LIMIT SHAPE OF PERFECT MATCHINGS ON CONTRACTING BIPARTITE GRAPHS

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Abstract. We consider random perfect matchings on a general class of contracting bipartite graphs by letting certain edge weights to be 0 on the contracting square-hexagon lattice in a periodic way. We obtain a deterministic limit shape in the scaling limit. The results can also be applied to prove the existence of multiple disconnected liquid regions for all the contracting square-hexagon lattices with certain edge weights, extending the results proved in [11] for contracting square-hexagon lattices where the number of square rows in each period is either 0 or 1.

1. Introduction

A dimer configuration, or a perfect matching of a graph is a choice of subset of edges such that each vertex is incident to exactly one edge. Dimer configurations on graphs form a natural mathematical model for the structure of matter, for example, the perfect matchings on the hexagonal lattice $\mathbb{H}$ can describe the double-bond configurations in graphite molecules, where carbon atoms are represented by vertices of $\mathbb{H}$, and each double bond corresponds to a present edge in the perfect matching. One may assign each present edge in a perfect matching a non-negative weight depending on the energy of the double bond, and define the probability of a configuration to be proportional to the product of weights of present edges.

The limit behaviors of probability measures for perfect matchings on an infinite graph have been studied extensively. Unlike the well-known ferromagnetic Ising model, the limit measures of perfect matchings on infinite periodic bipartite graphs strongly depend on the boundary conditions. Indeed, for each slope of boundary conditions, one can construct a unique ergodic measure such that the measure is uniform conditional on the boundary slope. See [9, 13].

In this paper, we consider perfect matchings on a large class of bipartite graphs with a special type of boundary conditions, such that perfect matchings on such graphs form a Schur process, and the partition function (weighted sum of perfect matchings) can be computed by a Schur polynomial depending on edge weights and the bottom boundary condition. The dimer configurations on the class of bipartite graphs discussed here include uniform lozenge tilings of trapezoid domains ([4]), uniform domino tilings of rectangular domains ([5]), and perfect matchings on contracting square-hexagon lattices ([1, 3, 11, 10, 12]) as special cases. We shall show that the limit shape of random perfect matchings is deterministic, and therefore obtain a 2D analogue of the Law of Large Numbers. The idea to study limit shape here is to analyze the asymptotics of the Schur polynomials at a general point using a formula obtained in [11]. Limit shape of perfect matchings can also be obtained by the variational principle; see [6, 8].

The existence of multiple disconnected liquid regions for the limit shape of perfect matchings with certain edge weights on a contracting square-hexagon lattice, in which there are at most one row of squares in each period, was proved in [11]. The results proved in this paper can be applied to prove the existence of multiple disconnected liquid regions for limit shape of perfect matchings on an arbitrary contracting square-hexagon lattice with certain edge weights and boundary conditions. When multiple disconnected liquid regions in the limit shape occur, one of them turns out to be exactly the liquid region of the limit shape of dimer configurations on a contracting bipartite graph studied in this paper, while all the others are the same as liquid regions of the limit shape of dimer configurations on a contracting hexagon lattice (lozenge tilings).

The organization of the paper is as follows. In Section 2, we define the contracting bipartite graph and the dimer model, and review related known results about the dimer partition function and the Schur polynomial. In Section 3, we prove an integral formula for the deterministic limit shape of dimer models on the contracting bipartitie graphs, as well as the equations of the frozen boundary separating different phases in the limit shape. In Section 4, we prove the existence of multiple disconnected liquid regions for the limit shape of perfect matchings on any contracting square-hexagon lattices with certain edge weights, extending the results in [11].

2. Background

In this section, we define the contracting bipartite graph and the dimer model, and review related known results about the dimer partition function and the Schur polynomial.

2.1. **Contracting Bipartite Graphs.**

For a positive integer K, let

$$[K] := \{1, 2, \ldots, K\}.$$

Consider a doubly-infinite binary sequence indexed by integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

$$\check{a} = (\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots) \in \{0, 1\}^{\mathbb{Z}}. \tag{2.1}$$

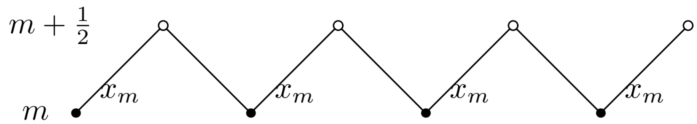
The *whole-plane square-hexagon lattice* $\mathrm{SH}(\check{a})$ associated with the sequence $\check{a}$ is defined as follows. The vertex set of $\mathrm{SH}(\check{a})$ is a subset of $\frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{2}$. Each vertex of $\mathrm{SH}(\check{a})$, represented by a point in the plane with coordinate in $\frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{2}$, is colored by either black or white. For $m \in \mathbb{Z}$, the black vertices have y-coordinate m; while the white vertices have y-coordinate $m - \frac{1}{2}$. We will label all the vertices with y-coordinate t $\left(t \in \frac{\mathbb{Z}}{2}\right)$ as vertices in the $(2t)$th row. We further require that for each $m \in \mathbb{Z}$,

- each black vertex on the $(2m)$th row is adjacent to two white vertices in the $(2m+1)$th row; and
- if $a_m = 1$, each white vertex on the $(2m-1)$th row is adjacent to exactly one black vertex in the $(2m)$th row; if $a_m = 0$, each white vertex on the $(2m-1)$th row is adjacent to two black vertices in the $(2m)$th row.

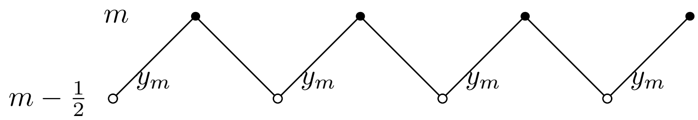
See Figure 2.1.

Note that in the graph $\mathrm{SH}(\check{a})$, either all the faces on a row are hexagons, or all the faces on a row are squares, depending on the corresponding entry of $\check{a}$.

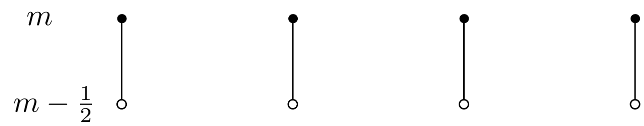
We assign edge weights to $\mathrm{SH}(\check{a})$ as follows.



(A) Structure of SH($\check{a}$) between the $(2m)$th row and the $(2m+1)$th row



(B) Structure of SH($\check{a}$) between the $(2m-1)$th row and the $(2m)$th row when $a_m = 0$



(C) Structure of SH($\check{a}$) between the $(2m-1)$th row and the $(2m)$th row when $a_m = 1$

FIGURE 2.1. Graph structures of the square-hexagon lattice on the $(2m-1)$th, $(2m)$th, and $(2m+1)$th rows depend on the values of (a_m). Black vertices are along the $(2m)$th row, while white vertices are along the $(2m-1)$th and $(2m+1)$th row.

Assumption 2.1. *(1) For $m \in \mathbb{Z}$, we assign weight $x_m \geq 0$ to each NE-SW edge joining the $(2m)$th row to the $(2m+1)$th row of* SH($\check{a}$)*. We assign weight $y_m > 0$ to each NE-SW edge joining the $(2m-1)$th row to the $(2m)$th row of* SH($\check{a}$)*, if such an edge exists. We assign weight 1 to all the other edges.*

(2) There exists a fixed positive integer n, such that for any $i, j \in \mathbb{Z}$ satisfying

$$[(i-j) \mod n] = 0,$$

we have

$$x_i = x_j; \qquad y_i = y_j$$

In other words, the edge weights are assigned periodically with period $1 \times n$.

(3) There exists $\gamma \in [0,1), J \subset [n]$, such that
- *(a) $\gamma n \in \{0, 1, 2, \ldots, n-1\}$; and*
- *(b) $|J| = \gamma n$; and*
- *(c) $x_j = 0$, for all $j \in J$;*
- *(d) $x_i = x > 0$, for all $i \in [n] \setminus J$.*

After removing all the edges with weight 0, we obtain a bipartite graph where each face has even number of edges. We use SH($\check{a}, X, Y, n$) to denote such a graph, where

$$X = (x_1, \ldots, x_N); \tag{2.2}$$

$$Y = (y_1, \ldots, y_N); \tag{2.3}$$

are edge weights.

A *contracting bipartite graph* is built from a whole-plane lattice as follows:

Definition 2.2. *Let $N \in \mathbb{N}$. Let $\Omega = (\Omega_1, \ldots, \Omega_N)$ be an N-tuple of positive integers, such that $1 = \Omega_1 < \Omega_2 < \cdots < \Omega_N$. Set $m = \Omega_N - N$. The contracting square-hexagon lattice $\mathcal{R}(\Omega, \check{a})$ is a subgraph of $\mathrm{SH}(\check{a})$ with $2N$ or $2N+1$ rows of vertices. We shall now enumerate the rows of $\mathcal{R}(\Omega, \check{a})$ inductively, starting from the bottom as follows:*

- *The first row consists of vertices (i, j) with $i = \Omega_1 - \frac{1}{2}, \ldots, \Omega_N - \frac{1}{2}$ and $j = \frac{1}{2}$. We call this row the boundary row of $\mathcal{R}(\Omega, \check{a})$.*
- *When $k = 2s$, for $s = 1, \ldots N$, the kth row consists of vertices (i, j) with $j = \frac{k}{2}$ and incident to at least one vertex in the $(2s-1)$th row of the whole-plane square-hexagon lattice $\mathrm{SH}(\check{a})$ lying between the leftmost vertex and rightmost vertex of the $(2s-1)$th row of $\mathcal{R}(\Omega, \check{a})$*
- *When $k = 2s+1$, for $s = 1, \ldots N$, the kth row consists of vertices (i, j) with $j = \frac{k}{2}$ and incident to two vertices in the $(2s)$th row of of $\mathcal{R}(\Omega, \check{a})$.*

Let $\mathcal{R}(\Omega, \check{a}, X, Y, n)$ be the corresponding weighted graph with edge weights satisfying Assumption 2.1. Again we remove all the weight-0 edges in $\mathcal{R}(\Omega, \check{a}, X, Y, n)$.

