}{12m(d-1)+d}-\frac{1}{12lm+1}+\frac{d}{12lm}}\\ B\left(m,\frac{m}{d}\right) &= e^{\frac{1}{12m+1}+\frac{d}{12lm(d-1)+d}-\frac{d}{12m}-\frac{d}{12m(d-1)}-\frac{1}{12lm}+\frac{d}{12lm+d}}\end{aligned}$$

Then by (2.7)

$$\mathbb{E}Z = \sum_{h=0}^{\frac{m}{d}}\mathbb{E}Z_h = e^{m\Phi_{d,l}(\beta_*)}\left(\sum_{h=1}^{\frac{m}{d}-1}\frac{e^{m\left[\Phi_{d,l}\left(\frac{h}{m}\right)-\Phi_{d,l}(\beta_*)\right]}}{\sqrt{2h\pi\left(1-\frac{dh}{m}\right)}}\right)\left(1+O\left(\frac{1}{m}\right)\right) \tag{2.8}$$

Let $K\geq 1$ be a positive integer. We shall divide the sum over $h$ in the right hand side of (2.8) into two parts

(1) Part I is the sum over $h$ when $\frac{h}{m}$ is in a $\frac{K}{\sqrt{m}}$ neighborhood of $\beta_*$.
(2) Part II is the sum over $h$ when $\frac{h}{m}$ is outside the $\frac{K}{\sqrt{m}}$ neighborhood of $\beta_*$.

More precisely, let

$$\mathcal{Q}:=\left\{h\in\mathbb{N}:1\leq h\leq\frac{m}{d}-1,\left|\frac{h}{m}-\beta_*\right|\leq\frac{K}{\sqrt{m}}\right\};$$

then

$$\sum_{h=1}^{\frac{m}{d}-1}\frac{e^{m\left[\Phi_{d,l}\left(\frac{h}{m}\right)-\Phi_{d,l}(\beta_*)\right]}}{\sqrt{2h\pi\left(1-\frac{dh}{m}\right)}}=\mathrm{I}+\mathrm{II}$$

where

$$\mathrm{I}=\sum_{h\in\mathcal{Q}}\frac{e^{m\left[\Phi_{d,l}\left(\frac{h}{m}\right)-\Phi_{d,l}(\beta_*)\right]}}{\sqrt{2h\pi\left(1-\frac{dh}{m}\right)}};\qquad \mathrm{II}=\sum_{1\leq h\leq\frac{m}{d}-1,h\notin\mathcal{Q}}\frac{e^{m\left[\Phi_{d,l}\left(\frac{h}{m}\right)-\Phi_{d,l}(\beta_*)\right]}}{\sqrt{2h\pi\left(1-\frac{dh}{m}\right)}}.$$

Taylor expansion of $\Phi_{d,l}(\cdot)$ gives us

$$\Phi_{d,l}\left(\frac{h}{m}\right)-\Phi_{d,l}(\beta_*)=\frac{\Phi''_{d,l}(\beta_*)}{2}\left(\frac{h}{m}-\beta_*\right)^2+O\left(\frac{h}{m}-\beta_*\right)^3$$

Hence we have

$$\mathrm{II}\leq e^{-\frac{K^2}{2}\left(\frac{1}{\beta_*}+\frac{ld-d\beta_*-(l-1)}{(1-\beta_*)(1-d\beta_*)}\right)}\cdot\text{convergent geometric series}\to 0,$$

as $K\to\infty$. Moreover

$$\begin{aligned}\lim_{K\to\infty}\lim_{m\to\infty} I &= \frac{1}{\sqrt{2\pi\beta_*(1-d\beta_*)}}\int_{-\infty}^{\infty}e^{-\frac{\left(\frac{1}{\beta_*}+\frac{ld-d\beta_*-(l-1)}{(1-\beta_*)(1-d\beta_*)}\right)x^2}{2}}dx\\ &= \sqrt{\frac{1-\beta_*}{1+(ld-d-l)\beta_*}}.\end{aligned}$$

Then the lemma follows. □

## 3. Second Moment

In this section, we compute the second moment of $Z_{m\beta}$ and find explicit conditions under which $\mathbb{E}Z_{m\beta}^2$ and $(\mathbb{E}Z_{m\beta})^2$ have the same exponential growth rate.

We can also compute the second moment of $Z_{m\beta}$ as follows

(1) A double matching configuration is the union of two matchings on the same graph. Consider $Z_{m\beta}^2$ as the partition function of double matching configurations on a random graph, in which each matching consists of exactly $m\beta$ edges.

(2) In each double matching configuration $M_1 \cup M_2$ on a random graph of $m$ edges such that each matching consists of exactly $m\beta$ edges; all the hyperedges in $[m]$ are divided into 4 types by the configuration

   (a) Type (a) hyperedges are in $M_1 \cap M_2$; assume there are $s$ Type (a) edges, where $0 \leq s \leq m\beta$. The $s$ Type (a) edges splits into $ls$ half edges, each one of which can not share a vertex with any half edge in $M_1 \cup M_2$. A half edge is of Type (a) if and only if it is part of a Type (a) hyperedge. The total number of Type (a) half-edges is $ls$.

   (b) Type (b) hyperedges are in $M_1 \cap M_2^c$; the total number of Type (b) edges is $m\beta - s$. The $(m\beta - s)$ Type (b) edges splits into $l(m\beta - s)$ half edges, each one of which cannot share a vertex with any half edge in $M_1$. A half edge is of Type (b) if and only if it is part of a Type (b) edge. The total number of Type (b) half edges is $lm\beta - ls$. Depending on whether they share a vertex with a half edge in $M_2 \cap M_1^c$, Type (b) half edges can be further divided into 2 sub-types:

      (i) Each Type (bi) half edge does not share a vertex with any edge in $M_1 \cup M_2$. The total number of Type (bi) half edges is $lm\beta - ls - t$.

      (ii) Each Type (bii) half edge shares a vertex with exactly one half-edge in $M_2 \cap M_1^c$. The total number of Type (bii) half edges is $t$..

   (c) Type (c) hyperedges are in $M_1^c \cap M_2$; the total number of Type (c) edges is $m\beta - s$. The $(m\beta - s)$ Type (c) edges splits into $l(m\beta - s)$ half edges, each one of which cannot share a vertex with any half edge in $M_2$. A half edge is of Type (c) if and only if it is part of a Type (c) edge. The total number of Type (c) half edges is $lm\beta - ls$. Depending on whether they share a vertex with a half edge in $M_1 \cap M_2^c$, Type (c) half edges can be further divided into 2 sub-types:

      (i) Each Type (ci) half edge does not share a vertex with any edge in $M_1 \cup M_2$. The total number of Type (ci) half edges is $lm\beta - ls - t$.

      (ii) Each Type (cii) half edge shares a vertex with exactly one half-edge in $M_2 \cap M_1^c$. The total number of Type (cii) half edges is $t$.

   (d) Type (d) edges are in $M_1^c \cap M_2^c$; the total number of Type (d) edges is $m - 2m\beta + s$. A half edge is of Type (d) if and only if it is part of a Type (d) edge. The total number of Type (d) half-edges is $l(m - 2m\beta + s)$.

Let

$$\rho := \frac{s}{m}; \qquad \theta := \frac{t}{m}. \tag{3.1}$$

It is straight forward to check the following lemma

**Lemma 3.1.**

$$\rho \in [0, \beta]; \qquad \theta \in [0, l\beta - l\rho]. \tag{3.2}$$

**Lemma 3.2.**

$$\left(2\beta - \rho - \frac{\theta}{l}\right) d \leq 1. \tag{3.3}$$

*Proof.* It suffices to show that

$$(lm(2\beta - \rho) - \theta m)d \leq lm$$

Note that $lm$ is the total number of half edges in the graph $\mathcal{G}_{m,d,l}$; $lm(2\beta - \rho) - 2\theta m$ is the total number of half edges of Type (a) (bi) and (ci); any two of these half edges cannot share a vertex; hence the total number of half edges sharing a vertex with these half edges is

$$(lm(2\beta - \rho) - 2\theta m)d \tag{3.4}$$

The total number of Type (bii) and (cii) half edges is $2t$; these half edges are incident to exactly $t$ vertices, and the total number of half edges incident to these $t$ vertices is

$$\theta m d \tag{3.5}$$

There are no common half edges in (3.4) and (3.5), therefore their sum is at most $ml$-the total number of half edges in the graph $\mathcal{G}_{m,d,l}$. Then the lemma follows. □

By (3.3), we obtain

$$t \geq \max\left\{0, 2lm\beta - ls - \frac{lm}{d}\right\}.$$

Let $K_{m,\beta,s,t}$ be the total number of possible half edge assignments for the first $(2lm\beta - ls - t)$ vertices. Then

$$\begin{aligned} K_{m,\beta,s,t} &= \prod_{i=0}^{2lm\beta-ls-t-1} \prod_{j=0}^{d-2} (lm - 1 - di - j) \\ &= \frac{(lm)!}{(lm - d(2lm\beta - ls - t))!} \frac{\left(\frac{lm}{d} - 2lm\beta + ls + t\right)!}{d^{2lm\beta - ls - t}\left(\frac{lm}{d}\right)!} \end{aligned}$$

$$\begin{aligned}
\mathbb{E}Z_{m\beta}^2 &= \sum_{s=0}^{m\beta}\binom{m}{s}\binom{m-s}{m\beta-s}\binom{m-m\beta}{m\beta-s}\prod_{j=0}^{d-2}\left[\prod_{i=0}^{ls-1} l(m-2m\beta+s)-(d-1)i-j\right]\\
&\quad\left\{\sum_{t=\max\left\{0,2lm\beta-ls-\frac{lm}{d}\right\}}^{lm\beta-ls}\binom{lm\beta-ls}{t}^2(d-1)^t t!\right.\\
&\quad\left[\prod_{k=0}^{lm\beta-ls-t-1} lm(1-2\beta)+ls-(d-1)(2s+k)-j\right]\\
&\quad\left.\times\left[\prod_{h=0}^{lm\beta-ls-t-1} lm(1-2\beta)+ls-(d-1)(lm\beta-t+h)-j\right]\right\}\\
&\quad\times\left\{\prod_{w=0}^{t-1}\prod_{r=0}^{d-3} 2m-d(2lm\beta-ls-2t+w)-(2t-2w)-r\right\}\frac{1}{K_{m,\beta,s,t}}\\
&= \sum_{s=0}^{m\beta}\sum_{t=\max\left\{0,2lm\beta-ls-\frac{lm}{d}\right\}}^{lm\beta-ls} F_\beta(s,t). \qquad (3.6)
\end{aligned}$$

where

$$\begin{aligned}
F_\beta(s,t) &= \frac{m!(d-1)^t}{s!(m\beta-s)!(m\beta-s)!(m-2m\beta+s)!}\frac{[(lm\beta-ls)!]^2}{t![(lm\beta-ls-t)!]^2}\\
&\quad\frac{[l(m-2m\beta+s)]!}{(lm)!}\frac{d^{2lm\beta-ls-t}\left(\frac{lm}{d}\right)!}{\left(\frac{lm}{d}-2lm\beta+ls+t\right)!} \qquad (3.7)
\end{aligned}$$

Now we assume

$$\rho\in(0,\beta); \qquad (3.8)$$

$$\theta\in(0,l\beta-l\rho). \qquad (3.9)$$

When $m\to\infty$, by Stirling's formula we obtain

$$F_\beta(\rho m,\theta m)\asymp m^{-2}e^{m\Psi_{d,l}(\beta,\rho,\theta)}$$

where

$$\begin{aligned}
\Psi_{d,l}(\beta,\rho,\theta) &= -\rho\ln\rho-\theta\ln\frac{\theta}{l}+(l-1)(1-2\beta+\rho)\ln(1-2\beta+\rho)+\theta\ln(d-1)\\
&\quad+2(l-1)(\beta-\rho)\ln(\beta-\rho)-2l\left(\beta-\rho-\frac{\theta}{l}\right)\ln\left(\beta-\rho-\frac{\theta}{l}\right)\\
&\quad-\frac{l}{d}\left(1-2\beta d+\rho d+\frac{\theta}{l}d\right)\ln\left(1-2\beta d+\rho d+\frac{\theta}{l}d\right) \qquad (3.10)
\end{aligned}$$

Then we have for all $\beta \in \left(0, \frac{1}{d}\right)$

$$\begin{aligned}
\frac{\partial \Psi_{d,l}(\beta,\rho,\theta)}{\partial \rho} &= -\ln \rho + (l-1)\ln(1-2\beta+\rho) - 2(l-1)\ln(\beta-\rho) \\
&\quad +2l\ln\left(\beta-\rho-\frac{\theta}{l}\right) - l\ln\left(1-2\beta d+\rho d+\frac{\theta d}{l}\right) \\
\frac{\partial \Psi_{d,l}(\beta,\rho,\theta)}{\partial \theta} &= -\ln\frac{\theta}{l} + 2\ln\left(\beta-\rho-\frac{\theta}{l}\right) - \ln\left(1-2\beta d+\rho d+\frac{\theta d}{l}\right) + \ln(d-1)
\end{aligned}$$

and

$$\frac{\partial^2 \Psi_{d,l}(\beta,\rho,\theta)}{\partial \rho^2} = -\frac{1}{\rho} + \frac{l-1}{1-2\beta+\rho} + \frac{2(l-1)}{\beta-\rho} - \frac{2l}{\beta-\rho-\frac{\theta}{l}} - \frac{ld}{1-2\beta d+\rho d+\frac{\theta d}{l}} \tag{3.11}$$

$$\frac{\partial^2 \Psi_{d,l}(\beta,\rho,\theta)}{\partial \theta^2} = -\frac{1}{\theta} - \frac{\frac{2}{l}}{\beta-\rho-\frac{\theta}{l}} - \frac{\frac{d}{l}}{1-2\beta d+\rho d+\frac{\theta d}{l}} < 0. \tag{3.12}$$

$$\frac{\partial^2 \Psi_{d,l}(\beta,\rho,\theta)}{\partial \theta \partial \rho} = -\frac{2}{\beta-\rho-\frac{\theta}{l}} - \frac{d}{1-2\beta d+\rho d+\frac{\theta d}{l}}$$

Now we solve the following system of equations

$$\begin{cases} \frac{\partial \Psi_{d,l}(\beta,\rho,\theta)}{\partial \rho} = 0; \\ \frac{\partial \Psi_{d,l}(\beta,\rho,\theta)}{\partial \theta} = 0. \end{cases} \tag{3.13}$$

and obtain

$$(d-1)\left(\beta-\rho-\frac{\theta}{l}\right)^2 = \frac{\theta}{l}\left(1-2\beta d+\rho d+\frac{\theta d}{l}\right). \tag{3.14}$$

$$\rho(d-1)^l(\beta-\rho)^{2(l-1)} = \left(\frac{\theta}{l}\right)^l (1-2\beta+\rho)^{l-1} \tag{3.15}$$

It is straightforward to check that

$$\rho = \beta^2; \qquad \theta = l(d-1)\beta^2. \tag{3.16}$$

is a pair of solutions for (3.14) and (3.15).

**Lemma 3.3.** *Assume (2.1),(3.8) (3.9) hold. Assume $d \geq 2$. For each $\beta \in \left(0, \frac{1}{d}\right)$, let*

$$\mathcal{R}_\beta := \left\{(\rho,\theta) : \rho \in (0,\beta), \theta \in \left(\max\left\{0, 2l\beta - l\rho - \frac{l}{d}\right\}, l\beta - l\rho\right)\right\}, \tag{3.17}$$

*Then for any $(\rho,\theta) \in \mathcal{R}_\beta$,*

$$\frac{\partial^2 \Psi_{d,l}(\beta,\rho,\theta)}{\partial \rho^2} < 0. \tag{3.18}$$

*Proof.* Note that $\theta \leq \frac{l}{d}$ by (3.9) when $0 < \beta < \frac{1}{d}$. By (3.9), (3.3), we obtain

$$\begin{aligned}
\frac{l-1}{1-2\beta+\rho} - \frac{ld}{1-2\beta d+\rho d+\frac{\theta d}{l}} &= \frac{-l(d-1) + \frac{\theta d(l-1)}{l} - (1-2\beta d+\rho d)}{(1-2\beta+\rho)\left(1-2\beta d+\rho d+\frac{\theta d}{l}\right)}; \\
&\leq \frac{-l(d-2) - \left(1-2\beta d+\rho d+\frac{\theta d}{l}\right)}{(1-2\beta+\rho)\left(1-2\beta d+\rho d+\frac{\theta d}{l}\right)} < 0.
\end{aligned}$$

Moreover,

$$\frac{2(l-1)}{\beta-\rho} - \frac{2(l-1)}{\beta-\rho-\frac{\theta}{2}} < 0.$$

Then (3.18) follows from (3.11). □

Define the Hessian matrix

$$H(\beta,\rho,\theta) := \begin{pmatrix} \frac{\partial^2\Psi_{d,l}(\beta,\rho,\theta)}{\partial\rho^2} & \frac{\partial^2\Psi_{d,l}(\beta,\rho,\theta)}{\partial\theta\partial\rho} \\ \frac{\partial^2\Psi_{d,l}(\beta,\rho,\theta)}{\partial\theta\partial\rho} & \frac{\partial^2\Psi_{d,l}(\beta,\rho,\theta)}{\partial\theta^2} \end{pmatrix}$$

**Lemma 3.4.** *Assume (2.1),(3.8) (3.9) hold.*

*(1) If*

$$l \le d \tag{3.19}$$

*Then (3.16) is a local maximizer of $\Psi_{d,l}(\beta,\rho,\theta)$ when $(\rho,\theta)\in\mathcal{R}_\beta$ for every $\beta\in\left(0,\frac{1}{d}\right)$;*

*(2) If*

$$\beta < \frac{1}{1+\sqrt{(d-1)(l-1)}} \tag{3.20}$$

*then (3.16) is a local maximizer of $\Psi_{d,l}(\beta,\rho,\theta)$ when $(\rho,\theta)\in\mathcal{R}_\beta$.*

*Proof.* Since (3.16) satisfies (3.13), it suffice to show that $H(\beta,\beta^2,l(d-1)\beta^2)$ is negative definite when (3.19) holds. By (3.18), (3.12),it suffices to show that

$$\det H(\beta,\beta^2,l(d-1)\beta^2) > 0. \tag{3.21}$$

when (3.19) holds. Note that

$$\begin{aligned} &H(\beta,\beta^2,l(d-1)\beta^2)\\ =&\begin{pmatrix} -\frac{1}{\beta^2}+\frac{l-1}{(1-\beta)^2}+\frac{2(l-1)}{\beta(1-\beta)}-\frac{2l}{\beta(1-d\beta)}-\frac{ld}{(1-\beta d)^2} & -\frac{2}{\beta(1-d\beta)}-\frac{d}{(1-d\beta)^2} \\ -\frac{2}{\beta(1-d\beta)}-\frac{d}{(1-d\beta)^2} & -\frac{1}{l(d-1)\beta^2}-\frac{2}{l\beta(1-d\beta)}-\frac{d}{l(1-\beta d)^2} \end{pmatrix}, \end{aligned}$$

and therefore

$$\det H(\beta,\beta^2,l(d-1)\beta^2) = \frac{\beta^2 d-2\beta+\beta^2 l-\beta^2 dl+1}{l\beta^4(\beta d-1)^2(\beta-1)^2(d-1)} \tag{3.22}$$

Hence (3.21) holds if and only if

$$g(\beta) := \beta^2(d+l-dl)-2\beta+1 > 0.$$

Note that when $d\ge 2, l\ge 2$, $d+l-dl\le 0$, and the identity holds if and only if $d=l=2$.