Definition 2.3. *Let I_1 (resp. I_2) be the set of indices j such that vertices of the $(2j-1)$th row are connected to one vertex (resp. two vertices) of the $(2j)$th row. In terms of the sequence $\check{a}$,*

$$I_1 = \{k \in [N] \mid a_k = 1\}, \quad I_2 = \{k \in [N] \mid a_k = 0\}.$$

The sets I_1 and I_2 form a partition of $[N]$, and we have $|I_1| = N - |I_2|$.

2.2. **Partitions, counting measure and Schur functions.**

Definition 2.4. *A* partition *of length N is a sequence of nonincreasing, nonnegative integers $\mu = (\mu_1 \geq \mu_2 \geq \ldots \geq \mu_N \geq 0)$. Each μ_k is a* component *of the partition μ. The length N of the partition μ is denoted by $l(\mu)$. The* size *of a partition μ is*

$$|\mu| = \sum_{i=1}^{N} \mu_i.$$

We denote by $\mathbb{GT}_N^+$ the subset of length-N partitions.

A graphic way to represent a partition μ is through its *Young diagram* Y_μ, a collection of $|\mu|$ boxes arranged on non-increasing rows aligned on the left: with μ_1 boxes on the first row, μ_2 boxes on the second row,... μ_N boxes on the Nth row. Some rows may be empty if the corresponding μ_k is equal to 0. The correspondence between partitions of length N and Young diagrams with N (possibly empty) rows is a bijection.

Definition 2.5. *We say that two partitions λ and μ* interlace*, or equivalently, their corresponding Young diagrams $Y_\lambda \subset Y_\mu$ differ by a horizontal strip, and write $\lambda \prec \mu$ if*

$$\lambda_i \geq \mu_i \geq \lambda_{i+1}, \ \forall i \in \mathbb{N},$$

where we assume $\lambda_i = 0$ for all $i \geq l(\lambda)$ and $\mu_j = 0$ or all $j \geq l(\mu)$. We say they co-interlace *and write $\lambda \prec' \mu$ if $\lambda' \prec \mu'$.*

Definition 2.6. *Let* $\lambda \in \mathbb{GT}_N^+$. *The* rational Schur function s_λ *associated to* λ *is the homogeneous symmetric function of degree* $|\lambda|$ *in* N *variables defined as follows*

(1) *If* $N = 1$, *and* $\lambda = (\lambda_1)$ *then*

$$s_\lambda(u_1) = u_1^{\lambda_1}.$$

(2) *If* $N \geq 2$, *and* $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0)$, *then*

$$s_\lambda(u_1, \ldots, u_N) = \frac{\det_{i,j=1,\ldots,N}(u_i^{\lambda_j+N-j})}{\prod_{1\leq i<j\leq N}(u_i - u_j)}.$$

(3) *Assume* $(0, \ldots, 0)$ *consists of* N *0's. Then*

$$s_{(0,\ldots,0)}(u_1, \ldots, u_N) = 1$$

It is straightforward to check that the Schur function defined above is a symmetric polynomial in the variables $(u_1, \ldots, u_N)$.

Let $\lambda \in \mathbb{GT}_N^+$ be a partition of length N. We define the *counting measure* $m(\lambda)$ corresponding to λ, which is a probability measure on $\left[0, 1 + \frac{\lambda_1}{N}\right)$, as follows.

$$m(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \delta\left(\frac{\lambda_i + N - i}{N}\right). \tag{2.4}$$

Let $\lambda(N) \in \mathbb{GT}_N^+$. Let Σ_N be the permutation group of N elements and let $\sigma \in \Sigma_N$. Assume that there exists a positive integer $n \in [N]$ such that $a_1, ..., a_n$ are pairwise distinct and $\{a_1, ..., a_n\} = \{x_1, ..., x_N\}$. For $j \in [N]$, let

$$\eta_j^\sigma(N) = |\{k \in [N] : k > j, x_{\sigma(k)} \neq x_{\sigma(j)}\}|. \tag{2.5}$$

For $i \in [s]$, let

$$\Phi^{(i,\sigma)}(N) = \{\lambda_j(N) + \eta_j^\sigma(N) : x_{\sigma(j)} = a_i, j \in [N]\} \tag{2.6}$$

and let $\phi^{(i,\sigma)}(N)$ be the partition with length $|\{j \in [N] : x_j = a_i\}|$ obtained by decreasingly ordering all the elements in $\Phi^{(i,\sigma)}(N)$. Let Σ_N^X be the subgroup Σ_N that preserves the value of X; more precisely

$$\Sigma_N^X = \{\sigma \in \Sigma_N : x_{\sigma(i)} = x_i, \text{ for } i \in [N]\}.$$

Let $[\Sigma/\Sigma_N^X]^r$ be the collection of all the right cosets of Σ_N^X in Σ_N. More precisely,

$$[\Sigma/\Sigma_N^X]^r = \{\Sigma_N^X \sigma : \sigma \in \Sigma_N\}, \tag{2.7}$$

where for each $\sigma \in \Sigma_N$

$$\Sigma_N^X \sigma = \{\xi\sigma : \xi \in \Sigma_N^X\}$$

and $\xi\sigma \in \Sigma_N$ is defined by

$$\xi\sigma(k) = \xi(\sigma(k)), \text{ for } k \in [N].$$

The following combinatorial formula, which relates the value of a Schur function at a general point to the values of Schur functions at $(1, \ldots, 1)$, was proved in [11].

Proposition 2.7. *Under the assumptions above, the Schur function can be computed by the following formula*

$$s_\lambda(x_1,\ldots,x_N) = \sum_{\overline{\sigma}\in[\Sigma_N/\Sigma_N^X]^r}\left(\prod_{i=1}^{n} s_{\phi^{(i,\sigma)}(N)}(a_i,\ldots,a_i)\right) \times\left(\prod_{i<j,x_{\sigma(i)}\neq x_{\sigma(j)}}\frac{1}{x_{\sigma(i)}-x_{\sigma(j)}}\right) \tag{2.8}$$

where $\sigma\in\overline{\sigma}\cap\Sigma_N$ is a representative.

Proof. See Theorem 2.4 of [11]. □

2.3. **Dimer model.**

Definition 2.8. *A dimer configuration, or a perfect matching M of a contracting bipartite lattice $\mathcal{R}(\Omega,\check{a},X,Y,n)$ is a set of edges $((i_1,j_1),(i_2,j_2))$, such that each vertex of $\mathcal{R}(\Omega,\check{a},X,Y,n)$ belongs to an unique edge in M. The set of perfect matchings of $\mathcal{R}(\Omega,\check{a},X,Y,n)$ is denoted by $\mathcal{M}(\Omega,\check{a},X,Y,n)$.*

Definition 2.9. *The partition function of the dimer model of a finite graph G with edge weights $(w_e)_{e\in E(G)}$ is given by*

$$Z=\sum_{M\in\mathcal{M}}\prod_{e\in M}w_e,$$

where $\mathcal{M}$ is the set of all perfect matchings of G. The Boltzmann dimer probability measure on M induced by the weights w is thus defined by declaring that probability of a perfect matching is equal to

$$\frac{1}{Z}\prod_{e\in M}w_e.$$

Note that the contracting bipartite lattice $\mathcal{R}(\Omega,\check{a},X,Y,n)$ has degree-2 vertices. One way to study dimer model on a graph G with degree-2 vertices is as follows. Let v be an arbitrary degree-2 vertex, and let u and w be the two neighboring vertices of v. Remove the vertex v and its two incident edges (u,v) and (v,w), then identify the two vertices u and w to obtain a new graph $\hat{G}$. It is straight forward to check that perfect matchings on G and $\hat{G}$ are in 1-1 correspondence. If both (u,v) and (v,w) has weight 1, then the partition function for dimer configurations on G and $\hat{G}$ are equal. For the contracting bipartite lattice $\mathcal{R}(\Omega,\check{a},X,Y,n)$, we may do the same thing of removing vertices, edges and identifying vertices as above, however, this will change the scaling limit of the graph and therefore obtain a different limit shape. In this paper, we shall always study perfect matchings on the graph $\mathcal{R}(\Omega,\check{a},X,Y,n)$ without the above manipulations and use the limit shape result to prove existence of multiple disconnected liquid regions on any contracting square-hexagon lattice with certain edge weights.