- When $d=l=2$, $1-2\beta>0$ given that $\beta\in\left(0,\frac{1}{d}\right)$.
- When $d\ge 2$, $l\ge 2$, and at least one of $d$ and $l$ is strictly greater than 2, $g(\beta)>0$ for all $\beta\in\left(0,\frac{1}{d}\right)$ if and only if $g\left(\frac{1}{d}\right)\ge 0$. This gives $l\le d$.

□

**Lemma 3.5.** *Assume (2.1),(3.8) (3.9) hold. If $l=2$, then for each $\beta\in\left(0,\frac{1}{d}\right)$, the Hessian matrix $H(\beta,\rho,\theta)$ is negative definite for all $(\rho,\theta)\in\mathcal{R}_\beta$.*

*Proof.* Again by (3.12) (3.18) it suffices to show that $\det H>0$ when $l=2$. Let

$$\begin{aligned}
A: &= -\frac{1}{\rho}+\frac{1}{1-2\beta+\rho}+\frac{2}{\beta-\rho};\\
B: &= -\frac{2}{\beta-\rho-\frac{\theta}{2}}-\frac{d}{1-2\beta d+\rho d+\frac{\theta d}{2}}<0\\
C: &= -\frac{1}{\theta}<0.\\
U: &= -\frac{1}{\rho}+\frac{1}{1-2\beta+\rho}<0;
\end{aligned}$$

Then

$$\det H = (A+2B)\left(C+\frac{B}{2}\right)-B^2=AC+B\left(\frac{A}{2}+2C\right)$$

Note that

$$AC=CU-\frac{2}{\theta(\beta-\rho)}\geq-\frac{2}{\theta(\beta-\rho)}$$

and

$$\begin{aligned}
B\left(\frac{A}{2}+2C\right) &= \frac{BU}{2}+B\left(\frac{1}{\beta-\rho}-\frac{2}{\theta}\right)\\
&= \frac{BU}{2}+\frac{4}{(\beta-\rho)\theta}+\left(-\frac{d}{1-2\beta d+\rho d+\frac{\theta d}{2}}\right)\left(\frac{1}{\beta-\rho}-\frac{2}{\theta}\right)\\
&\geq \frac{4}{(\beta-\rho)\theta}
\end{aligned}$$

where the last inequality follows from (3.9). Hence we have $\det H>0$, and the lemma follows. □

**Lemma 3.6.** *Assume (2.1),(3.8) (3.9) hold. Let $l\geq 3$. For each $\beta\in\left(0,\frac{1}{2d}\right]$, the set*

$$\mathcal{R}_\beta\cap\{(\rho,\theta):\det H(\beta,\rho,\theta)>0\}$$

*has at most one connected component in the $(\rho,\theta)$-plane.*

*Proof.* For simplicity let

$$\eta:=\frac{\theta}{l}.$$

For each fixed $\beta\in\left(0,\frac{1}{d}\right)$, explicit computations show that $\det H(\beta,\rho,l\eta)=0$ if and only if

$$\eta=(\beta-\rho)\left(1-\frac{2(1-\beta d)(\beta+\rho-2\beta^2-\rho l(1-\beta))}{D(\rho)}\right):=\xi(\rho)$$

where

$$D(\rho) = \beta + \rho - 2l\rho - d\rho^2 - l\rho^2 - 2\beta^2 - \beta d\rho + 3\beta l\rho + 2\beta^2 d\rho + dl\rho^2 + \beta dl\rho - 2\beta^2 dl\rho$$

Then for each fixed $(d, l, \beta)$ such that $\beta \in \left(0, \frac{1}{d}\right)$, the equation

$$\xi(\rho) = \beta - \rho$$

has at most one solution in $(0, \beta)$ given by

$$\rho_5 = \frac{\beta(1-2\beta)}{l(1-\beta)-1}. \tag{3.23}$$

Moreover,

$$\xi'(\rho) = -1 + \frac{F(\rho)}{D^2} \tag{3.24}$$

where

$$F(\rho) = 2(\beta d - 1)^2(l-1)\left[\left(2l(1-\beta)^2 - 1\right)\rho^2 - 2\beta(1-2\beta)\rho - \beta^2(1-2\beta)^2\right]$$

The equation

$$\xi'(\rho) = -1$$

has a unique positive solution given by

$$\rho_3 = \frac{\beta(1-2\beta)}{\sqrt{2l}(1-\beta)-1} \tag{3.25}$$

Moreover, when $\rho < \rho_3$, $F(\rho) < 0$ and when $\rho > \rho_3$, $F(\rho) > 0$. Note that

$$D(\rho) = (dl - l - d)\rho^2 + (3\beta l - \beta d - 2l + 2\beta^2 d + \beta dl - 2\beta^2 dl + 1)\rho - 2\beta^2 + \beta. \tag{3.26}$$

Hence

$$\begin{aligned} D(0) &= \beta(1-2\beta) > 0. \\ D(\beta) &= 2\beta(1-\beta d)(\beta-1)(l-1) < 0. \end{aligned}$$

Let $\rho_1$ be the unique point in $(0, \beta)$ such that $D(\rho_1) = 0$.

When $\beta d \leq \frac{1}{2}$ we have

$$\mathcal{R}_\beta = \{(\rho, \theta) : \rho \in (0, \beta), \theta \in (0, l\beta - l\rho)\},$$

then

$$\eta = \xi(\rho) = (\beta - \rho)\frac{J(\rho)}{D(\rho)};$$

where

$$\begin{aligned} J(\rho) &= (dl - l - d)\rho^2 + (\beta d + \beta l + 2\beta^2 d - \beta dl - 1)\rho + 2\beta^2 d - \beta - 4\beta^3 d + 2\beta^2 \\ &= -\rho(dl - l - d)(\beta - \rho) - \rho(1 - 2\beta^2 d) - \beta(1-2\beta)(1-2\beta d) \leq 0 \end{aligned} \tag{3.27}$$

where the identity holds if and only if $\rho = 0$ and $\beta d = \frac{1}{2}$.

$$J(\beta) = 2\beta(1-\beta d)(\beta - 1) < 0$$

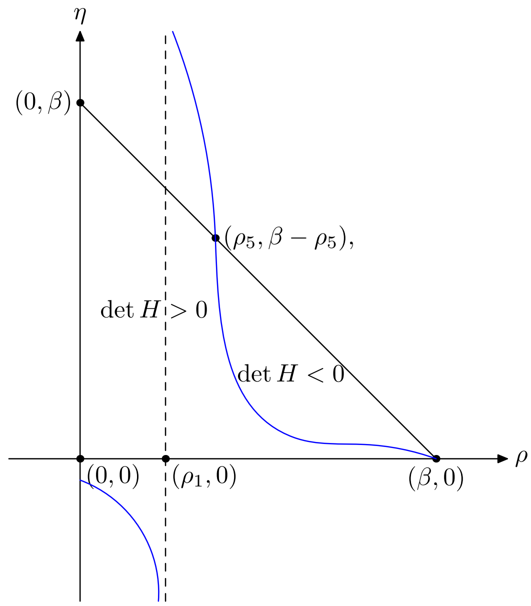


FIGURE 3.1. The curve $\det H = 0$ is represented by blue lines. When $\beta \in \left(0, \frac{1}{2d}\right]$, $R_\beta$ is represented by the triangular region in the $(\rho, \theta)$ plane with vertices $(\beta, 0)$, $(0,0)$ and $(0, \beta)$. The curve $\det H = 0$ intersects the line $\rho + \eta = \beta$ at a unique point $(\rho_5, \beta - \rho_5)$. The curve $\det H = 0$ divide the triangular region into two connected components - the left component satisfies $\det H > 0$ and the right component satisfies $\det H < 0$.

The when $\rho \in (0, \beta)$, the curve $\eta = \xi(\rho)$ has two components:

- When $\rho \in (0, \rho_1)$, $\xi(\rho) \leq 0$ and $\lim_{\rho \to \rho_1 -} \xi(\rho) = -\infty$.
- When $\rho \in (\rho_1, \rho)$, $\xi(\rho) > 0$, $\lim_{\rho \to \rho_1 +} \xi(\rho) = \infty$ and $\frac{J(\beta)}{D(\beta)} = \frac{1}{l-1} < 1$ when $l \geq 3$

Hence when $\rho \in (0, \rho_1)$, the curve $\theta = l\xi(\rho)$ does not intersect $\mathcal{R}_\beta$. When $\rho \in (\rho_1, \rho)$, the curve $\theta = l\xi(\rho)$ divide the triangular region $\mathcal{R}_\beta$ into two connected components, exactly one of them satisfy $\det H > 0$. It is straightforward to check that the component on the right (resp. left) satisfies $\det H < 0$ (resp. $\det H > 0$) given that $D(\rho) < 0$ for $\rho \in (\rho_5, \beta)$; see Figure 3.1. □

**Lemma 3.7.** *Assume (2.1),(3.8) (3.9) hold. Let $l \geq 3$. For each $\beta \in \left(\frac{1}{2d}, \frac{1}{d}\right)$, the set*

$$\mathcal{R}_\beta \cap \{(\rho, \theta) : \det H(\beta, \rho, \theta) > 0\}$$

*has at most two connected components in the $(\rho, \theta)$-plane.*

*Proof.* If $1 > \beta d > \frac{1}{2}$, then

$$\begin{aligned} \mathcal{R}_\beta : &= \left\{(\rho, \theta) : \rho \in \left(0, 2\beta - \frac{1}{d}\right), \theta \in \left(2\beta l - \rho l - \frac{l}{d}, l\beta - l\rho\right)\right\} \\ &\cup \left\{(\rho, \theta) : \rho \in \left(2\beta - \frac{1}{d}, \beta\right), \theta \in (0, l\beta - l\rho)\right\}, \end{aligned}$$

$$0 < J(0) = -\beta(1-2\beta)(1-2\beta d)$$

and

$$(\beta - 0)\frac{J(0)}{D(0)} = \beta(2\beta d - 1) < 2\beta - \frac{1}{d} \tag{3.28}$$

Moreover

$$J\left(2\beta - \frac{1}{d}\right) = \frac{l(d-1)(2\beta d-1)(\beta d-1)}{d^2} < 0.$$

From (3.27) we can see that the graph of $J(\rho)$ is a parabola opening upwards. Therefore there is a unique solution $\rho_2 \in \left(0, 2\beta - \frac{1}{d}\right)$ such that $J(\rho_2) = 0$; and $J(\rho) < 0$ for all $\rho \in \left(2\beta - \frac{1}{d}, \beta\right)$. Recall that $\rho_1$ is the unique solution for $D(\rho) = 0$ in $(0, \beta)$. The following cases might occur

(1) $\rho_1 \geq \rho_2$:
  (a) when $\rho \in (0, \rho_2]$: $D(\rho) > 0$, $J(\rho) > 0$ hence $\frac{J(\rho)}{D(\rho)} > 0$. We claim that the graph of $\eta = \xi(\rho)$ when $\rho \in (0, \rho_2)$ in this case is always below the straight line $\eta = 2\beta - \rho - \frac{1}{d}$ in the $(\rho, \eta)$ plane. By (3.28) we know this is true when $\rho = 0$. This is also true when $\rho = \rho_2$ since $\rho_2 < 2\beta - \frac{1}{d}$ and therefore $2\beta - \rho_2 - \frac{1}{d} > 0$ while $\frac{J(\rho_2)}{D(\rho_2)} = 0$. Therefore if $\eta = \xi(\rho)$ intersects $\eta = 2\beta - \rho - \frac{1}{d}$ when $\rho \in (0, \rho_2)$, then the intersection must be at least two points (counting multiplicities).
  By (3.24) we have

$$\frac{d\xi(\rho) - (2\beta - \rho - \frac{1}{d})}{d\rho} = \frac{F(\rho)}{D(\rho)^2}$$

  When $l \geq 3$ and $d \geq 2$, $F(\rho)$ a parabola opening upwards with $F(0) < 0$. Hence if there exists $0 < \rho_3 < \rho_2$ such that $\frac{F(\rho_3)}{D(\rho_3)^2} = 0$; then $\frac{F(\rho_3)}{D(\rho_3)} > 0$ for all $\rho \in (\rho_3, \rho_2)$, and $\frac{F(\rho_3)}{D(\rho_3)} < 0$ for all $\rho \in (0, \rho_3)$. Then $\eta = \xi(\rho)$ intersects $\eta = 2\beta - \rho - \frac{1}{d}$ in at most one point when $\rho \in (0, \rho_2)$. It follows that $\xi(\rho) < 2\beta - \rho - \frac{1}{d}$ for all $\rho \in (0, \rho_2)$.
  (b) When $\rho \in (\rho_2, \rho_1]$: $D(\rho) > 0$ and $J(\rho) < 0$, hence $\frac{J(\rho)}{D(\rho)} < 0$, and there for $\eta = \xi(\rho)$ is always below the curve $\eta = \max\{0, 2\beta - \rho - \frac{1}{d}\}$.
  (c) When $\rho \in (\rho_1, \beta)$. $D(\rho) < 0$ and $J(\rho) < 0$. Hence $\frac{J(\rho)}{D(\rho)} > 0$ and $\lim_{\rho \to \rho_1+} \frac{J(\rho)}{D(\rho)} = +\infty$.
  Note that $\xi(\rho) - (2\beta - \rho - \frac{1}{d})$ achieves its minimum in $(\rho_1, \beta)$ at $\rho_3$. Explicit computations show that

$$\lim_{\beta \to \frac{1}{d}-} \left[\xi(\rho_3) - \left(2\beta - \rho_3 - \frac{1}{d}\right)\right] \frac{d}{1-\beta d} = -1$$

  Therefore when $\beta \in \left(\frac{1}{d} - \epsilon, \frac{1}{d}\right)$ and $\epsilon$ is sufficiently small,

$$\xi(\rho_3) - \left(2\beta - \rho_3 - \frac{1}{d}\right) < 0.$$

Analyzing the derivative shows that $\xi(\rho) - \max\{\left(2\beta - \rho - \frac{1}{d}\right), 0\}$ can have at most two zeros; hence in this case $\mathcal{R}_\beta \cap \{(\rho, \theta), \det H(\beta, \rho, \theta) > 0\}$ has at most two connected components in the $(\rho, \theta)$-plane.

(2) $\rho_1 < \rho_2$: when $\rho \in (0, \rho_1)$, $D(\rho) > 0$, $J(\rho) > 0$, hence $\frac{J(\rho)}{D(\rho)} > 0$. By (3.28) we see that $\xi(0) < 2\beta - \frac{1}{d} < \beta$. Note also that $(\beta - \rho)\lim_{\rho \to \rho_1 -} \frac{J(\rho)}{D(\rho)} = +\infty > \beta - \rho_1 > \max\{2\beta - \rho_1 - \frac{1}{d}, 0\}$. Then the curve $\eta = \xi(\rho)$ must intersect the curve $\eta = \beta - \rho$ at $\rho_5 \in (0, \rho_1)$. Then somewhere in $(0, \rho_5)$ we have $\xi'(\rho) > -1$. Then we have $\rho_3 < \rho_5$. From (3.23) and (3.25) we see that if $\rho_3 < \rho_5$, then $l < 2$. But this is a contradiction to the fact that $l \geq 3$. Then we must have $\rho_1 \geq \rho_2$.

Then the lemma follows; see Figure 3.2. □

**Lemma 3.8.** *$\xi'(\rho)$ is strictly increasing when $\rho \in (\rho_5, \beta)$.*

*Proof.* It suffices to show that $\xi''(\rho) > 0$, $\forall \rho \in (\rho_5, \beta)$. Note that

$$\xi''(\rho) = \frac{F'(\rho)D(\rho) - 2F(\rho)D'(\rho)}{[D(\rho)]^3} = \frac{4(1-\beta d)^2(l-1)K(\rho)}{[D(\rho)]^3}$$

In particular we have

$$\xi''(\rho_3) = \frac{F'(\rho_3)}{[D(\rho_3)]^2} > 0 \tag{3.29}$$

and therefore

$$K(\rho_3) < 0. \tag{3.30}$$

Note that

$$K'(\rho) = 3(dl - d - l)L(\rho); \tag{3.31}$$

where

$$L(\rho) = -2l(1-\beta)^2\rho^2 + (\rho + \beta(1-2\beta))^2$$

In particular we have

$$L(\beta) = -\beta^2(1-\beta)^2(l-2) < 0$$

and

$$L(\rho_3) = 0.$$

Since $L$ is a parabola opening downwards, with exactly one negative root and one positive root $\rho_3$, we obtain that

$$L(\rho) < 0;\ \forall \rho \in (\rho_3, \beta);$$

and

$$L(\rho) > 0;\ \forall \rho \in (\rho_5, \rho_3).$$

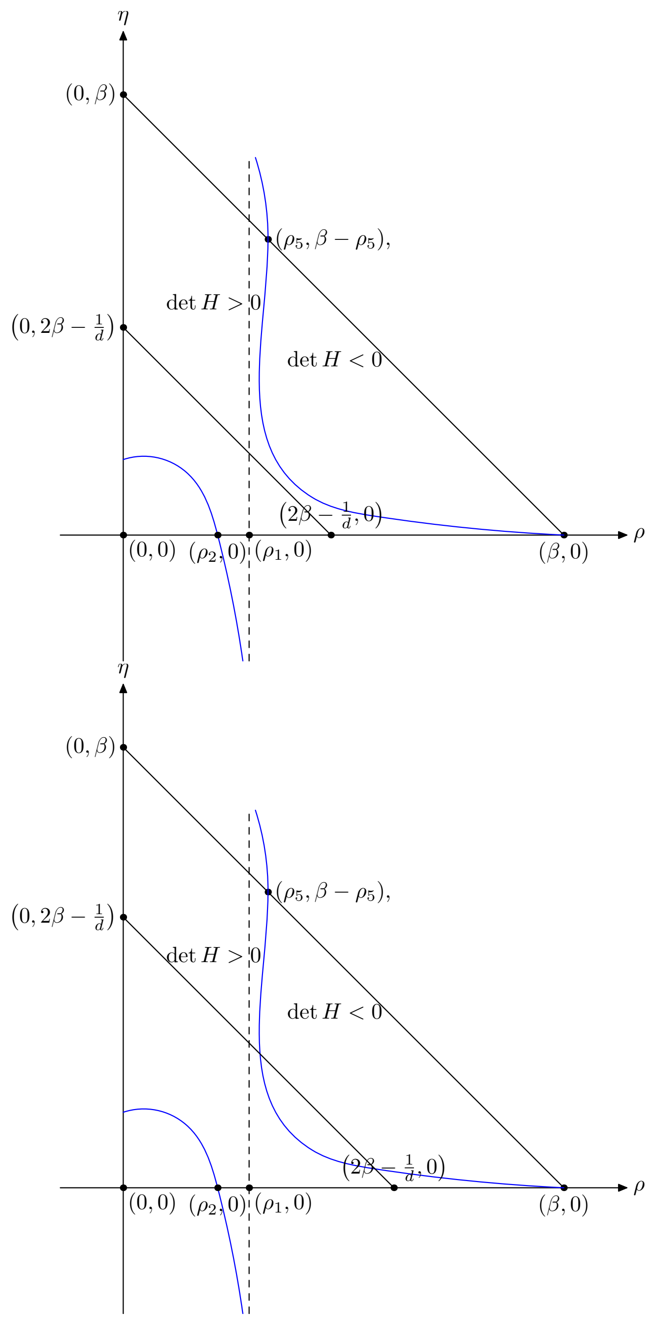


FIGURE 3.2. The top graph represents the case when $\det H > 0$ has a unique component in $R_\beta$; while the bottom graph represents the case when $\det H > 0$ has two components in $R_\beta$.

By (3.31) we obtain

$$K'(\rho) < 0;\ \forall \rho \in (\rho_3, \beta);$$
$$K'(\rho) > 0;\ \forall \rho \in (\rho_5, \rho_3).$$

Hence $K(\rho_3)$ is the maximal value of $K(\rho)$ when $\rho \in (\rho_5, \beta)$.By (3.30), we obtain

$$K(\rho) < 0,\ \forall \rho \in (\rho_5, \beta).$$

By (3.29) we obtain

$$\xi''(\rho) > 0,\ \forall \rho \in (\rho_5, \beta).$$

Then the lemma follows. □

**Lemma 3.9.** *Let $l \geq 3$ and $d \geq 2$, $\beta \in \left(0, \frac{1}{d}\right)$ such that (3.20) holds. Assume $R_\beta \cap \{(\rho, \theta) : \det H > 0\}$ is connected. If*

$$\xi'(\rho) < -\frac{(d-1)\beta}{1-\beta};\ \forall \rho \in (\rho_3, \beta) \tag{3.32}$$

*Then*

$$\max_{(\rho,\theta)\in R_\beta \cap\{(\rho,\theta):\det H \geq 0\}} \Psi_{d,l}(\beta, \rho, \theta) = 2\Phi_{d,l}(\beta).$$

*Moreover, $(\beta^2, l(d-1)\beta^2)$ is the unique maximizer for $\Psi_{d,l}(\beta, \rho, \theta)$ in the region $R_\beta \cap \{(\rho, \theta) : \det H \geq 0\}$.*

*Proof.* If $\beta \in \left(0, \frac{1}{2d}\right]$, we consider the corresponding region in the $(\rho, \eta)$-plane which is bounded by

(a) $\rho = 0$, $\eta \in [0, \beta]$; and
(b) $\eta = 0$, $\rho \in [0, \theta]$; and
(c) $\eta = \xi(\rho)$, $\rho \in [\rho_5, \beta]$; and
(d) $\eta = \beta - \rho$; $\rho \in [0, \rho_5]$.

See Figure 3.1.

If $\beta \in \left(\frac{1}{2d}, \frac{1}{d}\right)$, when the set $\mathcal{R}_\beta \cap \{(\rho, \theta : \det H(\beta, \rho, \theta)) > 0\}$ has exactly one connected component in the $(\rho, \theta)$-plane, its corresponding region in the $(\rho, \eta)$-plane is bounded by

(A) $\rho = 0$, $\eta \in \left[2\beta - \frac{1}{d}, \beta\right]$; and
(B) $\eta = 0$, $\rho \in \left[2\beta - \frac{1}{d}, \theta\right]$; and
(C) $\eta = \xi(\rho)$, $\rho \in [\rho_5, \beta]$; and
(D) $\eta = \beta - \rho$, $\rho \in [0, \rho_5]$;and
(E) $\rho + \eta = 2\beta - \frac{1}{d}$, $\eta \in \left[0, 2\beta - \frac{1}{d}\right]$.

See Figure 3.2(1).

In either case by Lemma 3.4, the point $z_0 := (\beta, (d-1)\beta^2)$ is a local maximizer for $\Psi_{d,l}(\beta, \rho, \eta l)$. If the following conditions hold

(i) for any point $z$ in the region or on the boundary of the region $\mathcal{R}_\beta(\rho, \eta l) \cap \{(\rho, \eta) : \det H(\beta, \rho, l\eta)) > 0\}$, the straight line joining $z_0$ and $z$ lies in the region $\mathcal{R}_\beta(\rho, \eta l) \cap \{(\rho, \eta) : \det H(\beta, \rho, l\eta)) > 0\}$ except the endpoint (in the case that $z$ is on the boundary).

Then for any point $z$ in the region $\mathcal{R}_\beta(\rho, \eta l) \cap \{(\rho, \eta) : \det H(\beta, \rho, l\eta)) > 0\}$, or on the boundary of the region; let $l_{z_0 z}$ be the straight line joining $z$ and $z_0$ starting from $z_0$ and ending at $z$; we have

$$\Psi_{d,l}(\beta,z)=\Psi_{d,l}(\beta,z_0)+\int_{l_{z_0z}}\partial_{l_{z_0z},w}\Psi_{d,l}(\beta,w)dw$$

where $\partial_{l_{z_0z},w}\Psi_{d,l}(\beta,w)$ is the directional derivative of $\Psi_{d,l}(\beta,w)$ with respect to $w$ along $l_{z_0z}$. Note that

$$\partial_{l_{z_0z}w}\Psi_{d,l}(\beta,w)\Big|_{w=z_0}=0.$$

Since $H$ is negative definite in the region $\mathcal{R}_\beta(\rho,\eta l)\cap\{(\rho,\eta):\det H(\beta,\rho,l\eta))>0\}$ , the second order directional derivative satisfies

$$\partial^2_{l_{z_0z},w}\Psi_{d,l}(\beta,w)=\mathbf{v}^TH\mathbf{v}<0$$

for all $w$ in the interior of the line $l_{z_0z}$; where $\mathbf{v}\in\mathbb{R}^{1\times 2}$ is the unit directional vector along the line $l_{z_0,z}$. Then we obtain

$$\partial_{l_{z_0z},w}\Psi_{d,l}(\beta,w)<0;$$

for each $w$ in the interior of the line $l_{z_0z}$; and therefore

$$\Psi_{d,l}(\beta,z)<\Psi_{d,l}(\beta,z_0)$$

for any point $z$ in the region $\mathcal{R}_\beta(\rho,\eta l)\cap\{(\rho,\eta):\det H(\beta,\rho,l\eta))>0\}$, or on the boundary of the region other than $z_0$.

Now it remains to prove condition (i). It suffices to check condition (i) for each boundary point on the boundary of the region $\mathcal{R}_\beta(\rho,\eta l)\cap\{(\rho,\eta):\det H(\beta,\rho,l\eta))>0\}$. The straight line in the $(\rho,\eta)$-plane joining $z_0$ and $(\beta,0)$ has equation

$$\eta=-\frac{(d-1)\beta}{1-\beta}(\rho-\beta)$$

If for each $\rho\in[\rho_5,\beta)$,

$$\xi'(\rho)<-\frac{(d-1)\beta}{1-\beta}\tag{3.33}$$

Then conditions (i) holds. Since when $\rho\le\rho_3$ and $d\beta<1$,

$$\xi'(\rho)\le-1<-\frac{(d-1)\beta}{1-\beta};$$

it suffices to check (3.33) for $\rho\in(\rho_3,\beta)$; which is equivalent to (3.32). □

**Lemma 3.10.** *Assume $d\ge 2$ and $l\ge 3$. If*

$$\beta\le\frac{1}{dl-d-l+2};\tag{3.34}$$

*then (3.32) holds.*

*Proof.* Then (3.32) holds if and only if

$$\xi'(\beta)\le-\frac{(d-1)\beta}{1-\beta};$$

which gives (3.34). □

**Lemma 3.11.** *Assume $d \geq 2$ and $l \geq 3$. If (3.34) holds, then $R_\beta \cap \{(\rho,\theta) : \det H > 0\}$ has exactly one connected component.*

*Proof.* We see that when (3.34) holds, the line segment passing through $(\beta^2, (d-1)\beta^2)$ and $(\beta, 0)$ in the $(\rho,\eta)$-plane intersects the curve $\det H = 0$ at a unique point $(\beta, 0)$, and the line segment lies below the curve, since

- both of them pass through the point $(\beta, 0)$; and
- The slope of the line segment is greater than or equal to the slope of the tangent line of the curve $\det H = 0$ at $(\beta, 0)$ in the $(\rho,\eta)$-plane; and
- The slope of the line segment is strictly greater than or the slope of the tangent line of the curve $\det H = 0$ at any point other than $(\beta, 0)$ in the $(\rho,\eta)$-plane.

Note that the line segment passing through $(\beta^2, (d-1)\beta^2)$ and $(\beta, 0)$ in the $(\rho,\eta)$-plane completely lies in the region $\{(\rho,\eta) : (\rho, \eta l) \in R_\beta\}$ since the region is convex. Hence the line segment does not cross the (possible) boundary $\eta = 2\beta - \rho - \frac{1}{d}$. Therefore the curve $\det H = 0$ does not cross the boundary $\eta = 2\beta - \rho - \frac{1}{d}$ either. Then the lemma follows. □

**Lemma 3.12.** *Assume $3 \leq l$, $2 \leq d$, and (3.34) holds. Then*

$$\max_{(\rho,\theta)\in R_\beta} \Psi_{d,l}(\beta,\rho,\theta) = 2\Phi_{d,l}(\beta)$$

*Proof.* By Lemmas 3.9 and 3.10, we obtain that under the assumptions of the lemma,

$$\max_{(\rho,\theta)\in R_\beta\cap\{\det H\geq 0\}} \Psi_{d,l}(\beta,\rho,\theta) = 2\Phi_{d,l}(\beta).$$

We now find

$$\max_{(\rho,\theta)\in R_\beta\cap\{\det H< 0\}} \Psi_{d,l}(\beta,\rho,\theta)$$

When the Hessian determinant is negative, $\Psi_{d,l}(\beta,\rho,\theta)$ as a function of $(\rho,\theta)$ cannot achieve a local maximal value at an interior point; it cannot achieve the global maximal value in $R_\beta$ along the boundary satisfying $\det H = 0$ either, since this is also the boundary of the region $\det H > 0$ where the function is strictly concave, and the maximal value in $\det H \geq 0$ is already achieved at the unique interior point $(\beta^2, l(d-1)\beta^2)$. Therefore if a global maximal value of $\Psi_{d,l}(\beta,\rho,\theta)$ in $R_\beta$ is achieved when $(\rho,\theta) \in R_\beta \cap \{\det H < 0\}$, then it must be achieved along the boundary $\theta = l(\beta - \rho)$; see Figure 3.1. More precisely, the only possible candidate for the global maximal value when $\det H < 0$ is

$$\max_{\rho\in[\rho_5,\beta]} \Psi_{d,l}(\beta,\rho,l(\beta-\rho))$$

Along the line $\theta = l(\beta - \rho)$, by (3.10) we have

$$\begin{aligned} T_{d,l,\beta}(\rho) := {} & \Psi_{d,l}(\beta,\rho,l(\beta-\rho)) \\ = {} & -\rho\ln\rho + (l-1)(1-2\beta+\rho)\ln(1-2\beta+\rho) + l(\beta-\rho)\ln(d-1) \\ & +(l-2)(\beta-\rho)\ln(\beta-\rho) - \frac{l}{d}(1-\beta d)\ln(1-\beta d) \end{aligned} \tag{3.35}$$

Then by (1.5),

$$\lim_{\rho\to\beta-}\Psi_{d,l}(\beta,\rho,l(\beta-\rho))=-\beta\ln\beta+(l-1)(1-\beta)\ln(1-\beta)-\frac{l}{d}(1-\beta d)\ln(1-\beta d)=\Phi_{d.l}(\beta)$$

Moreover,

$$\frac{dT_{d,l\beta}(\rho)}{d\rho}=-\ln\rho+(l-1)\ln(1-2\beta+\rho)-l\ln(d-1)-(l-2)\ln(\beta-\rho)$$

and

$$\frac{d^2T_{d,l\beta}(\rho)}{d\rho^2}=-\frac{1}{\rho}+\frac{l-1}{1-2\beta+\rho}+\frac{l-2}{\beta-\rho}=\frac{l\rho(1-\beta)-(\beta+\rho-2\beta^2)}{\rho(\beta-\rho)(1-2\beta+\rho)}. \tag{3.36}$$

When $\rho\in[\rho_5,\beta)$, $\frac{d^2T_{d,l\beta}(\rho)}{d\rho^2}\geq 0$, and the identity holds if and only if $\rho=\rho_5$. Therefore $\frac{dT_{d,l\beta}(\rho)}{d\rho}$ is monotone increasing when $\rho\in[\rho_5,\beta)$. The following cases might occur:

(1) $\left.\frac{dT_{d,l\beta}(\rho)}{d\rho}\right|_{\rho=\rho_5}<0$; in this case

$$\max_{\rho\in[\rho_5,\beta]}T_{d,l,\beta}(\rho)=\max\{T_{d,l,b}(\rho_5),T_{d,l,b}(\beta)\}.$$

But both $\rho_5$ and $\beta$ are along the boundary of the unique connected component of $R_\beta$ such that $\det H(\beta,\rho,\theta)>0$, where $\Psi_{d,l}(\beta,\rho,\theta)$ is strictly concave with a maximal value given by $2\Phi_d(\beta)$; hence we have

$$\max\{T_{d,l,\beta}(\rho_5),T_{d,l,b}(\beta)\}\leq 2\Phi_{d,l}(\beta).$$

Hence the lemma follows when $\left.\frac{dT_{d,l\beta}(\rho)}{d\rho}\right|_{\rho=\rho_5}<0$. Or

(2) $\left.\frac{dT_{d,l\beta}(\rho)}{d\rho}\right|_{\rho=\rho_5}\geq 0$. In this case $T_{d,l,\beta}(\rho)$ is increasing when $\rho\in[\rho_5,\beta)$. Hence we have

$$\max\{T_{d,l,\beta}(\rho_5),T_{d,l,b}(\beta)\}=T_{d,l,b}(\beta)\leq 2\Phi_{d,l}(\beta).$$

Then the lemma holds when $\left.\frac{dT_{d,l\beta}(\rho)}{d\rho}\right|_{\rho=\rho_5}\geq 0$ as well.

□

**Lemma 3.13.** *Assume one of the following conditions holds:*

- $l=2$, *and* $\beta\in\left(0,\frac{1}{d}\right)$; *or*
- *The assumptions of Lemma 3.12 hold.*

*Furthermore, we assume that*

$$\Psi_{d,l}(\beta,\beta^2,l(d-1)\beta^2)>0.$$

*we have*

$$\lim_{m\to\infty}\frac{\mathbb{E}Z_{m\beta}^2}{(\mathbb{E}Z_{m\beta})^2}=\frac{1-\beta}{\sqrt{\beta^2(d+l-dl)-2\beta+1}}$$

*Proof.* Let $L \geq 1$ be a positive integer. By Lemma 2.2 and (3.7), we obtain

$$F_\beta(m\rho, m\theta) = m^{-2} e^{m\Psi_{d,l}(\beta,\rho,\theta)} \frac{1}{4\pi^2\left(\beta-\rho-\frac{\theta}{l}\right)\sqrt{\rho\theta\left(1-2\beta d+\rho d+\frac{\theta d}{l}\right)}}\left(1+O\left(\frac{1}{m}\right)\right).$$

Then

$$\sum_{s=0}^{m\beta}\sum_{t=\max\left\{0,4m\beta-2s-\frac{2m}{d}\right\}}^{2m\beta-2s} F_\beta(s,t) = \left(1+O\left(\frac{1}{m}\right)\right)$$

$$\frac{e^{m\Psi_d\left(\beta,\beta^2,2(d-1)\beta^2\right)}}{4\pi^2 m^2}\sum_{s=1}^{m\beta-2}\sum_{t=\max\left\{0,4m\beta-2s-\frac{2m}{d}\right\}+1}^{2m\beta-2s-1}\frac{e^{-m\left[\Phi_d\left(\beta,\beta^2,2(d-1)\beta^2\right)-\Phi_d\left(\beta,\frac{s}{m},\frac{t}{m}\right)\right]}}{\left(\beta-\frac{s}{m}-\frac{t}{lm}\right)\sqrt{\frac{st}{m^2}\left(1-2\beta d+\frac{sd}{m}+\frac{td}{lm}\right)}}$$

We shall split the sum over $1 \leq s \leq m\beta-2$, $\max\left\{0,4m\beta-2s-\frac{2m}{d}\right\}+1 \leq t \leq 2m\beta-2s-1$ into two parts

- Part I is the sum over $\left(\frac{s}{m},\frac{t}{m}\right)$ in a $\frac{L}{\sqrt{m}}$ neighborhood of $(\beta^2, l(d-1)\beta^2)$; more precisely

$$(s,t)\in\mathcal{N} := \left\{(s,t)\in\mathbb{Z}^2 : \left|\frac{s}{m}-\beta^2\right| \leq \frac{L}{\sqrt{m}};\right.$$
$$\left.\left|\frac{t}{m}-l(d-1)\beta^2\right| \leq \frac{L}{\sqrt{m}}; 1\leq s\leq m\beta-2; \max\left\{0,2lm\beta-ls-\frac{lm}{d}\right\}+1\leq t\leq lm\beta-ls-1.\right\}$$

- Part II is the sum over $1 \leq s \leq m\beta-2$, $\max\left\{0,2lm\beta-ls-\frac{lm}{d}\right\}+1 \leq t \leq lm\beta-ls-1$, such that $\left(\frac{s}{m},\frac{t}{m}\right)$ is outside the $\frac{L}{\sqrt{m}}$ neighborhood of $(\beta^2, l(d-1)\beta^2)$.