Definition 2.10. *Let M be a perfect matching of the contracting bipartite graph $\mathcal{R}(\Omega,\check{a},X,Y,n)$ before removing weight 0-edges. We call a present edge $e=((i_1,j_1),(i_2,j_2))$ in M a V-edge if $\max\{j_1,j_2\}\in\mathbb{N}$ (i.e. if its higher extremity is black) and we call it a Λ-edge otherwise. In other words, the edges going upwards starting from an odd row are V-edges and those* *ones starting from an even row are Λ-edges. We also call the corresponding vertices-(i_1, j_1) and (i_2, j_2) of a V-edge (resp. Λ-edge) V-vertices (resp. Λ-vertices).*

There is a bijection between dimer configurations on the contracting bipartite graph $\mathcal{R}(\Omega, \check{a}, X, Y, n)$, before removing the weight-0 edges, and sequences of interlaces partitions. More precisely:

Lemma 2.11 ([3] Theorem 2.10, [2])**.** *For given $\Omega = (\Omega_1, \ldots, \Omega_N)$, $\check{a}$, let $\omega \in \mathbb{GT}_N^+$ be the partition associated to Ω given by*

$$\omega = (\Omega_N - N, \Omega_{N-1} - N + 1, \ldots, \Omega_1 - 1) \tag{2.9}$$

- *For $0 \le i \le (N-1)$ (resp. $1 \le j \le N$), let $\mu^{(i)} \in \mathbb{GT}_i^+$ (respectively. $\mu^{(j)}$) be the partition associated to the $(2i+1)$th row (resp. $2j$th row) of vertices counting from the top such that for $1 \le k \le i$, $\mu_k^{(i)}$ is the number of Λ-vertices on the left of the kth V-vertices, where the V-vertices are counting from the right.*

Then we obtain a bijection between the set of perfect matchings $\mathcal{M}(\Omega, \check{a})$ and the set $S(\omega, \check{a})$ of sequences of non-negative signatures

$$\{(\mu^{(N)}, \nu^{(N)}, \ldots, \mu^{(1)}, \nu^{(1)}, \mu^{(0)}\}$$

where the signatures satisfy the following properties:

- *All the parts of $\mu^{(0)}$ are equal to 0; and*
- *The signature $\mu^{(N)}$ is equal to ω; and*
- *The signatures satisfy the following (co)interlacement relations:*

$$\mu^{(N)} \prec' \nu^{(N)} \succ \mu^{(N-1)} \prec' \cdots \mu^{(1)} \prec' \nu^{(1)} \succ \mu^{(0)}.$$

Moreover, if $a_m = 1$, then $\mu^{(N+1-k)} = \nu^{(N+1-k)}$.

Proposition 2.12. *(Proposition 2.15 of [3])Let $\mathcal{R}(\Omega, \check{a}, X, Y, n)$ be a contracting bipartite graph, whose edge weights satisfy Assumption 2.1. Then the partition function for perfect matchings on $\mathcal{R}(\Omega, \check{a}, X, Y, n)$ is given by*

$$Z = \left[\prod_{i \in I_2} \Gamma_i\right] s_\omega(x_1, \ldots, x_N)$$

where $\omega \in \mathbb{GT}_N^+$ is the partition describing the bottom boundary condition of $\mathcal{R}(\Omega, \check{a}, X, Y, n)$ given by (2.9). Moreover, for $i \in I_2$, Γ_i is defined by

$$\Gamma_i = \prod_{t=i+1}^{N} (1 + y_i x_t)\,. \tag{2.10}$$

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3. Limit Shape

In this section, we prove an integral formula for the deterministic limit shape of dimer models on the contracting bipartitie graphs, as well as the equations of the frozen boundary separating different phases in the limit shape. In Section 3.1, we prove formulas to compute Schur polynomials with some variables equal to 0, which is related to the partition function of dimer configurations on a contracting bipartitie graph, obtained from a contracting square-hexagon lattice by assigning some edge weights to be 0. In Section 3.2, we prove an explicit integral formula for the moments of the limit measure of dimer configurations on each horizontal level of the domain covered by the contracting bipartite graph; see Theorem 3.11. In Section 3.3, we obtain the equation for the boundary curve separating different phases in the limit shape, and show that the boundary curve is an algebraic curve of a special type; more precisely (Theorem 3.14), it is a cloud curve characterized by the numbers of intersections of its dual curve with arbitrary straight lines in $\mathbb{RP}^2$ (Proposition 3.16).

3.1. Schur polynomials with vanishing variables.

Lemma 3.1. *Assume that* $x_1, \ldots, x_N \in \mathbb{C}$ *such that*

$$|\{i \in [N] : x_i = 0\}| = b \in [N] \tag{3.1}$$

Let $\lambda \in \mathbb{GT}_N^+$ *such that there are exactly* $a \in [N]$ *components of* λ *taking value 0. If* $a < b$ *Then*

$$s_\lambda(x_1, \ldots, x_N) = 0. \tag{3.2}$$

Proof. The proof is based on the combinatorial interpretation of the Schur polynomial $s_\lambda(x_1, \ldots, x_N)$ over sums of semi-standard Young tableaux (SSYT) of shape λ. Let Y_λ be a Young diagram of shape λ, drawn in the plane such that there are λ_1 squares on the top row, λ_2 squares on the 2nd top row, ..., and λ_N squares on the N-th top row (bottom row). The squares in Y_λ are indexed by (i, j) with i denoting the row number starting from the top and j denoting the column number starting from the left. We may consider the Young diagram as a set consisting of all the squares indexed by such (i, j)'s, i.e.,

$$Y_\lambda := \{(i, j) \in \mathbb{N}^2 : i \in [N], \lambda_i > 0, j \in [\lambda_i]\}.$$

A SSYT of shape λ is a map $T : Y_\lambda \to \mathbb{N}$, which assigns a unique positive integer to each square in Y_λ, such that

$$T(i, j) \leq T(i, j + 1); \qquad \text{and } T(i, j) < T(i + 1, j).$$

For a Young Tableaux T, let $sh(T)$ be the Young diagram (with 0 components removed) denoting the shape of T. Then

$$s_\lambda(x_1, \ldots, x_N) = \sum_{T:sh(T)=\lambda} \prod_{(i,j)\in Y_\lambda} x_{T(i,j)} \tag{3.3}$$

where we assume $x_j = 0$ for all $j > N$. When λ has exactly a components taking value 0, λ has exactly $N - a$ components which are strictly positive; and there are exactly $N - b$ numbers in $(x_1, x_2, \ldots, x_N)$ which are nonzero, denoted by $x_{j_1}, x_{j_2}, \ldots, x_{j_{N-b}}$ such that

$$j_1 < j_2 < \ldots < j_{N-b}.$$

When $a < b$, we have $N - a > N - b$. In the first column of the Young diagram with shape λ, there are exactly $N - a$ squares. The integers $1, 2, \ldots, N - b$ cannot fill the $N - a$ squares, hence in the strictly increasing sequence of integers in the first column of Y_λ, there must exist an integer j such that $x_j = 0$. This means that in the right hand side of (3.3), each summand is 0. Hence we obtain (3.2). □

Proposition 3.2. *Let $k \in [N]$ and*

$$w_i := \begin{cases} u_i & \text{if } 1 \leq i \leq k \\ x_i & \text{if } k+1 \leq i \leq N. \end{cases}$$

Assume that $(x_1, \ldots, x_N)$ takes n distinct values $a_1, \ldots, a_n$. For $j \in [n]$, let

$$I_j = \{i \in [N] : x_i = a_j\};$$

i.e., I_j consisting of all the indices $i \in [N]$ such that $x_i = a_j$. Let

$$U_j = \{u_l\}_{l \in [k] \cap I_j}$$

Then we have the following formula

$$\begin{aligned} &s_\lambda(w_1, \ldots, w_N) \\ &= \sum_{\overline{\sigma} \in [\Sigma_N / \Sigma_N^X]^r} \left(\prod_{j=1}^{n} s_{\phi^{(j,\sigma)}(N)}(U_j, a_j, \ldots, a_j) \right) \left(\prod_{i<j, x_{\sigma(i)} \neq x_{\sigma(j)}} \frac{1}{w_{\sigma(i)} - w_{\sigma(j)}} \right) \end{aligned} \tag{3.4}$$

where $\sigma \in \overline{\sigma} \cap \Sigma_N$ is a representative and

$$(U_j, a_j, \ldots, a_j) \in \mathbb{C}^{|I_j|}.$$

In particular, if $[k] \cap I_j = \emptyset$, then $U_j = \emptyset$, in this case by convention

$$(U_j, a_j, \ldots, a_j) = (a_j, \ldots, a_j) \in \mathbb{C}^{|I_j|}$$

Proof. When the $|I_j| = \frac{N}{n}$ for all $j \in [n]$, the formula (3.4) was proved in Proposition 3.4 of [11]. When $\{|I_j|\}_{j \in [n]}$ are not necessarily all equal, the proof follows from exactly the same arguments as the proof of Proposition 3.4 in [11]. □

Lemma 3.3. *Let n, N be positive integers such that N is a constant multiple of n. Let $b \in [N]$. Assume that*

$$(u_1, \ldots, u_{N-b}, 0^b) \in \mathbb{C}^N$$

where

$$0^b = (0, \ldots, 0) \in \mathbb{R}^b$$

Let $\lambda \in \mathbb{GT}_N^+$ such that there are exactly $a \in [N]$ components of λ taking value 0. If $a \geq b$, let

$$\tilde{\lambda} : = (\lambda_1, \lambda_2, \ldots, \lambda_{N-b}) \in \mathbb{GT}_{N-b} \tag{3.5}$$

Then

$$s_\lambda(u_1, \ldots, u_{N-b}, 0^b) = s_{\tilde{\lambda}}(u_1, \ldots, u_{N-b}).$$

Proof. Assume $(x_1, \ldots, x_N)$ satisfies Assumption 2.1 (2)(3) with $\gamma N = b$. Then there are exactly two distinct values in $(x_1, \ldots, x_N)$. Let

$$w_i = \begin{cases} u_i & \text{if } i \in [N-b] \\ 0 & \text{if } i \in \{N-b+1, \ldots, N\} \end{cases}.$$

By Proposition 3.2, we obtain that $s_\lambda(u_1, \ldots, u_{N-b}, 0, \ldots, 0)$ can be expressed as a sum over partitions of $[N]$ into two sets A_0 and A_1 such that

(1) $A_0 \cap A_1 = \emptyset$, $A_0 \cup A_1 = [N]$, $|A_0| = b$.