Then we have

$$\frac{1}{4\pi^2}\sum_{s=1}^{m\beta-2}\sum_{t=\max\left\{0,2lm\beta-ls-\frac{lm}{d}\right\}+1}^{lm\beta-ls-1} m^{-2}\frac{e^{-m\left[\Phi_d\left(\beta,\beta^2,l(d-1)\beta^2\right)-\Phi_d\left(\beta,\frac{s}{m},\frac{t}{m}\right)\right]}}{\left(\beta-\frac{s}{m}-\frac{t}{lm}\right)\sqrt{\frac{st}{m^2}\left(1-2\beta d+\frac{sd}{m}+\frac{td}{lm}\right)}}$$
$$= \mathrm{I}+\mathrm{II}$$

Let

$$\xi_{s,t} := \left(\frac{s}{m}-\beta^2, \frac{t}{m}-l(d-1)\beta^2\right)\in\mathbb{R}^{1\times 2}.$$

and $\xi_{s,t}^T \in \mathbb{R}^{2\times 1}$ be the transpose of $\xi_{s,t}$. Note that when $(s,t)$ is on the boundary of $\mathcal{N}$, from the Taylor expansion of $\Phi_d(\beta,\cdot,\cdot)$ at $(\beta^2, l(d-1)\beta^2)$, we obtain

$$\Phi_d\left(\beta,\beta^2,l(d-1)\beta^2\right)-\Phi_d\left(\beta,\frac{s}{m},\frac{t}{m}\right) = -\frac{\xi_{s,t}H(\beta,\beta^2,2(d-1)\beta^2)\xi_{s,t}^T}{2!}+O\left(\left(\frac{L}{\sqrt{m}}\right)^3\right)$$

Assume $-\lambda$ ($\lambda>0$) is the maximal eigenvalue of $H(\beta,\beta^2,l(d-1)\beta^2)$, then

$$e^{-m\left[\Phi_d\left(\beta,\beta^2,l(d-1)\beta^2\right)-\Phi_d\left(\beta,\frac{s}{m},\frac{t}{m}\right)\right]} = O\left(e^{-\frac{\lambda L^2}{2}}\right)$$

and

$$\sum_{s=1}^{m\beta-2}\sum_{t=\max\left\{0,2lm\beta-ls-\frac{lm}{d}\right\}+1}^{lm\beta-ls-1}\frac{e^{-m\left[\Phi_d\left(\beta,\beta^2,l(d-1)\beta^2\right)-\Phi_d\left(\beta,\frac{s}{m},\frac{t}{m}\right)\right]}}{\left(\beta-\frac{s}{m}-\frac{t}{lm}\right)\sqrt{\frac{st}{m^2}\left(1-2\beta d+\frac{sd}{m}+\frac{td}{lm}\right)}}$$

is approximately $e^{-\frac{\lambda L^2}{2}}$ multiplying a convergent geometric series. Hence when $m$ is large,

$$m\mathrm{II}\leq C''\frac{e^{-\frac{L^2}{2}}}{m}\rightarrow 0$$

as $L\rightarrow\infty$; where $C''>0$ is a constant independent of $m$.

Moreover,

$$\mathrm{I}=\frac{1}{4\pi^2m^2}\sum_{s,t\in\mathcal{N}}\frac{e^{m\left[\frac{\xi_{s,t}H(\beta,\beta^2,l(d-1)\beta^2)\xi_{s,t}^T}{2}+O\left(\left(\frac{r_m}{\sqrt{m}}\right)^3\right)\right]}}{\left(\beta-\frac{s}{m}-\frac{t}{lm}\right)\sqrt{\frac{st}{m^2}\left(1-2\beta d+\frac{sd}{m}+\frac{td}{lm}\right)}}\left(1+O\left(\frac{1}{m}\right)\right)$$

Hence we have

$$\begin{aligned}\lim_{I\rightarrow\infty}\lim_{m\rightarrow\infty}mI &= \frac{1}{4\pi^2\sqrt{l(d-1)}\beta^3(1-d\beta)^2}\int_{\mathbb{R}^2}e^{\frac{\mathbf{x}H(\beta,\beta^2,2(d-1)\beta^2)\mathbf{x}^T}{2}}dx_1dx_2\\ &= \frac{2\pi}{4\pi^2\sqrt{l(d-1)}\beta^3(1-d\beta)^2\sqrt{\det H(\beta,\beta^2,l(d-1)\beta^2)}} \qquad (3.37)\end{aligned}$$

where

$$\mathbf{x}=(x_1,x_2)\in\mathbb{R}^{1\times 2}$$

Then the lemma follows from (3.37), (3.22) and (2.7). □

## 4. Subgraph Conditioning

In this section, we show that conditional on a matching , the distributions of the number of cycles of different lengths on $\mathcal{G}_{m,d,l}$ converge to independent Poisson random variables as $m\rightarrow\infty$. Similar results when conditional on a perfect matching was proved in [5].

Let $k\geq 1$ be a positive integer. Let

$$[k]:=\{1,2,\ldots,k\}.$$

Define a $k$-cycle on $\mathcal{G}_{m,d,l}$ to be an alternating sequence of vertices and hyper-edges

$$v_0,e_1,v_1,\ldots,v_{i-1},e_i,v_i,\ldots,v_{k-1},e_k,v_k$$

such that

- each $v_i$ is a vertex for $0\leq i\leq k$;
- each $e_j$ is a hyper-edge for $1\leq j\leq k$;
- $v_0=v_k$; and
- $v_i\neq v_j$, for any $i\neq j$, $\{i,j\}\neq\{0,k\}$; and
- for each $i\in[k]$, $e_i$ is incident to both $v_{i-1}$ and $v_i$.

In particular, a 1-cycle is a hyper-edge, at least two half-edges of which share a vertex.

Let $C_k$ be the number of $k$-cycles on $\mathcal{G}_{m,d,l}$, and let $b \geq 1$ be a positive integer.

For $\mathbf{c} = (c_1, c_2, \ldots, c_b) \in \mathbb{N}^b$. Let

$$\Omega_{\mathbf{c}} := \{\mathcal{G}_{m,d,l} \in \Omega_{m,d,l} : C_k = c_k, k \in [b]\}.$$

For $\beta \in \left(0, \frac{1}{d}\right)$, define

$$E_{\beta,\mathbf{c}} := \mathbb{E}(Z_{m\beta} | \mathcal{G}_{m,d,l} \in \Omega_{\mathbf{c}}).$$

For each $\mathcal{G}_{m,d,l} = (V, E) \in \Omega_{m,d,l}$, let $Q_{\mathcal{G}_{m,d,l},\beta}$ be the set of all the matchings on $\mathcal{G}_{m,d,l}$ consisting of exactly $m\beta$ present hyperedges. Let $M_0 \subseteq E$, $|M_0| = m\beta$ be a fixed set of $m\beta$ hyperedges. Note that for any other set $M$ of $m\beta$ hyperedges, the probability that $M_0$ is matching on a random graph $\mathcal{G}_{m,d}$ is the same as the probability that $M$ is a matching on a random graph $\mathcal{G}_{m,d}$. More precisely,

$$\mathbb{P}(M \in \mathcal{Q}_{\mathcal{G}_{m,d,l},\beta}) = \mathbb{P}(M_0 \in \mathcal{Q}_{\mathcal{G}_{m,d,l},\beta}) = \frac{(lm(1-\beta))!}{(lm(1-d\beta))!} \frac{(lm(1-d\beta))!}{(lm)!} \frac{d^{lm\beta} \left(\frac{lm}{d}\right)!}{\left(\frac{lm}{d}(1-d\beta)\right)!}$$

and from the symmetry of hyperedges we can also see that

$$\mathbb{P}(\mathcal{G}_{m,d,l} \in \Omega_{\mathbf{c}} | M \in Q_{\mathcal{G}_{m,d,l},\beta}) = \mathbb{P}(\mathcal{G}_{m,d,l} \in \Omega_{\mathbf{c}} | M_0 \in Q_{\mathcal{G}_{m,d,l},\beta})$$

Then

$$\begin{aligned} E_{\beta,\mathbf{c}} &= \frac{1}{|\Omega_{\mathbf{c}}|} \sum_{\mathcal{G}_{m,d,l} \in \Omega_{\mathbf{c}}} \sum_{M \in Q_{\mathcal{G}_{m,d,l},\beta}} 1 \\ &= \frac{|\Omega|}{|\Omega_{\mathbf{c}}|} \sum_{M \subset E, |M| = m\beta} \mathbb{P}(M \in Q_{\mathcal{G}_{m,d,l},\beta}, \mathcal{G}_{m,d,l} \in \Omega_{\mathbf{c}}) \\ &= \frac{\mathbb{P}(\mathcal{G}_{m,d,l} \in \Omega_{\mathbf{c}} | M_0 \in Q_{\mathcal{G}_{m,d,l},\beta})}{\mathbb{P}(\mathcal{G}_{m,d,l} \in \Omega_{\mathbf{c}})} \sum_{M \subset E, |M| = m\beta} \mathbb{P}(M_0 \in Q_{G_{m,d,l}}, \beta) \\ &= \frac{\mathbb{E} Z_{m\beta} \mathbb{P}(\mathcal{G}_{m,d,l} \in \Omega_{\mathbf{c}} | M_0 \in Q_{\mathcal{G}_{m,d,l},\beta})}{\mathbb{P}(\mathcal{G}_{m,d,l} \in \Omega_{\mathbf{c}})} \end{aligned} \tag{4.1}$$

Let $E_t$, $t = 0, 1, \ldots, \lfloor \frac{k}{2} \rfloor$ be the expected number of $k$-cycles including exactly $t$ hyperedges from $M_0$. Then we may compute $E_t$ as follows

(1) Choose $t$ hyperedges from the $m\beta$ hyperedges in $M_0$; there are $\binom{m\beta}{t}$ choices. Cyclically order these edges, there are $\frac{(t-1)!}{2!}$ different orders. Choose ordered pairs of half-edges from each of these $t$ hyperedges to connect with half-edges not in $M_0$, there are $(l(l-1))^t$ choices. For $1 \leq i \leq t$, let $s_i \geq 1$ be the number of edges along the $k$-cycle between the $(i-1)$th and $i$th hyperedge in $M_0$ along the cycle. Then we need to choose positive integers $s_1, \ldots, s_k$, such that $s_1 + \ldots + s_k = k - t$. There are $\binom{k-t-1}{t-1}$ choices.

(2) Choose $k - t$ hyperedges from the $m - m\beta$ hyperedges not in $M_0$. Consider all the possible orderings of these hyperedges as well as choices of two ordered half-edges in each of these chosen hyperedge to place them along the cycle, there are

$(k-t)![l(l-1)]^{k-t}$ possibilities. The factor $(d-1)^k$ indicates the possible number of locations of two half edges incident to the same vertex along the $k$-cycle.

(3) In the $k-2t$ vertices along the cycle which are not incident to an edge in $M_0$ along the cycle, assume that $s$ of them are incident to an half-edge in $M_0$ but not along the cycle.

Hence we have

$$\begin{aligned} E_0 &= \left[\binom{m-m\beta}{k}\frac{(k-1)!}{2!}[l(l-1)]^k\right](d-1)^k\left[\sum_{s=0}^{k}(d-2)^s\binom{lm\beta}{s}\binom{k}{s}s!\right. \\ &\quad \left.\frac{(lm-lm\beta-2k)!}{(lm-(lm\beta+k-s)d)!}\frac{(lm-(lm\beta+k-s)d)!}{(lm)!}\frac{d^{lm\beta+k-s}\left(\frac{lm}{d}\right)!}{\left(\frac{lm}{d}-(lm\beta+k-s)\right)!}\right] \end{aligned}$$

If $t \geq 1$,

$$\begin{aligned} E_t &= \left[\binom{m\beta}{t}\frac{(t-1)!}{2!}[l(l-1)]^t\binom{k-t-1}{t-1}\right]\left[\binom{m-m\beta}{k-t}(k-t)![l(l-1)]^{k-t}\right](d-1)^k \\ &\quad \times\left[\sum_{s=0}^{k-2t}(d-2)^s\binom{lm\beta-2t}{s}\binom{k-2t}{s}s!\frac{(lm-lm\beta-2k+2t)!}{(lm-(lm\beta+k-2t-s)d)!}\right. \\ &\quad \left.\frac{(lm-(lm\beta+k-2t-s)d)!}{(lm)!}\frac{d^{lm\beta+k-2t-s}\left(\frac{lm}{d}\right)!}{\left(\frac{lm}{d}-(lm\beta+k-2t-s)\right)!}\right] \end{aligned}$$

Then

$$\mathbb{E}(C_k|M_0 \in Q_{\mathcal{G}_{m,d,l,\beta}}) = \sum_{t=0}^{\lfloor\frac{k}{2}\rfloor}\frac{E_t}{\mathbb{P}(M_0 \in Q_{\mathcal{G}_{m,d,l,\beta}})} = \sum_{t=0}^{\lfloor\frac{k}{2}\rfloor}\sum_{s=0}^{k-2t}E_{\beta,k}(t,s) \tag{4.2}$$

where when $t \geq 1$,

$$\begin{aligned} E_{\beta,k}(t,s) &= \frac{(m\beta)!(k-t-1)!}{2t!(m\beta-t)!}[l(l-1)]^k(d-1)^k(d-2)^s\frac{(m-m\beta)!}{(m-m\beta-k+t)!} \\ &\quad \frac{(lm\beta-2t)!}{(lm\beta-2t-s)!(k-2t-s)!s!}\frac{(lm-lm\beta-2k+2t)!}{(lm(1-\beta))!} \\ &\quad \frac{d^{k-2t-s}\left(\frac{lm}{d}(1-d\beta)\right)!}{\left(\frac{lm}{d}-(lm\beta+k-2t-s)\right)!} \end{aligned}$$

and

$$\begin{aligned} E_{\beta,k}(0,s) &= \frac{(m-m\beta)!}{2(m-m\beta-k)!}(k-1)![l(l-1)]^k(d-1)^k(d-2)^s\frac{(lm\beta)!}{s!(lm\beta-s)!(k-s)!} \\ &\quad \frac{(lm-lm\beta-2k)!}{(lm(1-\beta))!}\frac{d^{k-s}\left(\frac{lm}{d}(1-d\beta)\right)!}{\left(\frac{lm}{d}-(lm\beta+k-s)\right)!} \end{aligned}$$

**Lemma 4.1.** *When $k$ is fixed and $m \to \infty$,*

$$\sum_{t=0}^{\lfloor\frac{k}{2}\rfloor}\sum_{s=0}^{k-2t}E_{\beta,k}(t,s) \approx \frac{(d-1)^k(l-1)^k}{2k}\left(1+\frac{(-1)^k\beta^k}{(1-\beta)^k}\right)$$

*Proof.* When $k$ is fixed and $m\to\infty$, we have when $t\geq 1$,

$$E_{\beta,k}(t,s) \approx \frac{(d-1)^k(l-1)^k(1-d\beta)^k}{2(1-\beta)^k}\binom{k-t-1}{t-1}\binom{k-2t}{s}\frac{1}{t}\frac{\beta^{t+s}(1-\beta)^t(d-2)^s}{(1-d\beta)^{2t+s}}$$

and

$$\begin{aligned}\sum_{s=0}^{k-2t}E_{\beta,k}(t,s) &\approx \sum_{s=0}^{k-2t}\frac{(d-1)^k(l-1)^k(1-d\beta)^k}{2(1-\beta)^k}\frac{1}{k-t}\binom{k-t}{t}\binom{k-2t}{s}\frac{\beta^{t+s}(1-\beta)^t(d-2)^s}{(1-d\beta)^{2t+s}}\\ &= \frac{(d-1)^k(l-1)^k}{2(1-\beta)^k}\frac{1}{k-t}\binom{k-t}{t}\beta^t(1-\beta)^t\,(1-2\beta)^{k-2t}\end{aligned}$$

Moreover,

$$E_{\beta,k}(0,s)\approx\frac{(d-1)^k(l-1)^k(1-d\beta)^k}{2k(1-\beta)^k}\binom{k}{s}\frac{\beta^s(d-2)^s}{(1-d\beta)^s}$$

and

$$\sum_{s=0}^{k}E_{\beta,k}(0,s)\approx\frac{(d-1)^k(l-1)^k(1-2\beta)^k}{2k(1-\beta)^k}$$

Hence we have

$$\sum_{t=0}^{\lfloor\frac{k}{2}\rfloor}\sum_{s=0}^{k-2t}E_{\beta,k}(t,s) \approx \frac{(d-1)^k(l-1)^k(1-2\beta)^k}{2(1-\beta)^k}\sum_{t=0}^{\lfloor\frac{k}{2}\rfloor}\frac{1}{k-t}\binom{k-t}{t}\alpha^t$$

where

$$\alpha=\frac{\beta(1-\beta)}{(1-2\beta)^2}$$

If $f(x)$ is a Laurent series of $x$, for $i\in\mathbb{Z}$, let $[x^i]f(x)$ be the coefficient for $x^i$ in $f(x)$. Then we obtain,

$$\begin{aligned}\sum_{t=0}^{\lfloor\frac{k}{2}\rfloor}\frac{1}{k-t}\binom{k-t}{t}\alpha^t &= \alpha^k[x^k]\sum_{t=0}^{\lfloor\frac{k}{2}\rfloor}\left(\frac{x(1+x)}{\alpha}\right)^{k-t}\frac{1}{k-t}\\ &= \alpha^k[x^k]\sum_{j=\lceil\frac{k}{2}\rceil}^{k}\left(\frac{x(1+x)}{\alpha}\right)^{j}\frac{1}{j}\\ &= \alpha^k[x^k]\sum_{j=1}^{\infty}\left(\frac{x(1+x)}{\alpha}\right)^{j}\frac{1}{j}\\ &= -\alpha^k[x^k]\log\left(1-\frac{x(1+x)}{\alpha}\right)\\ &= -\alpha^k[x^k]\log\left(\left(1-\frac{x}{\frac{\sqrt{1+4\alpha}-1}{2}}\right)\left(1+\frac{x}{\frac{1+\sqrt{1+4\alpha}}{2}}\right)\right)\\ &= -\alpha^k\frac{(-1)^{k-1}}{k}\left[\left(\frac{2}{1-\sqrt{1+4\alpha}}\right)^k+\left(\frac{2}{\sqrt{1+4\alpha}+1}\right)^k\right]\\ &= \frac{(-1)^k}{k}\frac{\beta^k+(\beta-1)^k}{(1-2\beta)^k}\end{aligned}$$

Then the lemma follows. □

Hence by (4.2) and Lemma 4.1 we have

$$\mathbb{E}(C_k|M_0 \in Q_{\mathcal{G}_{m,d,l,\beta}}) \approx \frac{(d-1)^k(l-1)^k}{2k}\left(1+\frac{(-1)^k\beta^k}{(1-\beta)^k}\right) := \mu_{\beta,k}. \tag{4.3}$$

**Lemma 4.2.** *Let $g \geq 1$ be a positive integer; $k_1, \ldots, k_g \geq 1$ be positive integers, and $r_1, \ldots, r_g \geq 1$ be nonnegative integers. Let $\beta \in \left(0, \frac{1}{d}\right)$. Let $M_0 \subset E$ and $|M_0| = m\beta$. Let $N^*_{(r_1,k_1),(r_2,k_2),\ldots,(r_g,k_g),1}$ be the number of $(d,l)$-regular hypergraphs with $m$ hyperedges satisfying all the following conditions:*

- *$M_0$ is a matching; and*
- *The hypergraph contains $r_i$ labeled $k_i$-cycles; i.e., we can find $r_i$ $k_i$ cycles in the hypergraph and label them by $1, 2, \ldots, r_i$; different labelings of the same choice of $r_i$ $k$-cycles are counted differently; and*
- *Any two cycles among the $r_1$ $k_1$-cycles, $r_2$ $k_2$-cycles,..., $r_g$ $k_g$ cycles are disjoint.*

*Let $N_{M_0}$ be the number of $(d,l)$-regular hypergraphs such that $M_0$ is a matching. Then for fixed $g, (r_1,k_1), (r_2,k_2), \ldots, (r_g,k_g)$, let $m \to \infty$,*

$$\frac{N^*_{(r_1,k_1),(r_2,k_2),\ldots,(r_g,k_g),1}}{N_{M_0}} \approx \prod_{i=1}^{g} (\mu_{\beta,k_i})^{r_i}$$

*Proof.* The total number of $(d,l)$-regular hypergraphs such that $M_0$ is a matching is given by

$$N_{M_0} = \frac{(lm(1-\beta))!}{d^{\frac{lm}{d}(1-d\beta)}\left(\frac{lm}{d}(1-d\beta)\right)!}$$

We shall prove by induction. First we consider the case when $g=1$ and $r_1 = 1$. We have

$$N^*_{(1,k_1),1}\frac{d^{\frac{lm}{d}(1-d\beta)}\left(\frac{lm}{d}(1-d\beta)\right)!}{(lm(1-\beta))!} = \sum_{t=0}^{\lfloor\frac{k_1}{2}\rfloor}\sum_{s=0}^{k_1-2t} E_{\beta,k_1}(t,s)$$

where $t$ is the number of common hyperedges of the distinguished $k_1$ cycle and $M_0$, and $s$ is the number of common vertices of the distinguished $k$ cycle and $M_0$ not incident to a common hyperedge of the cycle and $M_0$.

Fix $k_1, d, l$ and let $m \to \infty$, by Lemma 4.1 we have

$$N^*_{(1,k_1),1}(s,t)\frac{d^{\frac{lm}{d}(1-d\beta)}\left(\frac{lm}{d}(1-d\beta)\right)!}{(lm(1-\beta))!} \approx \mu_{\beta,k_1} \tag{4.4}$$

with $\mu_{\beta,k}$ given by (4.3).

Note also that

$$N^*_{(r_1,k_1),(r_2,k_2),\ldots,(r_g+1,k_g),1} = \sum_{t_{1,1}=0}^{\lfloor\frac{k_1}{2}\rfloor}\sum_{s_{1,1}=0}^{k_1-2t_{1,1}}\cdots\sum_{t_{1,r_1}=0}^{\lfloor\frac{k_1}{2}\rfloor}\sum_{s_{1,r_1}=0}^{k_1-2t_{1,r_1}}\cdots\sum_{t_{g,1}=0}^{\lfloor\frac{k_g}{2}\rfloor}\sum_{s_{g,1}=0}^{k_g-2t_{g,1}}\cdots\sum_{t_{g,r_g+1}=0}^{\lfloor\frac{k_g}{2}\rfloor}\sum_{s_{g,r_g+1}=0}^{k_g-2t_{g,r_g+1}} N^*_{(r_1,k_1),(r_2,k_2),\ldots,(r_g,k_g),1}((s_{1,1},t_{1,1}),\ldots,(s_{g,r_g+1},t_{g,r_g+1}))$$

where

$$N^*_{(r_1,k_1),(r_2,k_2),\ldots,(r_g,k_g),1}((s_{1,1},t_{1,1}),\ldots,(s_{g,r_g+1},t_{g,r_g+1})) = N^*_{(r_1,k_1),(r_2,k_2),\ldots,(r_g,k_g),1}((s_{1,1},t_{1,1}),\ldots,(s_{g,r_g},t_{g,r_g}))$$
$$\frac{(lm-lm\beta d-\sum_{i=1}^{g}\sum_{j=1}^{r_i}(k_i-2t_{i,j}-s_{i,j})d)!}{(lm-lm\beta-\sum_{i=1}^{g}\sum_{j=1}^{r_i}(2k_i-2t_{i,j}))!}$$
$$\frac{d^{\frac{lm}{d}-lm\beta-\sum_{i=1}^{g}\sum_{j=1}^{r_i}(k_i-2t_{i,j}-s_{i,j})}(\frac{lm}{d}-lm\beta-\sum_{i=1}^{g}\sum_{j=1}^{r_i}(k_i-2t_{i,j}-s_{i,j}))!}{(lm-lm\beta d-\sum_{i=1}^{g}\sum_{j=1}^{r_i}(k_i-2t_{i,j}-s_{i,j})d)!}$$
$$\left[\binom{m\beta-\sum_{i=1}^{g}\sum_{j=1}^{r_i}t_{i,j}}{t_{g,r_g+1}}\frac{(t_{g,r_g+1}-1)!}{2!}[l(l-1)]^{t_{g,r_g+1}}\binom{k_g-t_{g,r_g+1}-1}{t_{g,r_g+1}-1}\right]$$
$$\left[\binom{m-m\beta-\sum_{i=1}^{g}\sum_{j=1}^{r_i}(k_i-t_{i,j})}{k_g-t_{g,r_g+1}}(k_g-t_{g,r_g+1})![l(l-1)]^{k_g-t_{g,r_g+1}}\right](d-1)^{k_g}(d-2)^{s_{g,r_g+1}}$$
$$\binom{lm\beta-\sum_{i=1}^{g}\sum_{j=1}^{r_i}(2t_{i,j}+s_{i,j})-2t_{g,r_g+1}}{s_{g,r_g+1}}\binom{k_g-2t_{g,r_g+1}}{s_{g,r_g+1}}(s_{g,r_g+1})!$$
$$\frac{(lm-lm\beta-\sum_{i=1}^{g}\sum_{j=1}^{r_i}(2k_i-2t_{i,j})-(2k_g-2t_{g,r_g+1}))!}{(Ad)!}\frac{(Ad)!}{d^A A!}$$

with

$$A:=\frac{lm}{d}-lm\beta-\sum_{i=1}^{g}\sum_{j=1}^{r_i}(k_i-2t_{i,j}-s_{i,j})-(k_g-2t_{g,r_g+1}-s_{g,r_g+1}).$$

Here for $1\leq i\leq g$ and $1\leq j\leq r_i+1$, $t_{i,j}$ is the number of common hyperedges of $k_i$ cycle labeled by $j$ and $M_0$, and $s_{i,j}$ is the number of common vertices of the $k_i$ cycle labeled by $j$ and $M_0$ not incident to a common hyperedge of the cycle and $M_0$.

Fix $r_1,k_1,\ldots,r_g,k_g,d,l$ and let $m\rightarrow\infty$, we obtain

$$\begin{aligned}
& N^*_{(r_1,k_1),(r_2,k_2),\ldots,(r_g,k_g),1}((s_{1,1},t_{1,1}),\ldots,(s_{g,r_g+1},t_{g,r_g+1}))\frac{d^{\frac{lm}{d}(1-d\beta)}\left(\frac{lm}{d}(1-d\beta)\right)!}{(lm(1-\beta))!}\\
\approx\ & N^*_{(r_1,k_1),(r_2,k_2),\ldots,(r_g,k_g),1}((s_{1,1},t_{1,1}),\ldots,(s_{g,r_g},t_{g,r_g}))\frac{d^{\frac{lm}{d}(1-d\beta)}\left(\frac{lm}{d}(1-d\beta)\right)!}{(lm(1-\beta))!}\\
& \times\frac{(d-1)^{k_g}(l-1)^{k_g}\beta^{t_{g,r_g+1}+s_{g,r_g+1}}(1-d\beta)^{k_g-2t_{g,r_g+1}-s_{g,r_g+1}}(d-2)^{s_{g,r_g+1}}}{2(k_g-t_{g,r_g+1})(1-\beta)^{k_g-t_{g,r_g+1}}}\\
& \binom{k_g-t_{g,r_g+1}}{t_{g,r_g+1}}\binom{k_g-2t_{g,r_g+1}}{s_{g,r_g+1}}\\
=\ & N^*_{(r_1,k_1),(r_2,k_2),\ldots,(r_g,k_g),1}((s_{1,1},t_{1,1}),\ldots,(s_{g,r_g},t_{g,r_g}))\frac{d^{\frac{lm}{d}(1-d\beta)}\left(\frac{lm}{d}(1-d\beta)\right)!}{(lm(1-\beta))!}E_{\beta,k_g}(t_{g,r_g+1},s_{g,r_g+1})
\end{aligned}$$

Moreover,

$$N^*_{(r_1,k_1),\ldots,(r_g,k_g),(1,k_{g+1})1}=\sum_{t_{1,1}=0}^{\lfloor\frac{k_1}{2}\rfloor}\sum_{s_{1,1}=0}^{k_1-2t_{1,1}}\cdots\sum_{t_{1,r_1}=0}^{\lfloor\frac{k_1}{2}\rfloor}\sum_{s_{1,r_1}=0}^{k_1-2t_{1,r_1}}\cdots\sum_{t_{g,1}=0}^{\lfloor\frac{k_g}{2}\rfloor}\sum_{s_{g,1}=0}^{k_g-2t_{g,1}}\cdots\sum_{t_{g,r_g}=0}^{\lfloor\frac{k_g}{2}\rfloor}\sum_{s_{g,r_g}=0}^{k_g-2t_{g,r_g}}$$
$$\sum_{t_{g+1,1}=0}^{\lfloor\frac{k_{g+1}}{2}\rfloor}\sum_{s_{g+1,1}=0}^{k_{g+1}-2t_{g+1,1}}N^*_{(r_1,k_1),\ldots,(r_g,k_g),(1,k_{g+1}),1}((s_{1,1},t_{1,1}),\ldots,(s_{g,r_g},t_{g,r_g})),(s_{g+1,1},t_{g+1,1})$$

where

$$N^*_{(r_1,k_1),\ldots,(r_g,k_g),(1,k_{g+1}),1}((s_{1,1},t_{1,1}),\ldots,(s_{g,r_g},t_{g,r_g}),(s_{g+1,1},t_{g+1,1}))$$
$$= N^*_{(r_1,k_1),(r_2,k_2),\ldots,(r_g,k_g),1}((s_{1,1},t_{1,1}),\ldots,(s_{g,r_g},t_{g,r_g}))\frac{(lm-lm\beta d-\sum_{i=1}^g\sum_{j=1}^{r_i}(k_i-2t_{i,j}-s_{i,j})d)!}{(lm-lm\beta-\sum_{i=1}^g\sum_{j=1}^{r_i}(2k_i-2t_{i,j}))!}$$
$$\frac{d^{\frac{lm}{d}-lm\beta-\sum_{i=1}^g\sum_{j=1}^{r_i}(k_i-2t_{i,j}-s_{i,j})}(\frac{lm}{d}-lm\beta-\sum_{i=1}^g\sum_{j=1}^{r_i}(k_i-2t_{i,j}-s_{i,j}))!}{(lm-lm\beta d-\sum_{i=1}^g\sum_{j=1}^{r_i}(k_i-2t_{i,j}-s_{i,j})d)!}$$
$$\left[\binom{m\beta-\sum_{i=1}^g\sum_{j=1}^{r_i}t_{i,j}}{t_{g+1,1}}\frac{(t_{g+1,1}-1)!}{2!}[l(l-1)]^{t_{g+1,1}}\binom{k_{g+1}-t_{g+1,1}-1}{t_{g+1,1}-1}\right]$$
$$\left[\binom{m-m\beta-\sum_{i=1}^g\sum_{j=1}^{r_i}(k_i-t_{i,j})}{k_{g+1}-t_{g+1,1}}(k_{g+1}-t_{g+1,1})![l(l-1)]^{k_{g+1}-t_{g+1,1}}\right](d-1)^{k_{g+1}}(d-2)^{s_{g+1,1}}$$
$$\binom{lm\beta-\sum_{i=1}^g\sum_{j=1}^{r_i}(2t_{i,j}+s_{i,j})-2t_{g+1,1}}{s_{g+1,1}}\binom{k_{g+1}-2t_{g+1,1}}{s_{g+1,1}}(s_{g+1,1})!$$
$$\frac{(lm-lm\beta-\sum_{i=1}^g\sum_{j=1}^{r_i}(2k_i-2t_{i,j})-(2k_{g+1}-2t_{g+1,1}))!}{(dB)!}\frac{(dB)!}{d^BB!}$$

with

$$B=\frac{lm}{d}-lm\beta-\sum_{i=1}^g\sum_{j=1}^{r_i}(k_i-2t_{i,j}-s_{i,j})-(k_{g+1}-2t_{g+1,1}-s_{g+1,1})$$

Similarly we obtain that for any fixed $r_1,k_1,\ldots,r_g,k_g,d,l$ and let $m\to\infty$,

$$N^*_{(r_1,k_1),(r_2,k_2),\ldots,(r_g,k_g),1}((s_{1,1},t_{1,1}),\ldots,(s_{g,r_g},t_{g,r_g}))\frac{d^{\frac{lm}{d}(1-d\beta)}\left(\frac{lm}{d}(1-d\beta)\right)!}{(lm(1-\beta))!}\approx\prod_{i=1}^g\prod_{j=1}^{r_i}E_{\beta,k_i}(t_{i,j},s_{i,j})$$

Therefore for any fixed $r_1,k_1,\ldots,r_g,k_g,d,l$ and let $m\to\infty$, by (4.4) and Lemma 4.1 we obtain

$$\begin{aligned}N^*_{(r_1,k_1),(r_2,k_2),\ldots,(r_g,k_g),1}\frac{d^{\frac{lm}{d}(1-d\beta)}\left(\frac{lm}{d}(1-d\beta)\right)!}{(lm(1-\beta))!}&\approx\prod_{i=1}^g\prod_{j=1}^{r_i}\left(\sum_{t_{i,j}=0}^{\lfloor\frac{k_i}{2}\rfloor}\sum_{s_{i,j}=1}^{k_i-2t_{i,j}}E_{\beta,k_i}(t_{i,j},s_{i,j})\right)\\&\approx\prod_{i=1}^g(\mu_{\beta,k_i})^{r_i}\end{aligned}$$

Then the lemma follows. □

**Lemma 4.3.** *Let $g,k_1,\ldots,k_g,r_1,\ldots,r_g,\beta,M_0$ be given as in Lemma 4.2.*

*Let*
$N^*_{(r_1,k_1),(r_2,k_2),\ldots,(r_g,k_g),2}$ *be the number of $(d,l)$-regular hypergraphs with $m$ hyperedges satisfying all the following conditions:*

- *$M_0$ is a matching; and*
- *The hypergraph contains precisely $r_i$ $k_i$-cycles.*

*Then for any fixed $r_1,k_1,\ldots,r_g,k_g,d,l$ and let $m\to\infty$, we have*

$$\lim_{m\to\infty}\frac{N^*_{(r_1,k_1),(r_2,k_2),\ldots,(r_g,k_g),2}}{N^*_{(r_1,k_1),(r_2,k_2),\ldots,(r_g,k_g),1}}=0.$$

*Proof.* Note that the total number of vertices in a $(d,l)$-regular hypergraph with $m$ hyperedges is $\frac{lm}{d}$. It is straightforward to check that for any fixed $r_1, k_1, \ldots, r_g, k_g, d, l$ and let $m \to \infty$,

$$\frac{N^*_{(r_1,k_1),(r_2,k_2),\ldots,(r_g,k_g),2}}{N^*_{(r_1,k_1),(r_2,k_2),\ldots,(r_g,k_g),1}} = O\left(\frac{d}{lm}\right).$$

Then the lemma follows. □

**Lemma 4.4.** *(1) When $m \to \infty$, for fixed $b \in \mathbb{N}$, $C_1, \ldots, C_b$ are independent Poisson random variables with means given by*

$$\mathbb{E}(C_k) = \frac{(d-1)^k(l-1)^k}{2k} := \lambda_k, \ \forall k \in [b]. \tag{4.5}$$

*(2) Let $M_0 \subset E$ and $|M_0| = m\beta$. Conditional on $M_0 \in Q_{\mathcal{G}_{m,d,l,\beta}}$, when $m \to \infty$, for fixed $b \in \mathbb{N}$, $C_1, \ldots, C_b$ are independent Poisson random variables with means given by (4.3) for $k \in [b]$.*

*Proof.* Note that Part (1) of the Lemma when $l = 2$ was given by Lemma 1 of [23]; see also [25]. We only prove Part(2) here; Part (1) can be proved similarly.

Let $g, k_1, \ldots, k_g, r_1, \ldots, r_g, N_{M_0}$ be given as in Lemma 4.2. Let $N_{(r_1,k_1),(r_2,k_2),\ldots,(r_g,k_g)}$ be the number of $(d,l)$-regular hypergraphs with $m$ hyperedges satisfying all the following conditions:

- $M_0$ is a matching; and
- The hypergraph contains $r_i$ labeled $k_i$-cycles; and
- at least two cycles among the $r_1$ $k_1$-cycles, $r_2$ $k_2$-cycles,..., $r_g$ $k_g$ cycles share a vertex.