Indeed, each partition (A_0, A_1) satisfying (1) corresponds to a right coset $\overline{\sigma} \in [\Sigma_N/\Sigma_N^X]^r$, such that for each $\sigma \in \Sigma_N$, if $\sigma \in \overline{\sigma}$, then

- $x_{\sigma(i)} = 0$ if and only if $i \in A_0$.
- $x_{\sigma(i)} = x$ if and only if $i \in A_1$.

Moreover, the correspondence between partitions (A_0, A_1) satisfying (1) and right cosets in $[\Sigma_N/\Sigma_N^X]^r$ is one-to-one.

We claim that the only summand that actually contributes to the sum on the right hand side of (3.4) is given by

$$A_1 = \{1, 2, \dots, N-b\}, \text{ and } A_0 = \{N-b+1, N-b+2, \dots, N\}.$$

To see why that is true, note that if A_0 is not the set consisting of the largest b numbers of $[N]$, then the summand on the right hand side of (3.4) corresponding to the partition (A_0, A_1) has a factor $s_\psi(0, \dots, 0)$, such that $|l(\psi)| = b$, $(0, 0, \dots, 0)$ consists of b 0's, and there exists at least one strictly positive component in ψ. By Lemma 3.1, $s_\psi(0, \dots, 0) = 0$. Then the claim follows.

Let

$$\phi : \;=\; (\lambda_1 + b, \lambda_2 + b, \dots, \lambda_{N-b} + b) \in \mathbb{GT}_{N-b}. \tag{3.6}$$

By Theorem 3.2, we obtain that under Assumption 2.1 (2)(3) and (3.1),

$$\begin{aligned} s_\lambda(u_1, \dots, u_{N-b}, 0, \dots, 0) &= \frac{s_\phi(u_1, \dots, u_{N-b}) s_{(0,\dots,0)}(0, \dots, 0)}{\left(\prod_{j=1}^{N-b}(u_j)\right)^b} \\ &= \frac{s_\phi(u_1, \dots, u_{N-b})}{\left(\prod_{j=1}^{N-b}(u_j)\right)^b} \\ &= s_{\tilde{\lambda}}(u_1, \dots, u_{N-b}) \end{aligned}$$

□

Assume $\psi \in \mathbb{GT}_m^+$, where m is a positive integer. Recall that $s_\psi(1, \dots, 1)$ can be computed by the Weyl character formula as follows

$$s_\psi(1, \dots, 1) = \prod_{1 \le i < j \le m} \frac{\psi_i - \psi_j + j - i}{j - i}. \tag{3.7}$$

Lemma 3.4. *Let n, N be positive integers such that N is a constant multiple of n. Assume that $x_1, \dots, x_N \in \mathbb{R}$ such that (3.1) and Assumption 2.1 (2)(3) hold with $b = \gamma N$. Let $\lambda \in \mathbb{GT}_N^+$ such that there are exactly $a \in [N]$ components of λ taking value 0. If $a \ge b$, let $\tilde{\lambda}, \phi \in \mathbb{GT}_{N-b}$ be defined as in (3.5) and (3.6), respectively. Then*

$$s_\lambda(x_1, \dots, x_N) = x^{|\lambda|} s_\phi(1, \dots, 1) = s_{\tilde{\lambda}}(x, \dots, x)$$

Proof. The identity that $s_\lambda(x_1, \dots, x_N) = s_{\tilde{\lambda}}(x, \dots, x)$ follows from Lemma 3.3 by letting $u_1 = u_2 = \dots = u_{N-b} = x$. Under the assumption that $a \ge b$, we obtain that

$$\lambda = (\tilde{\lambda}, 0, \dots, 0) \in \mathbb{GT}_N$$

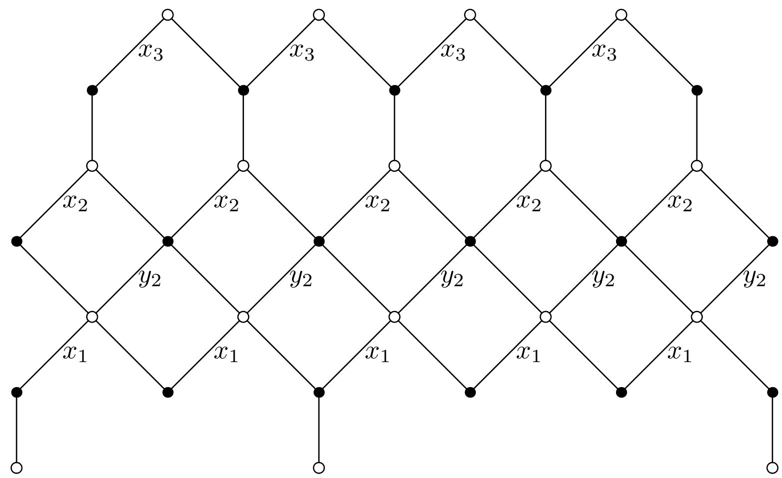

FIGURE 3.1. Contracting square-hexagon lattice with $N = 3$, $m = 3$, $\Omega = (1, 3, 6), (a_1, a_2, a_3) = (1, 0, 1)$.

therefore $|\lambda| = |\tilde{\lambda}|$. Therefore,

$$s_{\tilde{\lambda}}(x, \ldots, x) = x^{|\tilde{\lambda}|} s_{\tilde{\lambda}}(1, \ldots, 1) = x^{|\lambda|} s_{\tilde{\lambda}}(1, \ldots, 1).$$

By the Weyl formula (3.7), we obtain that

$$s_\phi(1, \ldots, 1) = s_{\tilde{\lambda}}(1, \ldots, 1).$$

Then the lemma follows. □

Example 3.5. *Figure 3.1 shows a contracting square-hexagon lattice with* $(a_1, a_2, a_3) = (1, 0, 1)$ *and* $\Omega = (1, 3, 6)$. *Then the boundary partition is given by*

$$\omega = (\Omega_3 - 3, \Omega_2 - 2, \Omega_1 - 1) = (3, 1, 0) \in \mathbb{GT}_3$$

By Proposition 2.12, the partition function of the dimer configurations on the graph as given in Figure 3.1 is

$$\begin{aligned} & (1 + y_2 x_3) s_{(3,1,0)}(x_1, x_2, x_3) \\ = \; & x_1^3 x_2 + x_1^3 x_3 + x_1^2 x_2^2 + 2x_1^2 x_2 x_3 + x_1^2 x_3^2 + x_1 x_2^3 + 2x_1 x_2^2 x_3 + 2x_1 x_2 x_3^2 + x_1 x_3^3 + x_2^3 x_3 + x_2^2 x_3^2 + x_2 x_3^3 \end{aligned}$$

When $x_2 = 0$, *and* $x_1 = x_3 = x > 0$, *let* $b = 1$, *and then*

$$\phi = (\lambda_1 + b, \lambda_2 + b) = (4, 2).$$

we obtain the partition function of dimer configurations on the graph as given in Figure 3.2 is

$$\begin{aligned} & (1 + y_2 x_3)(x_1^3 x_3 + x_1^2 x_3^2 + x_1 x_3^3) \\ = \; & 3(1 + y_2 x) x^4 = (1 + y_2 x) x^{|\lambda|} s_\phi(1, 1), \end{aligned}$$

where the last identity follows from (3.7).

When $x_2 = x_3 = 0$ *and* $x_1 = x > 0$, *then the partition function of dimer configurations on the graph as given in Figure 3.3 is 0. Indeed, Figure 3.3 does not admit a perfect matching.*

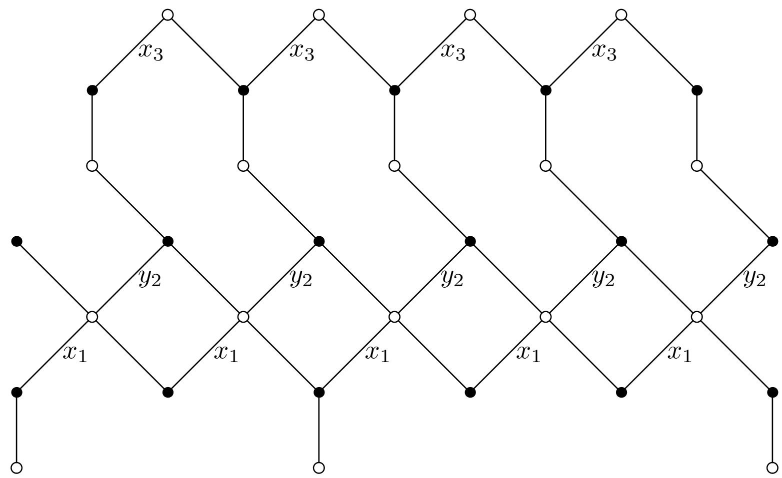

FIGURE 3.2. Contracting bipartite lattice with $N = 3$, $m = 3$, $\Omega = (1, 3, 6), (a_1, a_2, a_3) = (1, 0, 1)$, $x_2 = 0$.