Then by the inclusion-exclusion principle,

$$\frac{1}{N_{M_0}}\left|N_{(r_1,k_1),\ldots,(r_g,k_g)} - \sum_{s_1=r_1}^{u_1-1}\cdots\sum_{s_g=r_g}^{u_g-1}\left[\prod_{i=1}^{g}\frac{(-1)^{s_i-r_i}}{s_i!}\binom{s_i}{r_i}\right]\left(N^*_{(s_1,k_1),\ldots,(s_g,k_g),1} + N^*_{(s_1,k_1),\ldots,(s_g,k_g),2}\right)\right|$$
$$\leq \frac{1}{N_{M_0}}\left[\prod_{i=1}^{g}\frac{(-1)^{s_i-t_i}}{u_i!}\binom{u_i}{r_i}\right]\left(N^*_{(u_1,k_1),\ldots,(u_g,k_g),1} + N^*_{(u_1,k_1),\ldots,(u_g,k_g),2}\right).$$

By Lemmas 4.2 and 4.3, for any fixed $r_1, k_1, \ldots, r_g, k_g, d, l$ and let $m \to \infty$, by (4.4) we have

$$\mathbb{P}(C_{k_1} = r_1, \ldots, C_{k_g} = r_g | M_0 \in Q_{\mathcal{G}_{m,d,l,\beta}}) = \frac{N_{(r_1,k_1),\ldots,(r_g,k_g)}}{N_{M_0}} \approx \prod_{i=1}^{g}\frac{e^{-\mu_{\beta,k_i}}\mu_{\beta,k_i}^{r_i}}{r_i!}$$

Then the lemma follows. □

Let $\mathbf{c} = (c_1, \ldots, c_b) \in \mathbb{Z}^b$ be a $b$ tuple of non-negative integers. Define

$$p_{\mathbf{c}} := \mathbb{P}((C_1, \ldots, C_b) = \mathbf{c});$$

i.e., $p_{\mathbf{c}}$ is the probability that in a random $(d,l)$-regular hypergraph with $m$ hyperedges, there are exactly $c_k$ $k$-cycles for all $k \in [b]$. Then by Lemma 4.4(1), when $m \to \infty$,

$$p_{\mathbf{c}} \approx \prod_{k=1}^{b} \frac{e^{-\lambda_k}(\lambda_k)^{c_k}}{(c_k)!}$$

Then by Lemma 4.4, when $m$ is sufficiently large

$$E_{\beta,\mathbf{c}} \approx \left(\prod_{k=1}^{b} e^{\lambda_k-\mu_{\beta,k}}\left(\frac{\mu_{\beta,k}}{\lambda_k}\right)^{c_k}\right)\mathbb{E}Z_{m\beta} \tag{4.6}$$

Recall the following lemma about Poisson distributions:

**Lemma 4.5.** *Let $X$ has Poisson distritubion with mean $\mu$. Then*

$$\mathbb{P}\left(X < \mu(1-\epsilon)\right) \leq e^{-\frac{\mu\epsilon^2}{2}}$$
$$\mathbb{P}\left(X > \mu(1+\epsilon)\right) \leq \left[e^{\epsilon}(1+\epsilon)^{-(1+\epsilon)}\right]^{\mu}$$

*Proof.* See Theorem A.15 of [3]. □

**Lemma 4.6.** *Let $a > 0$ be a positive constant. Then for any positive constant $W > 2$,*

$$\left(\frac{e^{\frac{y}{a\lambda_k^{\frac{1}{2}}}}}{\left(1+\frac{y}{a\lambda_k^{\frac{1}{2}}}\right)^{1+\frac{y}{a\lambda_k^{\frac{1}{2}}}}}\right)^{\lambda_k} \leq e^{-\frac{y^2}{4a^2}} + e^{-\left|\phi\left(\frac{1}{2}\right)\right|\frac{y^2}{a^2W^2}} + e^{-(\log(1+W)-1)\frac{y\lambda_k^{\frac{1}{2}}}{a}}; \tag{4.7}$$

*where*

$$\phi(z) := z - (1+z)\log(1+z).$$

*Proof.* We use $L$ to denote the left hand side of (4.7). Note that

$$\frac{e^{\frac{y}{a\lambda_k^{\frac{1}{2}}}}}{\left(1+\frac{y}{a\lambda_k^{\frac{1}{2}}}\right)^{1+\frac{y}{a\lambda_k^{\frac{1}{2}}}}} = e^{\frac{y}{a\lambda_k^{\frac{1}{2}}}-\left(1+\frac{y}{a\lambda_k^{\frac{1}{2}}}\right)\log\left(1+\frac{y}{a\lambda_k^{\frac{1}{2}}}\right)}. \tag{4.8}$$

Consider the following cases:

(1) $\frac{y}{a\lambda_k^{\frac{1}{2}}} \in \left(0, \frac{1}{2}\right]$: note that

$$\log\left(1+z\right) \geq z - \frac{z^2}{2}, \forall z \in \left(0, \frac{1}{2}\right)$$

Let $z = \frac{y}{a\lambda_k^{\frac{1}{2}}}$, we obtain

$$e^{z-(1+z)\log(1+z)} \leq e^{-\left(\frac{z^2}{2}-\frac{z^3}{2}\right)}$$

and therefore

$$L \leq e^{-\left(\frac{y^2}{2a^2}-\frac{y^3}{2a^3\lambda_k^{\frac{1}{2}}}\right)} \leq e^{-\frac{y^2}{4a^2}}$$

where the last inequality follows from the fact that $\frac{y}{a\lambda_k^{\frac{1}{2}}} \in \left(0, \frac{1}{2}\right]$.

(2) $\frac{y}{a\lambda_k^{\frac{1}{2}}} \in \left(\frac{1}{2}, W\right]$, where $W > 2$ is a positive constant: note that

$$\phi'(z) = -\log(z+1) < 0, \ \forall z > 0$$

Therefore

$$\sup_{z\in\left(\frac{1}{2},2\right]} \phi(z) = \phi\left(\frac{1}{2}\right) = \frac{1}{2} - \frac{3}{2}\log\left(\frac{3}{2}\right) < 0$$

Then by (4.8) we obtain

$$L \leq e^{-\left|\phi\left(\frac{1}{2}\right)\right|\frac{y^2}{a^2W^2}}$$

(3) $\frac{y}{a\lambda_k^{\frac{1}{2}}} > W > 2$: we have

$$z - (1+z)\log(1+z) \leq -z(\log(1+z)-1) < 0, \ \forall z > 2$$

Again by (4.8), we have

$$L \leq e^{-(\log(1+W)-1)\frac{y\lambda_k^{\frac{1}{2}}}{a}}$$

□

**Lemma 4.7.** *Assume $\beta \in \left(0, \frac{1}{d}\right)$, $l \leq d$ and $d \geq 3$ For $y > 0$, define*

$$S(y) := \{\mathbf{c} \in \mathbb{N}^b : |c_k - \lambda_k| \leq y\lambda_k^{\frac{1}{2}}, \ \forall k \in [b]\}. \tag{4.9}$$

*Then for all sufficiently large $y$, and any constant $W > 2$*

$$\begin{aligned}\sum_{\mathbf{c}\in S(y)} p_{\mathbf{c}} E_{\beta,\mathbf{c}}^2 \ \geq \ & \left[1 - be^{-\left(\frac{d-2}{d-1}\right)^2\frac{y^2}{8}} - be^{-\frac{y^2}{16}\left(\frac{d-2}{d-1}\right)^2} - be^{-\frac{\left|\phi\left(\frac{1}{2}\right)\right|y^2}{4W^2}\left(\frac{d-2}{d-1}\right)^2} - be^{-(\log(1+W)-1)\frac{y}{2}\left(\frac{d-2}{d-1}\right)^2}\right] \\ & \times \left(1 - \left(\frac{(l-1)(d-1)\beta^2}{(1-\beta)^2}\right)^b\right)\frac{1-\beta}{\sqrt{1-2\beta-(dl-d-l)\beta^2}}(\mathbb{E}Z_{m\beta})^2\end{aligned}$$

*Proof.* Note that by (4.6),

$$\begin{aligned}
\sum_{\mathbf{c}\in S(y)} p_{\mathbf{c}} E_{\beta,\mathbf{c}}^2 &\approx (\mathbb{E}Z_{m\beta})^2 \sum_{\mathbf{c}\in S(y)} \prod_{k=1}^{b} \left(\frac{\mu_{\beta,k}^2}{\lambda_k}\right)^{c_k} \frac{e^{\lambda_k - 2\mu_{\beta,k}}}{(c_k)!} \\
&= (\mathbb{E}Z_{m\beta})^2 \prod_{k=1}^{b} \left[\sum_{c_k:|c_k-\lambda_k|\le y\lambda_k^{\frac{1}{2}}} \left(\frac{\mu_{\beta,k}^2}{\lambda_k}\right)^{c_k} \frac{e^{\lambda_k-2\mu_{\beta,k}}}{(c_k)!}\right] \\
&= (\mathbb{E}Z_{m\beta})^2 \prod_{k=1}^{b} e^{\frac{(\mu_{\beta,k}-\lambda_k)^2}{\lambda_k}} \left[1 - e^{-\frac{\mu_{\beta,k}^2}{\lambda_k}} \sum_{c_k:|c_k-\lambda_k|>y\lambda_k^{\frac{1}{2}}} \left(\frac{\mu_{\beta,k}^2}{\lambda_k}\right)^{c_k} \frac{1}{(c_k)!}\right] \qquad (4.10)
\end{aligned}$$

From (4.3), (4.5) we see that

$$\mu_{\beta,k} = \lambda_k \left(1 + \frac{(-1)^k\beta^k}{(1-\beta)^k}\right)$$

Hence

$$\frac{\mu_{\beta,k}^2}{\lambda_k} = \lambda_k \left(1 + \frac{(-1)^k\beta^k}{(1-\beta)^k}\right)^2 = \mu_{\beta,k}\left(1 + \frac{(-1)^k\beta^k}{(1-\beta)^k}\right) \qquad (4.11)$$

Let $X_k$ be a Poisson random variable with parameter $\frac{\mu_{\beta,k}^2}{\lambda_k}$. Assume

$$\lambda_k - y\lambda_k^{\frac{1}{2}} = (1-u_k)\left(\frac{\mu_{\beta,k}^2}{\lambda_k}\right); \qquad \lambda_k + y\lambda_k^{\frac{1}{2}} = (1+v_k)\left(\frac{\mu_{\beta,k}^2}{\lambda_k}\right).$$

Here

$$u_k, v_k > \frac{y}{2\lambda_k^{1/2}} \qquad (4.12)$$

when $y$ is sufficiently large. Then by Lemma 4.5

$$\sum_{c_k=0}^{(1-u_k)\left(\frac{\mu_{\beta,k}^2}{\lambda_k}\right)} \left(\frac{\mu_{\beta,k}^2}{\lambda_k}\right)^{c_k} \frac{e^{-\frac{\mu_{\beta,k}^2}{\lambda_k}}}{c_k!} = \mathbb{P}\left(X_k \le (1-u_k)\mathbb{E}X_k\right) \le e^{-\frac{u_k^2\mu_{\beta,k}^2}{2\lambda_k}}$$

By (4.11) and (4.12) and $\beta \in \left(0, \frac{1}{d}\right)$ we obtain

$$\sum_{c_k=0}^{(1-u_k)\left(\frac{\mu_{\beta,k}^2}{\lambda_k}\right)} \left(\frac{\mu_{\beta,k}^2}{\lambda_k}\right)^{c_k} \frac{e^{-\frac{\mu_{\beta,k}^2}{\lambda_k}}}{c_k!} \le e^{-\frac{y^2}{8}\left(1+\frac{(-1)^k\beta^k}{(1-\beta)^k}\right)^2} \le e^{-\frac{y^2}{8}\left(\frac{d-2}{d-1}\right)^2}$$

Similarly by Lemma 4.5,

$$\sum_{c_k=(1+v_k)\left(\frac{\mu_{\beta,k}^2}{\lambda_k}\right)}^{\infty} \left(\frac{\mu_{\beta,k}^2}{\lambda_k}\right)^{c_k} \frac{e^{-\frac{\mu_{\beta,k}^2}{\lambda_k}}}{c_k!} = \mathbb{P}(X_k \ge (1+v_k)\mathbb{E}X_k) \le \left(\frac{e^{v_k}}{(1+v_k)^{1+v_k}}\right)^{\frac{\mu_{\beta,k}^2}{\lambda_k}}$$

When $v_k$ is sufficiently large,

$$\frac{d}{dv_k}\frac{e^{v_k}}{(1+v_k)^{v_k}} < 0, \qquad \frac{e^{v_k}}{(1+v_k)^{1+v_k}} < 1.$$

Again by (4.12) and (4.11) we have

$$\sum_{c_k=(1+v_k)\left(\frac{\mu_{\beta,k}^2}{\lambda_k}\right)}^{\infty} \left(\frac{\mu_{\beta,k}^2}{\lambda_k}\right)^{c_k} \frac{e^{-\frac{\mu_{\beta,k}^2}{\lambda_k}}}{c_k!} \leq \left(\frac{e^{\frac{y}{2\lambda_k^{\frac{1}{2}}}}}{\left(1+\frac{y}{2\lambda_k^{\frac{1}{2}}}\right)^{1+\frac{y}{2\lambda_k^{\frac{1}{2}}}}}\right)^{\lambda_k\left(1+\frac{(-1)^k\beta^k}{(1-\beta)^k}\right)^2} \tag{4.13}$$

We shall apply (4.7) with $a=2$.
By (4.10) we obtain

$$\sum_{\mathbf{c}\in S(y)} p_{\mathbf{c}} E_{\beta,\mathbf{c}}^2 \geq \left[1 - be^{-\left(\frac{d-2}{d-1}\right)^2\frac{y^2}{8}} - be^{-\frac{y^2}{16}\left(\frac{d-2}{d-1}\right)^2} - be^{-\frac{\left|\phi\left(\frac{1}{2}\right)\right|y^2}{4W^2}\left(\frac{d-2}{d-1}\right)^2} - be^{-(\log(1+W)-1)\frac{y}{2}\left(\frac{d-2}{d-1}\right)^2}\right]$$

$$\times(\mathbb{E}Z_{m\beta})^2\prod_{k=1}^{b} e^{\frac{(\mu_{\beta,k}-\lambda_k)^2}{\lambda_k}}$$

Moreover, since $\beta\in\left(0,\frac{1}{d}\right)$, when $l\leq d$ we have

$$\prod_{k=b+1}^{\infty} e^{\frac{(\mu_{\beta,k}-\lambda_k)^2}{\lambda_k}} = e^{\sum_{k=b+1}^{\infty}\frac{\beta^{2k}}{(1-\beta)^{2k}}\frac{(d-1)^k(l-1)^k}{2k}} \leq \left(1-\left(\frac{(l-1)(d-1)\beta^2}{(1-\beta)^2}\right)^b\right)^{-1}$$

Let

$$\Theta := 1 - be^{-\left(\frac{d-2}{d-1}\right)^2\frac{y^2}{8}} - be^{-\frac{y^2}{16}\left(\frac{d-2}{d-1}\right)^2} - be^{-\frac{\left|\phi\left(\frac{1}{2}\right)\right|y^2}{4W^2}\left(\frac{d-2}{d-1}\right)^2} - be^{-(\log(1+W)-1)\frac{y}{2}\left(\frac{d-2}{d-1}\right)^2}$$

Hence

$$\begin{aligned}
\sum_{\mathbf{c}\in S(y)} p_{\mathbf{c}} E_{\beta,\mathbf{c}}^2 &\geq (\mathbb{E}Z_{m\beta})^2\,\Theta\prod_{k=1}^{\infty} e^{\frac{(\mu_{\beta,k}-\lambda_k)^2}{\lambda_k}}\left(\prod_{k=b+1}^{\infty} e^{\frac{(\mu_{\beta,k}-\lambda_k)^2}{\lambda_k}}\right)^{-1}\\
&\geq (\mathbb{E}Z_{m\beta})^2\,\Theta\left(1-\left(\frac{(l-1)(d-1)\beta^2}{(1-\beta)^2}\right)^b\right)e^{\sum_{k=1}^{\infty}\frac{\beta^{2k}(d-1)^k(l-1)^k}{2k(1-\beta)^{2k}}}\\
&= (\mathbb{E}Z_{m\beta})^2\,\Theta\left(1-\left(\frac{(l-1)(d-1)\beta^2}{(1-\beta)^2}\right)^b\right)\frac{1-\beta}{\sqrt{1-2\beta-(dl-d-l)\beta^2}}
\end{aligned}$$

Then the lemma follows. □

**Lemma 4.8.** *Let $S(y)$ be defined as in (4.9); and $C_1,\ldots,C_b$ be random variables denoting the number of 1-cycles, ..., $b$-cycles in a random $(d,l)$-regular hypergraph with $m$ hyperedges. Let*

$$\overline{p} := \mathbb{P}\left((C_1,\ldots,C_b)\notin S(y)\right)$$

*Then there exists an absolute constant $\theta > 0$, such that for all sufficiently large $y\lambda_k^{-\frac{1}{2}}$ and $m$, and for any constant $W > 2$,*

$$\overline{p} \leq be^{-\frac{y^2}{2}} + be^{-\frac{y^2}{4}} + be^{-\left|\phi\left(\frac{1}{2}\right)\right|\frac{y^2}{W^2}} + be^{-(\log(1+W)-1)y\lambda_k^{\frac{1}{2}}}$$

*Proof.* Recall that as $m \to \infty$, $C_1, \ldots, C_b$ are independent Poisson random variables with parameters $\lambda_1, \ldots, \lambda_b$, respectively. When $m$ is sufficiently large, by Lemma 4.5,

$$\mathbb{P}\left(C_k < \lambda_k\left(1 - \frac{y}{\lambda_k^{\frac{1}{2}}}\right)\right) \leq e^{-\frac{y^2}{2}}$$

Also by Lemma 4.5,

$$\mathbb{P}\left(C_k > \lambda_k\left(1 + \frac{y}{\lambda_k^{\frac{1}{2}}}\right)\right) \leq \left[\frac{e^{y\lambda_k^{-\frac{1}{2}}}}{(1 + y\lambda_k^{-\frac{1}{2}})^{1+y\lambda_k^{-\frac{1}{2}}}}\right]^{\lambda_k}$$

By (4.7) when $a = 1$, we obtain

$$\left[\frac{e^{y\lambda_k^{-\frac{1}{2}}}}{(1 + y\lambda_k^{-\frac{1}{2}})^{1+y\lambda_k^{-\frac{1}{2}}}}\right]^{\lambda_k} \leq e^{-\frac{y^2}{4}} + e^{-\left|\phi\left(\frac{1}{2}\right)\right|\frac{y^2}{W^2}} + e^{-(\log(1+W)-1)y\lambda_k^{\frac{1}{2}}}$$

Then the lemma follows. □

**Lemma 4.9.** *Assume $\beta \in \left(0, \frac{1}{\sqrt{(d-1)(l-1)}+1}\right)$. There exist constants $A, B > 0$ independent of $y$, such that for each $\mathbf{c} \in S(y)$ and sufficiently large $y$,*

$$E_{\beta,\mathbf{c}} \geq e^{-(A+By)}\mathbb{E}(Z_{m\beta})$$

*Proof.* By (4.6), we obtain

$$E_{\beta,\mathbf{c}} \approx \left[\prod_{k=1}^{b} e^{\frac{(-1)^{k+1}\beta^k}{(1-\beta)^k}\frac{(d-1)^k(l-1)^k}{2k}}\left(1 + \frac{(-1)^k\beta^k}{(1-\beta)^k}\right)^{c_k}\right]\mathbb{E}Z_{m\beta} \geq \mathbb{E}Z_{m\beta}UV^y$$

where

$$\begin{aligned}
U &= \prod_{k=1}^{b} e^{\frac{(-1)^{k+1}\beta^k}{(1-\beta)^k}\frac{(d-1)^k(l-1)^k}{2k}}\left(1 + \frac{(-1)^k\beta^k}{(1-\beta)^k}\right)^{\lambda_k} \\
V &= \prod_{k\in[b],k\ \mathit{odd}}\left(1 - \frac{\beta^k}{(1-\beta)^k}\right)^{\lambda_k^{\frac{1}{2}}}\prod_{k\in[b],k\ \mathit{even}}\left(1 + \frac{\beta^k}{(1-\beta)^k}\right)^{-\lambda_k^{\frac{1}{2}}}
\end{aligned}$$

Then

$$\begin{aligned}
U &= \prod_{k=1}^{b} e^{\frac{(-1)^{k+1}\beta^k(d-1)^k(l-1)^k}{2k(1-\beta)^k} + \lambda_k\sum_{j=1}^{\infty}\frac{(-1)^{jk+j-1}\beta^{jk}}{j(1-\beta)^{jk}}} \\
&\geq e^{-\sum_{k=1}^{\infty}\frac{\beta^{2k}(d-1)^k(l-1)^k}{4k(1-\beta)^{2k}}} = \left(\frac{1 - 2\beta - (dl - d - l)\beta^2}{(1-\beta)^2}\right)^{\frac{1}{4}} := e^{-A}
\end{aligned}$$

and

$$V \geq \prod_{k=1}^{\infty}\left(1-\frac{\beta^k}{(1-\beta)^k}\right)^{\lambda_k^{\frac{1}{2}}} \geq e^{-\sum_{k=1}^{\infty}\frac{\lambda_k^{\frac{1}{2}}}{\frac{(1-\beta)^k}{\beta^k}-1}} := e^{-B}$$

When $0<\beta<\frac{1}{\sqrt{(d-1)(l-1)}+1}$, we have

$$\frac{\beta\sqrt{(d-1)(l-1)}}{(1-\beta)}<1,$$

Hence $\sum_{k=1}^{\infty}\frac{\lambda_k^{\frac{1}{2}}}{\frac{(1-\beta)^k}{\beta^k}-1}$ is convergent. Then the lemma follows. □

## 5. Free Energy

In this section, we show the convergence in probability of the free energy to an explicit limit when $(\mathbb{E}Z_{m\beta_*})^2$ and $\mathbb{E}Z_{m\beta_*}^2$ have the same exponential growth rate, where $\beta_*$ is the unique root of (1.2).

**Lemma 5.1.** *Let $\beta\in\left(0,\frac{1}{d}\right)$, $l\geq 2$, $d\geq 2$ such that one of the following conditions holds*

*(1) $l=2$; or*
*(2) $l\geq 3$ and $\beta\leq L_1$.*

*Then as $m\rightarrow\infty$*

$$\frac{1}{m}\log Z_{m\beta}\longrightarrow\Phi_{d,l}(\beta)$$

*in probability.*

*Proof.* By (2.3), it suffices to show that

$$\lim_{m\rightarrow\infty}\mathbb{P}\left(\frac{1}{m}\left|\log\frac{Z_{m\beta}}{\mathbb{E}Z_{m\beta}}\right|>\epsilon\right)=0. \tag{5.1}$$

for all $\epsilon>0$.

Let

$$\mathbf{C}=(C_1,C_2,\ldots,C_b)$$

be the random variable denoting the number of 1-cycles, 2-cycles, ..., $b$-cycles in a random $(d,l)$-regular hypergraph with $m$ hyperedges. Let $E_{\beta,\mathbf{C}}$ be the conditional expectation given by

$$E_{\beta,\mathbf{C}}:=\mathbb{E}(Z_{m\beta}|\mathbf{C})$$

Then

$$\frac{1}{m}\log\frac{Z_{m\beta}}{\mathbb{E}Z_{m\beta}}=\frac{1}{m}\log\frac{Z_{m\beta}}{E_{\beta,\mathbf{C}}}+\frac{1}{m}\log\frac{E_{\beta,\mathbf{C}}}{\mathbb{E}Z_{m\beta}}$$

By Chebyshev's inequality, for any $t > 0$

$$\mathbb{P}(|Z_{m\beta} - E_{\beta,\mathbf{C}}| \geq t) \leq \frac{\mathbb{E}\left(Z_{m\beta} - E_{\beta,\mathbf{C}}\right)^2}{t^2} \tag{5.2}$$

By Lemmas 4.7 and 3.13, we obtain

$$\begin{aligned}
\mathbb{E}\left(Z_{m\beta} - E_{\beta,\mathbf{C}}\right)^2 &= \mathbb{E}Z_{m\beta}^2 - \mathbb{E}E_{\beta,\mathbf{C}}^2 \\
&= \mathbb{E}Z_{m\beta}^2 - \sum_{\mathbf{c}\in S(y)} p_{\mathbf{c}} E_{\beta,\mathbf{c}}^2 - \sum_{\mathbf{c}\notin S(y)} p_{\mathbf{c}} E_{\beta,\mathbf{c}}^2 \\
&\leq \mathbb{E}Z_{m\beta}^2 \left( be^{-\left(\frac{d-2}{d-1}\right)^2 \frac{y^2}{8}} + be^{-\frac{y\lambda_k^{\frac{1}{2}}}{4}\log\left(\frac{y}{2\lambda_k^{\frac{1}{2}}}\right)\left(\frac{d-2}{d-1}\right)^2} + \left(\frac{l-1}{d-1}\right)^b \right) \qquad (5.3)
\end{aligned}$$

Then by Lemma 4.8 we have for any constant $W > 2$,

$$\begin{aligned}
\mathbb{P}\left(\frac{1}{m}\left|\log\frac{Z_{m\beta}}{E_{\beta,\mathbf{C}}}\right| > \epsilon\right) &\leq \mathbb{P}(\mathbf{C}\notin S(y)) + \mathbb{P}\left(\frac{1}{m}\left|\log\frac{Z_{m\beta}}{E_{\beta,\mathbf{C}}}\right| > \epsilon \text{ and } \mathbf{C}\in S(y)\right) \\
&\leq be^{-\frac{y^2}{2}} + be^{-\frac{y^2}{4}} + be^{-\left|\phi\left(\frac{1}{2}\right)\right|\frac{y^2}{W^2}} + be^{-(\log(1+W)-1)y\lambda_k^{\frac{1}{2}}} + P_1 + P_2
\end{aligned}$$

where

$$\begin{aligned}
P_1 &= \mathbb{P}\left(Z_{m\beta} < E_{\beta,\mathbf{C}}e^{-m\epsilon} \text{ and } \mathbf{C}\in S(y)\right) \\
&\leq \mathbb{P}\left(|Z_{m\beta} - E_{\beta,\mathbf{C}}| > (1-e^{-m\epsilon})E_{\beta,\mathbf{C}} \text{ and } \mathbf{C}\in S(y)\right) := P_3
\end{aligned}$$

$$\begin{aligned}
P_2 &= \mathbb{P}\left(Z_{m\beta} > E_{\beta,\mathbf{C}}e^{m\epsilon} \text{ and } \mathbf{C}\in S(y)\right) \\
&\leq \mathbb{P}\left(|Z_{m\beta} - E_{\beta,\mathbf{C}}| > (e^{m\epsilon}-1)E_{\beta,\mathbf{C}} \text{ and } \mathbf{C}\in S(y)\right)
\end{aligned}$$

Note that

$$P_2 \leq P_3$$

when $m$ is sufficiently large. By Lemma 4.9, we have

$$P_3 \leq \mathbb{P}\left(|Z_{m\beta} - E_{\beta,\mathbf{C}}| > (1-e^{-m\epsilon})e^{-(A+By)}\mathbb{E}Z_{m\beta} \text{ and } \mathbf{C}\in S(y)\right) := P_4$$

Then by (5.2), (5.3) we obtain

$$P_4 \leq \frac{e^{2A+2By}\mathbb{E}Z_{m\beta}^2}{(\mathbb{E}Z_{m\beta})^2(1-e^{-m\epsilon})^2}\left(1 - \Theta + \left(\frac{(l-1)(d-1)\beta^2}{(1-\beta)^2}\right)^b\right) \tag{5.4}$$

where

$$1-\Theta = be^{-\left(\frac{d-2}{d-1}\right)^2\frac{y^2}{8}} + be^{-\frac{y^2}{16}\left(\frac{d-2}{d-1}\right)^2} + be^{-\frac{\left|\phi\left(\frac{1}{2}\right)\right|y^2}{4W^2}\left(\frac{d-2}{d-1}\right)^2} - be^{-(\log(1+W)-1)\frac{y}{2}\left(\frac{d-2}{d-1}\right)^2}$$

Note that when $\beta \in \left(0, \frac{1}{d}\right)$, and $l \leq d$,

$$\frac{(l-1)(d-1)\beta^2}{(1-\beta)^2} < 1$$

Let

$$b := -\frac{3By}{\log\left(\frac{(l-1)(d-1)\beta^2}{(1-\beta)^2}\right)}$$

and

$$(\log(1+W)-1)\frac{1}{2}\left(\frac{d-2}{d-1}\right)^2 \geq 3B$$

By Lemma 3.13, as $y \to \infty$, the right hand side of (5.4) goes to 0. Moreover,

$$\mathbb{P}\left(\frac{1}{m}\left|\log\frac{E_{\beta,\mathbf{C}}}{\mathbb{E}Z_{m\beta}}\right| > \epsilon\right) \quad \leq \quad P_5 + P_6;$$

where

$$P_5 = \mathbb{P}\left(\frac{E_{\beta,\mathbf{C}}}{\mathbb{E}Z_{m\beta}} > e^{m\epsilon}\right) \leq e^{-m\epsilon} \to 0,$$

as $m \to \infty$; and

$$P_6 \quad = \quad \mathbb{P}\left(\frac{E_{\beta,\mathbf{C}}}{\mathbb{E}Z_{m\beta}} < e^{-m\epsilon}\right) \leq P_7 + P_8.$$

where

$$\begin{aligned} P_7 &= \mathbb{P}(\mathbf{C} \notin S(y)); \\ P_8 &= \mathbb{P}(\mathbf{C} \in S(y))\mathbb{P}(e^{-(A+By)} \leq e^{-m\epsilon}|\mathbf{C} \in S(y)). \end{aligned}$$

We obtain $P_7 \to 0$ as $y \to \infty$ by Lemma 4.8 and $P_8 \to 0$ as $m \to 0$ by Lemma 4.9. Then the lemma follows. □

It is straightforward to check the following lemma.

**Lemma 5.2.** *Let $L_1$ be given by (3.34). If*

$$\Phi'_{d,l}(L_1) \leq 0;$$

*Then $\beta_* \leq L_1$.*

**Proof of Theorem 1.1(2).** By Theorem 1.1(1), it suffices to show that

$$\lim_{m\to\infty}\mathbb{P}\left(\frac{1}{m}\left|\log\frac{Z}{\mathbb{E}Z_{m\beta_*}}\right| > \epsilon\right) = 0.$$

for all $\epsilon > 0$. Note that

$$\mathbb{P}\left(\frac{1}{m}\left|\log\frac{Z}{\mathbb{E}Z_{m\beta_*}}\right| > \epsilon\right) \leq R_1 + R_2$$

where

$$R_1 \quad = \quad \mathbb{P}\left(\frac{Z}{\mathbb{E}Z_{m\beta_*}} > e^{m\epsilon}\right); \qquad R_2 = \mathbb{P}\left(\frac{Z}{\mathbb{E}Z_{m\beta_*}} < e^{-m\epsilon}\right)$$

Note that

$$R_1 = \mathbb{P}\left(\frac{\sum_{j=1}^{\frac{m}{d}} Z_j}{\mathbb{E}Z_{m\beta_*}} > e^{m\epsilon}\right) \leq \sum_{j=1}^{\frac{m}{d}} \mathbb{P}\left(\frac{Z_j}{\mathbb{E}Z_{m\beta_*}} > \frac{1}{m}e^{m\epsilon}\right) \leq \sum_{j=1}^{\frac{m}{d}} \frac{m}{e^{m\epsilon}} \frac{\mathbb{E}Z_j}{\mathbb{E}Z_{m\beta_*}} \to 0,$$

as $m \to \infty$, where the limit follows from the fact that

$$\lim_{m\to\infty} \frac{1}{m} \log \mathbb{E}Z_{m\beta_*} = \max\left\{\beta \in \left(0, \frac{1}{d}\right) : \lim_{m\to\infty} \frac{1}{m} \log \mathbb{E}Z_{m\beta}\right\}.$$

Moreover

$$R_2 = \mathbb{P}\left(\frac{\sum_{j=1}^{\frac{m}{d}} Z_j}{\mathbb{E}Z_{m\beta_*}} < e^{-m\epsilon}\right) \leq \mathbb{P}\left(\frac{Z_{m\beta_*}}{\mathbb{E}Z_{m\beta_*}} < e^{-m\epsilon}\right) \to 0,$$

as $m \to \infty$, where the last limit follows from (5.1). Then the lemma follows. □

## 6. Weighted Free Energy

In this section, we show the convergence in probability of the weighted energy to an explicit limit when each hyperedge in the matching is given a sufficiently small weight $x > 0$.

Let $x > 0$. Define the weighted partition $Z(x)$ function for matchings on a $(d, l)$-regular graph as in (1.6). When $\beta \in \left(0, \frac{1}{d}\right)$, we have

$$Z_{m\beta} x^{m\beta} \asymp m^{-\frac{1}{2}} e^{m\left(\Delta_{d,l}(\beta,x)\right)}$$

where

$$\Delta_{d,l}(\beta, x) = \Phi_{d,l}(\beta, x) + \beta \ln(x).$$

Hence we have

$$\frac{\partial \Delta_{d,l}(\beta, x)}{\partial \beta} = \Phi'_{d,l}(\beta) + \ln(x)$$

and

$$\frac{\partial^2 \Delta_{d,l}(\beta, x)}{\partial \beta^2} = \Phi''_{d,l}(\beta) < 0$$

Since

$$\lim_{\beta\to 0+} \frac{\partial \Delta_{d,l}(\beta, x)}{\partial \beta} = \ln(x) + \lim_{\beta\to 0+} \Phi'_{d,l}(\beta) = +\infty$$
$$\lim_{\beta\to \frac{1}{d}-} \frac{\partial \Delta_{d,l}(\beta, x)}{\partial \beta} = \ln(x) + \lim_{\beta\to \frac{1}{d}-} \Phi'_{d,l}(\beta) = -\infty$$

We obtain that there exists a unique $\beta_*(x) \in \left(0, \frac{1}{d}\right)$, such that

$$\left.\frac{\partial \Delta_{d,l}(\beta, x)}{\partial \beta}\right|_{\beta=\beta_*(x)} = 0$$

and

$$\Delta_{d,l}(\beta_*(x),x)=\max_{\beta\in\left(0,\frac{1}{d}\right)}\Delta_{d,l}(\beta,x)$$

and moreover $\beta_*(x)$ is the unique maximizer.

**Proof of Theorem 1.1(3b).** Note that

$$\mathbb{P}\left(\frac{1}{m}\left|\log\frac{Z(x)}{e^{m[\Phi_{d,l}(\beta_*(x))+\beta_*(x)\ln(x)]}}\right|>\epsilon\right)\leq Q_1+Q_2$$

where

$$Q_1 = \mathbb{P}\left(\frac{Z(x)}{e^{m[\Phi_{d,l}(\beta_*(x))+\beta_*(x)\ln(x)]}}>e^{m\epsilon}\right);\qquad Q_2=\mathbb{P}\left(\frac{Z(x)}{e^{m[\Phi_{d,l}(\beta_*(x))+\beta_*(x)\ln(x)]}}<e^{-m\epsilon}\right)$$

Note that

$$\begin{aligned} Q_1 &= \mathbb{P}\left(\frac{\sum_{j=1}^{\frac{m}{d}}Z_j(x)}{e^{m[\Phi_{d,l}(\beta_*(x))+\beta_*(x)\ln(x)]}}>e^{m\epsilon}\right)\leq\sum_{j=1}^{\frac{m}{d}}\mathbb{P}\left(\frac{Z_j(x)}{e^{m[\Phi_{d,l}(\beta_*(x))+\beta_*(x)\ln(x)]}}>\frac{1}{m}e^{m\epsilon}\right)\\ &\leq \sum_{j=1}^{\frac{m}{d}}\frac{m}{e^{m\epsilon}}\frac{\mathbb{E}Z_j(x)}{e^{m[\Phi_{d,l}(\beta_*(x))+\beta_*(x)\ln(x)]}}\to 0, \end{aligned}$$

as $m\to\infty$, Where the limit follows from the fact that

$$\lim_{m\to\infty}\frac{1}{m}\log\mathbb{E}Z_{m\beta_*(x)}(x)=\max\left\{\beta\in\left(0,\frac{1}{d}\right):\lim_{m\to\infty}\frac{1}{m}\log\mathbb{E}Z_{m\beta}(x)\right\}.$$

If $\Phi'_{d,l}(L_1)+\ln(x)\leq 0$, then

$$\beta_*(x)\leq L_1. \tag{6.1}$$

Moreover

$$\begin{aligned} Q_2 &= \mathbb{P}\left(\frac{\sum_{j=1}^{\frac{m}{d}}Z_j(x)}{e^{m[\Phi_{d,l}(\beta_*(x))+\beta_*(x)\ln(x)]}}<e^{-m\epsilon}\right)\leq\mathbb{P}\left(\frac{Z_{m\beta_*(x)}(x)}{e^{m[\Phi_{d,l}(\beta_*(x))+\beta_*(x)\ln(x)]}}<e^{-m\epsilon}\right)\\ &= \mathbb{P}\left(\frac{Z_{m\beta_*(x)}}{e^{m[\Phi_{d,l}(\beta_*(x))]}}<e^{-m\epsilon}\right)\to 0, \end{aligned}$$

as $m\to\infty$, where the last limit follows from (5.1) and (6.1). Then the lemma follows. $\square$

## 7. Another Criterion to Guarantee Global Maxima

**Lemma 7.1.** *Assume $d\geq 2$, $l\geq 3$ and $0<\beta d<1$. For any straight line in the $(\rho,\eta)$-plane passing through $(\beta^2,(d-1)\beta^2)$ with slope in $s\in(-\infty,-1]$, the second order directional derivative of $\Psi_{d,l}$ along the line is strictly negative when $\rho\in\left(\beta^2,\frac{(d-1-s)\beta^2}{-s}\right)$ and*

$$\beta<\frac{dl+l^2-2l-d+1}{2dl^2-dl} \tag{7.1}$$

*Proof.* Let $s \in (-\infty, -1]$. It suffices to show that under the assumptions of the lemma,

$$Y(s) := \frac{\partial^2 \Psi_{d,l}}{\partial \rho^2} + s^2 \frac{\partial^2 \Psi_{d,l}}{\partial \eta^2} + 2s \frac{\partial^2 \Psi_{d,l}}{\partial \rho \partial \eta} < 0$$

By (3.11) (3.12) and the fact that $\theta = \eta l$ we obtain

$$\begin{aligned} Y(s) &= -\frac{1}{\rho} + \frac{l-1}{1-2\beta+\rho} + \frac{2(l-1)}{\beta-\rho} - \frac{s^2 l}{\eta} \\ &\quad -(s+1)^2 \left( \frac{2l}{\beta-\rho-\eta} + \frac{ld}{1-2\beta d+\rho d+\eta d} \right) \end{aligned}$$

When $(\rho, \eta)$ is along the straight line passing through $(\beta^2, (d-1)\beta^2)$ with slope $s$, they satisfy the following equation

$$\eta = s(\rho - \beta^2) + (d-1)\beta^2$$

When $\beta^2 \leq \rho \leq \frac{(d-1-s)\beta^2}{-s}$ and $s \in (-\infty, -1]$, we have

$$0 \leq \eta \leq -\rho + d\beta^2.$$

and therefore

$$Y(s) \leq -\frac{1}{\rho} + \frac{l-1}{1-2\beta+\rho} + \frac{2(l-1)}{\beta-\rho} - \frac{l}{d\beta^2-\rho} = Y(-1)$$

Hence it suffices to show that $Y(-1) < 0$ when $\rho \in (\beta^2, d\beta^2)$ and (7.1) holds.