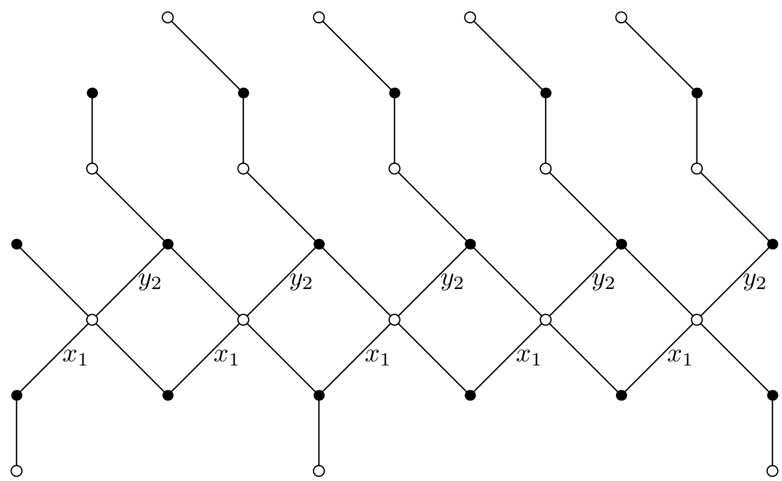

FIGURE 3.3. Contracting bipartite lattice with $N = 3$, $m = 3$, $\Omega = (1, 3, 6), (a_1, a_2, a_3) = (1, 0, 1)$, $x_2 = x_3 = 0$.

3.2. **Limit counting measure.**

Definition 3.6 ([7])**.** *A sequence of signatures $\lambda(N) \in \mathbb{GT}_N$ is called* regular*, if there exists a piecewise continuous function $f(t)$ and a constant $C > 0$ such that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \left| \frac{\lambda_j(N)}{N} - f\left(\frac{j}{N}\right) \right| = 0, \quad \text{and} \quad \sup_{1\leq j\leq N} \left| \frac{\lambda_j(N)}{N} - f\left(\frac{j}{N}\right) \right| < C \text{ for all } N \geq 1.$$

If $(\lambda(N))$ is a regular sequence of signatures, then the sequence of counting measures $m(\lambda(N))$ converges weakly to a measure $\mathbf{m}$ with compact support. By Theorem 3.6 of [5] that there exists an explicit function $H_{\mathbf{m}}$, analytic in a neighborhood of 1, depending on the weak limit $\mathbf{m}$ such that

$$\lim_{N\to\infty}\frac{1}{N}\log\left(\frac{s_{\lambda(N)}(u_1,\dots,u_k,1,\dots,1)}{s_{\lambda(N)}(1,\dots,1)}\right)=H_{\mathbf{m}}(u_1)+\dots+H_{\mathbf{m}}(u_k), \tag{3.8}$$

and the convergence is uniform when $(u_1,\dots,u_k)$ is in a neighborhood of $(1,\dots,1)$. Precisely, $H_{\mathbf{m}}$ is constructed as follows: let $S_{\mathbf{m}}(z)=z+\sum_{k=1}^{\infty}M_k(\mathbf{m})z^{k+1}$ be the moment generating function of the measure $\mathbf{m}$, where $M_k(\mathbf{m})=\int x^k d\mathbf{m}(x)$, and $S_{\mathbf{m}}^{(-1)}$ be the inverse function of $S_{\mathbf{m}}$. Let $R_{\mathbf{m}}(z)$ be the *Voiculescu R-transform* of $\mathbf{m}$ defined as

$$R_{\mathbf{m}}(z)=\frac{1}{S_{\mathbf{m}}^{(-1)}(z)}-\frac{1}{z}.$$

Then

$$H_{\mathbf{m}}(u)=\int_0^{\ln u}R_{\mathbf{m}}(t)dt+\ln\left(\frac{\ln u}{u-1}\right). \tag{3.9}$$

In particular, $H_{\mathbf{m}}(1)=0$, and

$$H'_{\mathbf{m}}(u)=\frac{1}{uS_{\mathbf{m}}^{(-1)}(\ln u)}-\frac{1}{u-1}.$$

Definition 3.7. *Let X be defined as in (2.2). Let ρ be a probability measure on $\mathbb{GT}_N^+$. The* Schur generating function $\mathcal{S}_{\rho,\mathbf{X}}(u_1,\dots,u_N)$ *with respect to parameters X is the symmetric Laurent series in $(u_1,\dots,u_N)$ given by*

$$\mathcal{S}_{\rho,\mathbf{X}}(u_1,\dots,u_N)=\sum_{\lambda\in\mathbb{GT}_N}\rho(\lambda)\frac{s_\lambda(u_1,\dots,u_N)}{s_\lambda(X)},$$

Assumption 3.8. *Let $\mathcal{R}(\Omega,\check{a},X,Y,n)$ be a contracting bipartite graph with edge weights satisfying Assumption 2.1. Assume the bottom boundary partition $\omega^{(N)}=(\omega_1^{(N)},\omega_2^{(N)},\dots,\omega_N^{(N)})\in\mathbb{GT}_N^+$ satisfies*

- *there exists $\alpha\in[\gamma,1)$ and $\alpha N\in[N]$, such that*
$$\omega_{(1-\alpha)N+1}^{(N)}=\omega_{(1-\alpha)N+2}^{(N)}=\ldots=\omega_N^{(N)}=0.$$
- *Let let*
$$\tilde{\omega}^{(N)}=(\omega_1^{(N)},\omega_2^{(N)},\dots,\omega_{(1-\gamma)N}^{(N)}),$$
then $\{\tilde{\omega}^{(N)}\}_{N\in\mathbb{N}}$ form a regular sequence of partitions with counting measures converging to $\tilde{\mathbf{m}}$ as $N\to\infty$.

Lemma 3.9. *(Lemma 3.6 of [3]) Assume the edge weights of $\mathcal{R}(\Omega,\check{a},X,Y,n)$ satisfy Assumption 2.1 (1)(2). For any k between 0 and $2N-1$, define $t=\lfloor k/2\rfloor$, and let*

$$X^{(N-t)}=(x_{\overline{t+1}},\dots,x_{\bar{N}}),\quad \textit{and}\quad Y^{(t)}=(x_{\bar{1}},\dots,x_{\bar{t}}).$$

and ρ^k be the probability measure of the partitions corresponding to dimer configurations on the $(2N-k)$th row, counting from the top. Then the generating Schur function $\mathcal{S}_{\rho^k,X^{(N-t)}}$ is given by:

$$\mathcal{S}_{\rho^k,X^{(N-t)}}(u_1,\dots,u_{N-t})=\frac{s_\omega\left(u_1,\dots,u_{N-t},Y^{(t)}\right)}{s_\omega(X^{(N)})}\prod_{i\in\{1,\dots,t\}\cap I_2}\prod_{j=1}^{N-t}\left(\frac{1+y_{\bar{i}}u_j}{1+y_{\bar{i}}x_{\overline{t+j}}}\right).$$

where ω is given by (2.9).

When the edge weights are assigned periodically as in Assumption 2.1 (1)(2)(3), and the boundary partition ω satisfies Assumption 3.8, let

$$\tilde{\omega} = (\omega_1, \omega_2, \ldots, \omega_{(1-\gamma)N})$$

For an integer k between 0 and $2N-1$, the following cases might occur

(1) If $k = 2t-1$ for some $l \in [N]$, then the partition corresponding to the dimer configuration on the $(2N-2t+1)$th row satisfies $\mu^{(N-t)} \in \mathbb{GT}^+_{N-t}$ and

$$\mu^{(N-t)}_{\lfloor(1-\gamma)(N-t)\rfloor+1} = \mu^{(N-t)}_{\lfloor(1-\gamma)(N-t)\rfloor+2} = \ldots = \mu^{(N-t)}_{N-t} = 0.$$

Let

$$\tilde{\mu}^{(N-t)} := \left(\mu^{(N-t)}_1, \mu^{(N-l)}_2, \ldots, \mu^{(N-t)}_{\lfloor(1-\gamma)(N-t)\rfloor}\right) \in \mathbb{GT}^+_{\lfloor(1-\gamma)(N-t)\rfloor}.$$

(2) If $k = 2t-2$ for some $t \in [N]$, then the partition corresponding to the dimer configuration on the $(2N-2t+2)$th row satisfies $\nu^{(N-t)} \in \mathbb{GT}^+_{N-t}$ and

$$\nu^{(N-t)}_{\lfloor(1-\gamma)(N-t)\rfloor+1} = \nu^{(N-t)}_{\lfloor(1-\gamma)(N-t)\rfloor+2} = \ldots = \nu^{(N-t)}_{N-t} = 0.$$

Let

$$\tilde{\nu}^{(N-t)} := \left(\nu^{(N-t)}_1, \nu^{(N-t)}_2, \ldots, \nu^{(N-t)}_{\lfloor(1-\gamma)(N-t)\rfloor}\right) \in \mathbb{GT}^+_{\lfloor(1-\gamma)(N-t)\rfloor}.$$

Let $\tilde{\rho}^k$ (resp. ρ^k) be the probability measure on $\tilde{\mu}^{(N-t)}$ or $\tilde{\nu}^{(N-t)}$ (resp. $\mu^{(N-t)}$ or $\nu^{(N-t)}$).
Assume

$$u_{(1-\gamma)(N-t)+1} = u_{(1-\gamma)(N-t)+2} = \ldots = u_{N-t} = 0.$$

By Lemmas 3.4 and 3.9 we obtain

$$\begin{aligned}
&\mathcal{S}_{\rho^k, X^{(N-t)}}(u_1, \ldots, u_{(1-\gamma)(N-t)}, 0, \ldots, 0) \\
= \quad &\frac{s_{\tilde{\omega}}\left(u_1, \ldots, u_{(1-\gamma)(N-t)}, x, \ldots, x\right)}{s_{\tilde{\omega}}(x, \ldots, x)} \prod_{i \in \{1, \ldots, t\} \cap I_2} \prod_{j=1}^{(1-\gamma)(N-t)} \left(\frac{1 + y_{\bar{i}} u_j}{1 + y_{\bar{i}} x}\right).
\end{aligned}$$

Let $\tilde{X}^{(N-t)}$ consist of all the components of $X^{(N-t)}$ which are nonzero. Let

$$1^{\lfloor(1-\gamma)(N-t)\rfloor} = (1, \ldots, 1) \in \mathbb{R}^{\lfloor(1-\gamma)(N-t)\rfloor}$$

Then

$$\begin{aligned}(3.10)\qquad & \mathcal{S}_{\tilde{\rho}^k,1^{\lfloor(1-\gamma)(N-t)\rfloor}}\left(u_1,\ldots,u_{\lfloor(1-\gamma)(N-t)\rfloor}\right)\\ =\ & \sum_{\tilde{\lambda}\in\mathbb{GT}^+_{\lfloor(N-t)(1-\gamma)\rfloor}} \tilde{\rho}^k(\tilde{\lambda})\frac{s_{\tilde{\lambda}}\left(u_1,\ldots,u_{\lfloor(1-\gamma)(N-t)\rfloor}\right)}{s_{\tilde{\lambda}}(1^{\lfloor(1-\gamma)(N-t)\rfloor})}\\ =\ & \sum_{\tilde{\lambda}\in\mathbb{GT}^+_{\lfloor(N-t)(1-\gamma)\rfloor}} \tilde{\rho}^k(\tilde{\lambda})\frac{s_{\tilde{\lambda}}\left(u_1x,\ldots,u_{\lfloor(1-\gamma)(N-t)\rfloor}x\right)}{s_{\tilde{\lambda}}(\tilde{X}^{(N-t)})}\\ =\ & \sum_{\lambda\in\mathbb{GT}^+_{N-t}} \rho^k(\lambda)\frac{s_{\lambda}\left(u_1x,\ldots,u_{\lfloor(1-\gamma)(N-t)\rfloor}x,0,\ldots,0\right)}{s_{\lambda}(X^{(N-t)})}\\ =\ & \mathcal{S}_{\rho^k,X^{(N-t)}}(u_1x,\ldots,u_{(1-\gamma)(N-t)}x,0,\ldots,0)\end{aligned}$$

In particular we have $\rho^k(\lambda)=0$ if $\lambda\in\mathbb{GT}^+_{N-t}$ is not an extension of $\tilde{\lambda}\in\mathbb{GT}^+_{\lfloor(1-\gamma)(N-t)\rfloor}$ by 0's, according to Lemma 3.1. The 2nd identity also follows from Lemma 3.3.

Letting $N\to\infty$, $\frac{t}{N}\to\kappa\in[0,1)$, by (3.8) we have

$$\begin{aligned}\lim_{(1-\kappa)N\to\infty}\frac{1}{(1-\kappa)N}\log\mathcal{S}_{\rho^k,X^{(N-t)}}(u_1x,\ldots,u_lx,x,\ldots,x,0,\ldots,0)\\ =\frac{1-\gamma}{1-\kappa}\sum_{1\le i\le l}\left[Q_\kappa(u_i)-Q_\kappa(1)\right],\end{aligned}$$

where

$$Q_\kappa(u)=H_{\tilde{\mathbf{m}}}(u)+\frac{\kappa}{n(1-\gamma)}\sum_{i\in[n]\cap I_2}\log(1+y_ixu).$$

By (3.10), we obtain

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{(1-\gamma)(1-\kappa)N}\log\mathcal{S}_{\tilde{\rho}^k,1^{\lfloor(1-\gamma)(N-t)\rfloor}}(u_1,\ldots,u_l,1,\ldots,1)\\ =\frac{1}{1-\kappa}\sum_{1\le i\le l}\left[Q_\kappa(u_i)-Q_\kappa(x)\right],\end{aligned}$$

Lemma 3.10 ([4], Theorem 5.1). *Let $(\rho_N)_{N\ge1}$ be a sequence of measures such that for each N, ρ_N is a probability measure on $\mathbb{GT}^+_N$, and for every j, the following convergence holds uniformly in a complex neighborhood of $(1,\ldots,1)\in\mathbb{C}^j$*

$$(3.11)\qquad \lim_{N\to\infty}\frac{1}{N}\log\mathcal{S}_{\rho_N,1^N}(u_1,\ldots,u_j,1,\ldots,1)=Q(u_1)+\cdots+Q(u_j),$$

with Q an analytic function in a neighborhood of 1. Then the sequence of random measures $(m(\rho_N))_{N\ge1}$ converges as $N\to\infty$ in probability in the sense of moments to a deterministic measure $\mathbf{m}$ on $\mathbb{R}$, whose moments are given by

$$\int_{\mathbb{R}}x^j\mathbf{m}(dx)\ =\ \sum_{l=0}^{j}\frac{j!}{l!(l+1)!(j-l)!}\frac{\partial^l}{\partial u^l}\left(u^jQ'(u)^{j-l}\right)\Big|_{u=1}.$$

By Lemma 3.10, we obtain the following theorem about limit shape of perfect matchings on the contracting bipartite graph $\mathcal{R}(\Omega, \check{a}, X, Y, n)$.

Theorem 3.11. *Let $k \in \{0, 1, \ldots, 2N-1\}$, such that $\lim_{N\to\infty} \frac{k}{2N} = \kappa$. Assume the edge weights and boundary partition of the contraction bipartite graph $\mathcal{R}(\Omega, \check{a}, X, Y, n)$ satisfy Assumptions 2.1 and 3.8. Then the sequence of random measures $(m(\tilde{\rho}^k))_{N\in\mathbb{N}}$ converges as $N \to \infty$ in probability in the sense of moments to a deterministic measure $\tilde{\mathbf{m}}^\kappa$ on $\mathbb{R}$, whose moments are given by*

$$\int_{\mathbb{R}} x^j \tilde{\mathbf{m}}^\kappa(dx) = \frac{1}{2(j+1)\pi\mathbf{i}} \oint_1 \frac{dz}{z} \left[F_\kappa(z)\right]^{j+1}$$

where

$$F_\kappa(z) = \frac{z}{1-\kappa} H'_{\tilde{\mathbf{m}}}(z) + \frac{\kappa z}{n(1-\kappa)(1-\gamma)} \sum_{i\in[n]\cap I_2} \frac{y_i x}{1+y_i x z} + \frac{z}{z-1}$$

and the integration goes over a small positively oriented contour around 1.

3.3. **Frozen boundary.** The frozen region is defined to be the region where the density of the limit of the counting measures for ρ^k is 0 or 1, while the liquid region is the region where this density is strictly between 0 or 1. The frontier between the the frozen region and the liquid region is called the frozen boundary.

We consider a special case of bottom boundary conditions.

Assumption 3.12. *(1) Let*

$$\Omega = (A_1, A_1+1, \ldots, B_1-1, B_1, A_2, A_2+1, \ldots, B_2, \ldots, A_s, A_s+1, \ldots, B_s), \tag{3.12}$$

where

$$\sum_{i=1}^{s} (B_i - A_i + 1) = N.$$

In other words, Ω is an N-tuple of integers whose entries take values of all the integers in $\cup_{i=1}^s [A_i, B_i]$.

We shall consider $\Omega(N)$ changing with N. Suppose that for each N, $\Omega(N)$ has corresponding $A_i(N)$, $B_i(N)$, for a fixed s. Assume also that $A_i(N), B_i(N), \Omega(N)_N - N$ have the following asymptotic growth:

$$A_i(N) = a_i N + o(N), \quad B_i(N) = b_i N + o(N), \;\; \Omega(N)_N - N = \mu N + o(N) \tag{3.13}$$

where $0 = a_1 < b_1 < \ldots < a_s < b_s$ are new parameters such that $\sum_{i=1}^s (b_i - a_i) = 1$.

(2) Suppose Part (1) of the assumption holds with $b_1 - a_1 \geq \gamma$ for $\gamma \in [0,1)$ as given in Assumption 2.1 (3).

Recall that the Stieljes transform of $\tilde{\mathbf{m}}$ is given by

$$\mathrm{St}_{\tilde{\mathbf{m}}}(t) = \int_{\mathbb{R}} \frac{\tilde{\mathbf{m}}(ds)}{t-s}$$

Following similar computations as in Section 4 of [3], we obtain that

Proposition 3.13. *Suppose Assumptions 2.1, 3.12 hold. Let*

$$F_\kappa(z,t) := \frac{t}{1-\kappa} - \frac{\kappa z}{(z-1)(1-\kappa)} + \frac{\kappa z}{n(1-\kappa)(1-\gamma)} \sum_{i\in[n]\cap I_2} \frac{y_i x}{1+y_i x z} \tag{3.14}$$

Then the following system of equations

$$\begin{cases} F_\kappa(z,t) = \frac{\chi}{(1-\kappa)(1-\gamma)} - \frac{\gamma}{1-\gamma} \\ \mathrm{St}_{\tilde{\mathbf{m}}}(t) = \log(z) \end{cases} \tag{3.15}$$

have at most one pair of complex conjugate (non real) roots in z. A point (χ,κ) is in the liquid region if and only if the above system of equations have one pair of complex conjugate roots.