Explicit computations show that

$$Y(-1) = -\frac{U(\rho)}{\rho(d\beta^2-\rho)(\beta-\rho)(\rho-2\beta+1)}$$

where

$$U(\rho) = (l(1-d\beta^2)-1)\rho^2 + \beta[l-1+\beta(d-2l+2-2dl)+3\beta^2 dl]\rho + d\beta^3(1-2\beta)$$

Then $U(\rho)$ is a parabola opening upwards. The following cases might occur

(1) $U(\rho) = 0$ has no real roots, then $U(\rho) > 0$ for all $\rho \in \mathbb{R}$, in particular this implies that $Y(-1) < 0$.
(2) $U(\rho) = 0$ has two real roots $\rho_m \leq \rho_M$. If $\rho_M < \beta^2$, then $U(\rho) > 0$ for $\rho \in (\beta^2, d\beta^2)$, and therefore $Y(-1) < 0$.

Let $T$ be the discriminant of $U(\rho)$. In case (1), $T < 0$. In case (2), we need $T \geq 0$ and

$$\rho_M = \frac{\sqrt{T} - \beta[l-1+\beta(d-2l+2-2dl)+3\beta^2 dl]}{2(l(1-d\beta^2)-1)} < \beta^2 \tag{7.2}$$

Explicit computations show that (7.2) holds if and only if

$$-d^2l^2\beta^4 + 2d^2l^2\beta^3 - d^2l\beta^2 + (-2dl^2+2dl)\beta + dl - 2l - d + l^2 + 1 > 0 \tag{7.3}$$

It is straightforward to check that (7.3) holds whenever (7.1) holds, $d \geq 3$ and $l \geq 2$. □

**Lemma 7.2.** *Assume $d \geq 2$, $l \geq 3$ and $\beta d < 1$. Then along the line segment joining $(\beta^2, (d-1)\beta^2)$ and $(d\beta^2, 0)$ in the $(\rho, \eta)$-plane, there exists $\rho_8 \in [\beta^2, d\beta^2]$, such that*

- *When* $\rho \in (\beta^2, \rho_8)$, $\frac{\partial \Psi_{d,l}}{\partial \rho}$ *is decreasing; and*
- *When* $\rho \in (\rho_8, d\beta^2)$, $\frac{\partial \Psi_{d,l}}{\partial \rho}$ *is increasing.*

*Proof.* Let

$$X(\rho) := \frac{\partial^2 \Psi_{d,l}}{\partial \rho^2} - \frac{\partial^2 \Psi_{d,l}}{\partial \rho \partial \eta}$$

It suffices to show that

- $X(\rho) \leq 0$ when $\rho \in (\beta^2, \rho_8)$; and
- $X(\rho) \geq 0$ when $\rho \in (\rho_8, d\beta^2)$.

By (3.11) (3.12) and the fact that $\theta = \eta l$ we obtain

$$\begin{aligned} X(\rho) &= -\frac{1}{\rho} + \frac{l-1}{1-2\beta+\rho} + \frac{2(l-1)}{\beta-\rho} \\ &= \frac{V(\rho)}{\rho(\beta-\rho)(\rho-2\beta+1)} \end{aligned}$$

where

$$V(\rho) = \rho^2 + (l - 2\beta - \beta l)\rho + 2\beta^2 - \beta$$

Note that $V(\rho)$ is a parabola opening upwards with exactly one positive root and one negative root. Moreover,

$$V(0) = -\beta(1-2\beta) < 0.$$

Then the lemma follows. □

**Lemma 7.3.** *Let* $d \geq 3$, $l \geq 2$ *and* $\beta \in \left(0, \frac{1}{d}\right)$. *Assume (7.3) and (3.20) hold. If*

$$\left.\frac{\partial \Psi_{d,l}}{\partial \rho}(\beta, \rho, \theta)\right|_{(\rho,\theta)=(d\beta^2,0)} \leq 0 \tag{7.4}$$

*Then* $(\rho, \theta) = (\beta^2, (d-1)l\beta^2)$ *is the unique global maximum for* $\Psi_{d,l}(\beta, \rho, \theta)$ *when* $(\rho, \theta) \in R_\beta$.

*Proof.* Let $z = (\rho, \eta)$ be a point along the line segment joining $z_0 := (\beta^2, (d-1)\beta^2)$ and $z_1 := (d\beta^2, 0)$ in the $(\rho, \eta)$-plane. By Lemma 7.2, we obtain

$$\frac{\partial \Psi_{d,l}}{\partial \rho}(\beta, \rho, l\eta) < \max\left\{\frac{\partial \Psi_{d,l}}{\partial \rho}(\beta, \beta^2, (d-1)l\beta^2), \frac{\partial \Psi_{d,l}}{\partial \rho}(\beta, d\beta^2, 0)\right\} \leq 0.$$

By Lemma 3.3, if we consider the region $R_{\beta,2} \subset R_\beta$ in the $(\rho, \eta)$-plane bounded by

- $\eta = d\beta^2 - \rho$; and
- $\eta = \beta - \rho$; and
- $\eta = 0$; and
- $\eta = (d-1)\beta^2$;

we have

$$\frac{\partial \Psi_{d,l}}{\partial \rho}(\beta, \rho, l\eta) < 0, \ \forall (\rho, \eta) \in R_{\beta,2}.$$

Hence

$$\max_{(\rho,\eta)\in R_{\beta,2}} \Psi_{d,l}(\beta,\rho,l\eta) = \max_{(\rho,d\beta\rho^2):\beta^2\leq\rho\leq d\beta^2} \Psi_{d,l}(\beta,\rho,l\eta) = \Psi_{d,l}(\beta,\beta^2,l(d-1)\beta^2);$$

where the last identity follows from Lemma 7.1 and the fact that all the derivatives at $\Psi_{d,l}(\beta,\rho,\theta)$ at $(\rho,\theta) = (\beta^2,(d-1)l\beta^2)$ vanish. It is straightforward to check that the maximizer in $R_{\beta,2}$ for $\Psi_{d,l}$ is unique by the strict negativity of the derivatives.

Let $R_{\beta,1} \subset \mathcal{R}_\beta$ be the region in the $(\rho,\eta)$-plane bounded by

- $\rho = \beta^2$; and
- $\eta = 0$; and
- $\eta = d\beta^2 - \rho$; and
- $\eta = 2\beta - \rho - \frac{1}{d}$ if $2\beta - \frac{1}{d} > \beta^2$.

Then by Lemma 7.1 and the fact that all the derivatives at $\Psi_{d,l}(\beta,\rho,\theta)$ at $(\rho,\theta) = (\beta^2,(d-1)l\beta^2)$ vanish, we obtain

$$\max_{(\rho,\eta)\in R_{\beta,1}} \Psi_{d,l}(\beta,\rho,l\eta) = \Psi_{d,l}(\beta,\beta^2,l(d-1)\beta^2);$$

and that the maximizer in $R_{\beta,1}$ for $\Phi_{d,l}(\beta,\rho,l\eta)$ is unique.

Let

$$\begin{aligned} R_{\beta,3} &= \{\mathcal{R}_\beta \setminus [R_{\beta_1} \cup R_{\beta,2}]\} \cap \{(\rho,\eta) : \det H(\beta,\rho,\eta l) \geq 0\} \\ R_{\beta,4} &= \{\mathcal{R}_\beta \setminus [R_{\beta_1} \cup R_{\beta,2}]\} \cap \{(\rho,\eta) : \det H(\beta,\rho,\eta l) < 0\}. \end{aligned}$$

For any point $(\rho,\eta) \in R_{\beta,3}$, there exists a line segment joining $(\beta^2,(d-1)\beta^2)$ and $(\rho,\eta)$ such that every point along the line segment is in $R_{\beta,3}$. Note that $\det H > 0$ at each interior point of the line segment, therefore the second order directional derivative of $\Psi_{d,l}(\beta,\rho,l\eta)$ along the line segment is strictly negative at each interior point. Since the first order derivatives of $\Psi_{d,l}(\beta,\rho,l\eta)$ at $(\beta^2,(d-1)\beta^2)$ are 0, the first order directional derivative of $\Psi_{d,l}(\beta,\rho,l\eta)$ along the line segment is strictly negative at each interior point. Hence we have

$$\Psi_{d,l}(\beta,\rho,l\eta) < \Psi_{d,l}(\beta,\beta^2,l(d-1)\beta^2),\ \forall(\rho,\eta) \in R_{\beta,3} \setminus \{\beta^2,(d-1)\beta^2\}.$$

Now we consider $R_{\beta,4}$. When the Hessian determinant is negative, $\Psi_{d,l}(\beta,\rho,\theta)$ as a function of $(\rho,\theta)$ cannot achieve a local maximal value at an interior point; it cannot achieve the global maximal value in $R_\beta$ along the boundary satisfying $\det H = 0$ either, since this is also the boundary of the region $\det H > 0$ where the function is strictly concave, and the maximal value in $\det H \geq 0$ is already achieved at the unique interior point $(\beta^2, l(d-1)\beta^2)$. Therefore if a global maximal value of $\Psi_{d,l}(\beta,\rho,\theta)$ in $R_\beta$ is achieved when $(\rho,\theta) \in R_\beta \cap \{\det H < 0\}$, then it must be achieved along the boundary $\theta = l(\beta-\rho)$; see Figure 3.1. More precisely, the only possible candidate for the global maximal value when $\det H < 0$ is

$$\max_{\rho\in[\rho_5,\beta-(d-1)\beta^2]} \Psi_{d,l}(\beta,\rho,l(\beta-\rho))$$

Let $T_{d,l,\beta}(\rho)$ be defined as in (3.36). As in (3.36), we obtain that $\rho \in [\rho_5, \beta)$, $\frac{d^2 T_{d,l\beta}(\rho)}{d\rho^2} \geq 0$, and the identity holds if and only if $\rho = \rho_5$. Therefore $\frac{dT_{d,l\beta}(\rho)}{d\rho}$ is monotone increasing when $\rho \in [\rho_5, \beta)$. Then

$$\max_{\rho \in [\rho_5, \beta-(d-1)\beta^2]} T_{d,l,\beta}(\rho) \leq \max\{T_{d,l,\beta}(\rho_5), T_{d,l,\beta}(\beta)\}.$$

But $(\rho_5, \beta - \rho_5) \in R_{\beta,3}$, hence

$$T_{d,l,\beta}(\rho_5) < \Psi_{d,l}(\beta, \beta^2, l(d-1)\beta^2).$$

Moreover, $(\beta, 0) \in R_{\beta,2}$, hence

$$T_{d,l,\beta}(\rho_5) < \Psi_{d,l}(\beta, \beta^2, l(d-1)\beta^2).$$

Then the lemma follows. □

**Lemma 7.4.** *Let $d \geq 3$, $l \geq 2$ and $\beta \in \left(0, \frac{1}{d}\right)$. Assume (7.3) and (3.20) hold. If*

$$\beta \leq \frac{1}{d}\left(1 - \sqrt{\frac{d-1}{d^{\frac{l}{l-1}} - 1}}\right) \tag{7.5}$$

*Then $(\rho, \theta) = (\beta^2, (d-1)l\beta^2)$ is the unique global maximum for $\Psi_{d,l}(\beta, \rho, \theta)$ when $(\rho, \theta) \in R_\beta$.*

*Proof.* It is straightforward to check that (7.5) implies (7.4). Then the lemma follows from Lemma 7.3. □

**Proof of Theorem 1.1(3c).** Theorem 1.1(3c) follows from the similar arguments as the proof of Theorem 1.1(3b). □

## 8. Maximal Matching

In this section, we discuss the implications of our results on maximal matchings.

**Proposition 8.1.** *Let $\beta \in \left(0, \frac{1}{d}\right)$. Assume $f_l\left(\frac{1}{d}\right) < 0$. Define*

$$K_m := m\beta_0 + \frac{\ln m}{2\Phi'_{d,l}(\beta_0)} \tag{8.1}$$

*Then*

$$\lim_{m\to\infty} \mathbb{E}Z_{K_m} = \frac{1}{\sqrt{2\pi\beta_0(1-d\beta_0)}}$$

*Proof.* By (2.7), we obtain

$$\mathbb{E}Z_K = \frac{1}{\sqrt{2\pi\beta_0(1-d\beta_0)}\sqrt{m}} e^{m\Phi_{d,l}\left(\beta_0 + \frac{\ln m}{2m\Phi'_{d,l}(\beta_0)}\right)}\left(1 + O\left(\frac{(\ln m)^2}{m}\right)\right)$$

Using Taylor expansion at $\beta_0$ to approximate $\Phi_{d,l}\left(\beta_0 + \frac{\ln m}{2m\Phi'_{d,l}(\beta_0)}\right)$, the lemma follows. □

**Proposition 8.2.** *Let $\beta \in \left(0, \frac{1}{d}\right)$. Assume $f_l\left(\frac{1}{d}\right) < 0$. Let $K_m$ be given as in (8.1). Let $Z_{\geq K_m + C}$ be the total number of matchings containing at least $K_m + C$ hyperedges in a* *random regular graph* $\mathcal{G}_{m,d,l} \in \Omega_{m,d,l}$. *Then*

$$\lim_{C\to\infty} \mathbb{E}Z_{\geq K_m+C} = 0,$$

*where the convergence is uniform for all sufficiently large* $m \geq C$.

*Proof.* Let

$$C_m := -\frac{5\ln m}{2\Phi'_{d,l}(\beta_0)},$$

Note that $C_m > 0$ since $\Phi_{d,l}(\beta)$ is strictly decreasing in $\left(\beta_*, \frac{1}{d}\right)$, and $\beta_0 \in \left(\beta_*, \frac{1}{d}\right)$. Then

$$\mathbb{E}Z_{\geq K_m+C} = W_1 + W_2;$$

where

$$W_1 = \sum_{C\leq\delta\leq C_m-1} \mathbb{E}Z_{K_m+\delta}; \qquad W_2 = \sum_{\delta\geq C_m} \mathbb{E}Z_{K_m+\delta}.$$

By (2.7) we obtain

$$\mathbb{E}Z_{K_m+\delta} = \frac{1}{\sqrt{2\pi\beta_0(1-d\beta_0)}\sqrt{m}} e^{m\Phi_{d,l}\left(\beta_0+\frac{\ln m}{2m\Phi'_{d,l}(\beta_0)}+\frac{\delta}{m}\right)}\left(1+O\left(\frac{(\ln m)^2}{m}\right)\right)$$

When $\delta \geq C_m$, since $\Phi_{d,l}(\beta)$ is strictly decreasing when $\beta \in \left(\beta_*, \frac{1}{d}\right)$, we have

$$\begin{aligned}\mathbb{E}Z_{K_m+\delta} &\leq \frac{1}{\sqrt{2\pi\beta_0(1-d\beta_0)}\sqrt{m}} e^{m\Phi_{d,l}\left(\beta_0-\frac{2\ln m}{m\Phi'_{d,l}(\beta_0)}\right)}\left(1+O\left(\frac{(\ln m)^2}{m}\right)\right)\\ &\leq \frac{1}{\sqrt{2\pi\beta_0(1-d\beta_0)}m^{\frac{5}{2}}}\left(1+O\left(\frac{(\ln m)^2}{m}\right)\right) \leq \frac{1}{m^2}\end{aligned}$$

Since there are at most $m$ summands in $W_2$, we obtain $W_2 \leq \frac{1}{m} \leq \frac{1}{C}$, since $m \geq C$.

Again using the Taylor expansion of $\Phi_{d,l}$ at $\beta_0$ to approximate $\Phi_{d,l}(K_m+\delta)$ we obtain

$$\begin{aligned}W_1 &= \sum_{C\leq\delta\leq C_m-1} \frac{1}{\sqrt{2\pi\beta_0(1-d\beta_0)}} e^{\Phi'_{d,l}(\beta_0)\delta}\left(1+O\left(\frac{(\ln m)^2}{m}\right)\right)\\ &\leq \left(1+O\left(\frac{(\ln m)^2}{m}\right)\right)\frac{1}{\sqrt{2\pi\beta_0(1-d\beta_0)}}\frac{e^{\Phi'_{d,l}(\beta_0)C}}{1-e^{\Phi'_{d,l}(\beta_0)}}\end{aligned}$$

Then the lemma follows since $\Phi'_{d,l}(\beta_0) < 0$. ☐

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