When the bottom boundary condition satisfies Assumption 3.12. Let

$$\begin{aligned} \tilde{a}_1 &= \frac{a_1}{1-\gamma} \\ \tilde{a}_i &= \frac{a_i-\gamma}{1-\gamma}, \ \forall i \in [s]\setminus\{1\} \\ \tilde{b}_j &= \frac{b_j-\gamma}{1-\gamma}, \ \forall j \in [s]. \end{aligned}$$

It is straightforward to check that

$$\sum_{i=1}^{s} (\tilde{b}_s - \tilde{a}_s) = 1.$$

Note that $\tilde{\mathbf{m}}$, the limit counting measure for the bottom boundary partitions, has density 1 in each of the interval $[\tilde{a}_i, \tilde{b}_i]_{i\in[s]}$, and 0 everywhere else. The Stieltjes transform can be computed explicitly from the definition:

$$\mathrm{St}_{\tilde{\mathbf{m}}}(t) = \log \prod_{i=1}^{s} \frac{t-\tilde{a}_i}{t-\tilde{b}_j}. \tag{3.16}$$

Theorem 3.14. *Let $c_i = \frac{1}{y_i x}$, for $i \in I_2 \cap [n]$. The frozen boundary of the limit of a contracting bipartite graph $\mathcal{R}(\Omega, X, Y, n)$ satisfying Assumptions 2.1 and 3.12 is a rational algebraic curve C with an explicit parametrization $(\chi(t), \kappa(t))$ defined as follows:*

$$\begin{aligned} \chi(t) &= \left(t - \frac{J(t)}{J'(t)}\right)(1-\gamma) + \gamma\left(1 - \frac{1}{J'(t)}\right), \\ \kappa(t) &= \frac{1}{J'(t)} \end{aligned}$$

where

$$J(t) = \Phi_s(t)\left[\frac{1}{\Phi_s(t)-1} - \frac{1}{n(1-\gamma)} \sum_{i\in I_2\cap[n]} \frac{1}{\Phi_s(t)+c_i}\right] \tag{3.17}$$

and

$$\Phi_s(t) = \frac{(t-\tilde{a}_1)(t-\tilde{a}_2)\cdots(t-\tilde{a}_s)}{(t-\tilde{b}_1)(t-\tilde{b}_2)\cdots(t-\tilde{b}_s)}$$

Proof. By (3.14), the first equation of (3.15) is linear in t. Solving t from the first equation of (3.15), we obtain

$$t = \frac{\kappa z}{z-1} - \frac{\kappa z}{n(1-\gamma)} \sum_{i\in[n]\cap I_2} \frac{1}{z+c_i} + \frac{\chi - \gamma(1-\kappa)}{1-\gamma} \tag{3.18}$$

By (3.16), we may write the second equation of (3.15) as follows:

$$z = \prod_{i=1}^{s} \frac{t-\tilde{a}_i}{t-\tilde{b}_i} \tag{3.19}$$

Hence (3.15) is equivalent to:

$$\begin{cases} \Phi_s(t) & = z; \\ (1-\kappa)F_\kappa(z,t) & = t - \left(\frac{\kappa z}{z-1} - \frac{\kappa z}{n(1-\gamma)} \sum_{i\in[n]\cap I_2} \frac{1}{z+c_i}\right) = \frac{\chi-\gamma(1-\kappa)}{1-\gamma} \end{cases} \tag{3.20}$$

We plug the expression of z from the first equation into the second equation, and note that the condition that the resulting equation has a double root is equivalent to the following system of equations

$$\begin{cases} \frac{\chi-\gamma(1-\kappa)}{1-\gamma} = t - \kappa J(t), \\ 1 = \kappa J'(t). \end{cases}$$

where $J(t)$ is defined by (3.17). Then the parametrization of the frozen boundary follows. □

Definition 3.15. *([8]) A degree d real algebraic curve $C \subset \mathbb{R}P^2$ is winding if:*

(1) it intersects every line $L \subset \mathbb{R}P^2$ in at least $d-2$ points counting multiplicity; and
(2) there exists a point $p_0 \in \mathbb{R}P^2$ called center, such that every line through p_0 intersects C in d points.

The dual curve of a winding curve is called a cloud curve.

Proposition 3.16. *Let s be the number of segments on the bottom boundary, and m be the number of distinct values of $c_i = \frac{1}{xy_i}$ in one period. Then*

(1) The frozen boundary C is a cloud curve of class $(m+1)s$, if $\gamma < b_1$
(2) The frozen boundary C is a cloud curve of class $(m+1)(s-1)$, if $\gamma = b_1$.

The result about the frozen boundary being a cloud curve extends the result of [3] for the contracting square-hexagon lattice.

Proof. Recall that the class of a curve is the degree of its dual curve. So we need to show that the dual curve $C^\vee$ has degree $(m+1)s$ and is winding.

We apply the classical formula to obtain from a parametrization $(x(t), y(t))$ of the curve C defining the frozen boundary, another one for its dual $C^\vee$, $(x^\vee(t), y^\vee(t))$:

$$x^\vee = \frac{y'}{yx'-xy'}, \quad y^\vee = -\frac{x'}{yx'-xy'}.$$

and obtain that the dual curve $C^\vee$ is given in the following parametric form:

$$C^\vee = \left\{ \left(-\frac{1}{(1-\gamma)t+\gamma}, -\frac{(1-\gamma)J(t)+\gamma}{(1-\gamma)t+\gamma} \right) \; ; \; t \in \mathbb{C} \cup \{\infty\} \right\}. \tag{3.21}$$

from which we can see that its degree is $(m+1)s$ (resp. $(m+1)(s-1)$) if $\tilde{a}_s < \tilde{b}_s$ (resp. $\tilde{a}_s = \tilde{b}_s$). To show that $C^\vee$ is winding, we need to look at real intersections with straight lines.

Let

$$\tilde{t} = (1-\gamma)t + \gamma; \qquad \tilde{J}(\tilde{t}) = (1-\gamma)J(t) + \gamma.$$

Then we can reparametrize the curve $C^\vee$ by

$$C^\vee = \left\{ \left(-\frac{1}{\tilde{t}}, -\frac{\tilde{J}(\tilde{t})}{\tilde{t}} \right) \; ; \; t \in \mathbb{C} \cup \{\infty\} \right\}. \tag{3.22}$$

First, from Equation (3.22), one sees that the first coordinate $x^\vee$ of the dual curve $C^\vee$ and the parameter $\tilde{t}$ are linked by the simple relation $x^\vee \tilde{t} = -1$.

Using this relation to eliminate t from the expression of the second coordinate, we obtain that the points $(x^\vee, \tilde{t})$ on the dual curve satisfy the following implicit equation:

$$y^\vee = x^\vee \tilde{J}\left(-\frac{1}{x^\vee}\right).$$

The points of intersection $(x^\vee(\tilde{t}), y^\vee(\tilde{t}))$ of the dual curve with a straight line of the form $y^\vee = cx^\vee + d$ have a parameter t satisfying:

$$(c - d\tilde{t}) = \tilde{J}(\tilde{t}) \tag{3.23}$$

Note that

$$\Phi_s(t) = \frac{(\tilde{t}-\gamma)(\tilde{t}-a_2)\cdot\ldots\cdot(\tilde{t}-a_s)}{(\tilde{t}-b_1)(\tilde{t}-b_2)\cdot\ldots\cdot(\tilde{t}-b_s)} := \tilde{\Phi}_s(\tilde{t})$$

and

$$\tilde{J}(\tilde{t}) = \frac{|I_1 \cap [n]|}{n} + \frac{1-\gamma}{\tilde{\Phi}_s(\tilde{t}) - 1} + \frac{1}{n} \sum_{i \in I_2 \cap [n]} \frac{c_i}{\tilde{\Phi}_s(\tilde{t}) + c_i}$$

Then the conclusion that the frozen boundary C is a cloud curve follows from the same arguments as in the proof of Proposition 5.4 in [3]. □

Example 3.17. *Suppose that we have a sequence of contracting bipartite graphs satisfying:*

- *the boundary condition as in Assumption 3.12 satisfies* $s = 2$, $a_1 = 0$, $b_1 = \frac{2}{3}$, $a_2 = 1$, $b_2 = \frac{4}{3}$;
- *the edge weights as in Assumption 2.1 satisfy*
  - $n = 3$, $\gamma = \frac{1}{3}$; *and*
  - $x_1 = x_2 = 1$,$x_3 = 0$; *and*
  - $y_1 = 0$, $y_2 = 1$, $y_3 = 2$.

Then by Theorem 3.14,

$$\Phi_2(t) = \frac{t(t-1)}{\left(t-\frac{1}{2}\right)\left(t-\frac{3}{2}\right)}.$$

$$J(t) = \Phi_2(t)\left[\frac{1}{\Phi_2(t)-1} - \frac{1}{2}\left(\frac{1}{\Phi_2(t)+1} + \frac{1}{\Phi_2(t)+\frac{1}{2}}\right)\right].$$

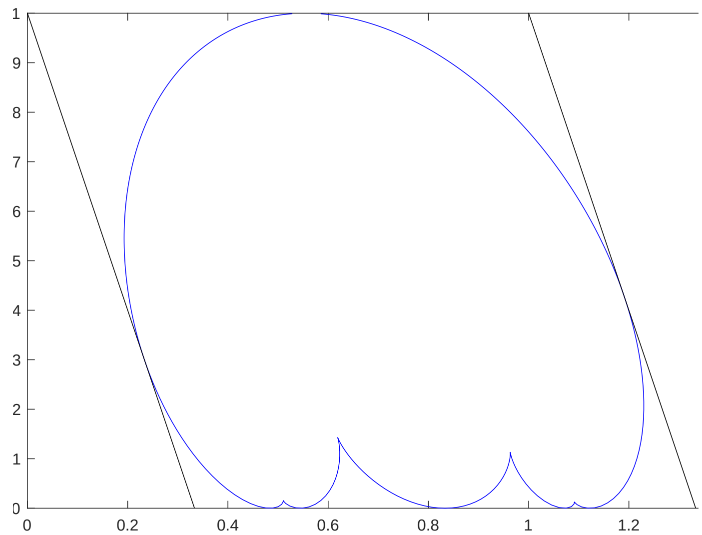

FIGURE 3.4. Frozen boundary of Example 3.17

The parametric equation for the frozen boundary is then given by

$$\chi(t) = \frac{2}{3}\left(t - \frac{J(t)}{J'(t)}\right) + \frac{1}{3}\left(1 - \frac{1}{J'(t)}\right),$$
$$\kappa(t) = \frac{1}{J'(t)}$$

See Figure 3.4 for the frozen boundary of the limit shape of perfect matchings. The region covered by the contracting bipartite graphs is the trapezoid region (χ, κ) bounded by $\kappa = 0$, $\kappa = 1$, $\chi = 0$, $\kappa = -3\chi + 4$. From the analysis above we see that the triangular region (χ, κ) bounded by $\kappa = 0$, $\chi = 0$, $\kappa = -3\chi + 1$ must be frozen.

4. Disconnected Liquid Regions

In this section, we prove the existence of multiple disconnected liquid regions for the limit shape of perfect matchings on any contracting square-hexagon lattices with certain edge weights, extending the results in [11]. The main theorem proved in this section is Theorem 4.4. We shall start with a few assumptions.

Assumption 4.1. *Suppose the edge weights $(x_1, \ldots, x_N)$ satisfies Assumption 2.1 (1)(2). Moreover,*

- $x_1 > x_2 > \ldots > x_n > 0$*; and*
- N *is an integral multiple of* n*.*

Recall that $[\Sigma_N/\Sigma_N^X]^r$ was defined in (2.7). Let $\overline{\sigma}_0 \in [\Sigma_N/\Sigma_N^X]^r$ be the unique element in $[\Sigma_N/\Sigma_N^X]^r$ satisfying the condition that for any representative $\sigma_0 \in \overline{\sigma}_0$, we have

$$x_{\sigma_0(1)} \geq x_{\sigma_0(2)} \geq \ldots \geq x_{\sigma_0(N)}. \tag{4.1}$$

Assumption 4.2. *Assume $x_1, \ldots, x_N$ satisfy Assumption 2.1 (1)(2) and Assumption 4.1.*

Let $s \in [N]$. Assume there exists positive integers $K_1, K_2, \ldots K_s$, such that

(1) $\sum_{t=1}^{s} K_t = N$;

(2)

$$\mu_1 > \ldots > \mu_s \tag{4.2}$$

are all the distinct elements in $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$.

(3)

$$\lambda_1 = \lambda_2 = \ldots = \lambda_{K_s} = \mu_1;$$
$$\lambda_{K_s+1} = \lambda_{K_s+2} = \ldots = \lambda_{K_s+K_{s-1}} = \mu_2;$$
$$\ldots$$
$$\lambda_{\sum_{t=2}^{s} K_t} = \lambda_{1+\sum_{t=2}^{s} K_t} = \ldots = \lambda_{\sum_{t=1}^{s} K_t} = \mu_s;$$

(4) *Let*

$$J_i = \{t \in [s] : \exists p \in [n], \text{ s.t. } x_{\sigma_0(p)} = x_i, \text{and } \lambda_p = \mu_t\} \tag{4.3}$$

(a) *If* $1 \le i < j \le n$, $l \in J_i$, *and* $t \in J_j$, *then* $l < t$.

(b) *For any* p, q *satisfying* $1 \le p \le s$ *and* $1 \le q \le s$, *and* $q > p$

$$L_1 N \le \mu_p - \mu_q \le L_2 N$$

where L_1, L_2 *are sufficiently large constants independent of* N.

(c) s *and* n *are fixed as* $N \to \infty$.

Assumption 4.3. *Assume* $x_{1,N} = x_1 > 0$ *and* $(x_{2,N}, \ldots, x_{n,N})$ *changes with* N. *Assume that for each fixed* N, $(x_{1,N}, \ldots, x_{n,N})$ *satisfy Assumption 4.1. Suppose that Assumption 4.2 holds. Moreover, assume that*

$$\liminf_{N\to\infty} \frac{\log\left(\min_{1\le i<j\le n} \frac{x_{i,N}}{x_{j,N}}\right)}{N} \ge \alpha > 0,$$

where α *is a sufficiently large positive constant independent of* N.

Theorem 4.4. *Suppose Assumptions 3.12(1), 4.1, 4.2 and 4.3 hold. Then the frozen boundary consists of* n *disjoint cloud curves.*

The conclusion of Theorem 4.4 under the further condition that $|I_2 \cap [n]| \in \{0, 1\}$ was proved in Theorem 2.20 of [11]. Here we prove the theorem for all the contracting square-hexagon lattice when edge weights satisfy the Assumptions 4.1, 4.3 and bottom boundary conditions satisfy Assumptions 3.12(1) and 4.2.

For $i \in [n]$, let J_i be defined as in (4.3). Under Assumptions 4.1 and 4.2, assume that

$$J_i = \begin{cases} \{d_i, d_i + 1, \ldots, d_{i+1} - 1\} & \text{if } 1 \le i \le n-1 \\ \{d_n, d_n + 1, \ldots, s\} & \text{if } i = n \end{cases}$$

where $d_1, \ldots, d_n$ are positive integers satisfying

$$1 = d_1 < d_2 < \ldots < d_n \le s$$

Let

$$d_{n+1} := s + 1.$$

For $i \in [n]$

$$\Psi_i(t_i) = \frac{(t_i - \beta_{i,0})(t_i - \beta_{i,1}) \cdots (t_i - \beta_{i,D_i})}{(t_i - \gamma_{i,0})(t_i - \gamma_{i,1}) \cdots (t_i - \gamma_{i,D_i})}; \tag{4.4}$$

and for $j \in [s]$, let a_j, b_j be given by (3.13).

- If $i \in [n]$, for $0 \leq k \leq d_{i+1} - d_i - 1 := D_i$, let

$$\beta_{i,k} = n\left(a_1 + \sum_{l=2}^{s-d_i-k+1}(a_l - b_{l-1})\right) + n - i + 1 - n\left(\sum_{l=s-d_i-k+1}^{s-d_i+1}(b_l - a_l)\right) \tag{4.5}$$

$$\gamma_{i,k} = n\left(a_1 + \sum_{l=2}^{s-d_i-k+1}(a_l - b_{l-1})\right) + n - i + 1 - n\left(\sum_{l=s-d_i-k+2}^{s-d_i+1}(b_l - a_l)\right). \tag{4.6}$$

For $\kappa \in (0,1)$, $2 \leq i \leq n$, let:

$$F_{i,\kappa}(z, t_i) := \frac{z}{(1-\kappa)n}\left(\frac{t_i}{z} - \frac{1}{z-1} - \frac{n-i}{z}\right) + \frac{z}{n(z-1)} + \frac{n-i}{n}.$$

and

$$F_{1,\kappa}(z, t_1) := \frac{z}{(1-\kappa)n}\left(\frac{t_1}{z} - \frac{1}{z-1} - \frac{n-1}{z} + \kappa \sum_{r \in I_2 \cap [n]} \frac{y_r x_1}{1 + y_r x_1 z}\right) + \frac{z}{n(z-1)} + \frac{n-1}{n}.$$

Let C_1 be the curve consisting of all the points $(\chi, \kappa) \in [0, \infty) \times [0, 1]$ such that the following system of equations has a double root in z

$$\begin{cases} \Psi_1(t_1) = z; \\ n(1-\kappa)F_{1,\kappa}(z) = t_1 - \kappa\left[\frac{1}{z-1} + (n-l) + \sum_{j=1}^{m} \frac{n_j \gamma_j}{z+\gamma_j}\right] = n\chi. \end{cases} \tag{4.7}$$

where $\gamma_1, \ldots, \gamma_m$ are all the distinct values in $\left\{\frac{1}{y_i x}\right\}_{i \in I_2 \cap [n]}$; for $j \in [m]$, n_j is the number of $i \in I_2 \cap [n]$ such that $\frac{1}{y_i x} = \gamma_j$; and

$$l = I_2 \cap [n].$$

For $i \in \{2, 3, \ldots, n\}$, let C_i be the curve consisting of all the points $(\chi, \kappa) \in [0, \infty) \times [0, 1]$ such that the following system of equations has a double root in z

$$\begin{cases} \Psi_i(t_i) = z; \\ n(1-\kappa)F_{i,\kappa}(z) = t_i - \kappa\left[(n-i+1) + \frac{1}{z-1}\right] = n\chi. \end{cases} \tag{4.8}$$

Then we have the following lemma

Lemma 4.5. *Let $i \in [n]$. The curve C_i has an explicit parametrization given by*

$$\chi_i(t_i) = \frac{1}{n}\left[t_i - \frac{J_i(t_i)}{J_i'(t_i)}\right], \quad \kappa_i(t_i) = \frac{1}{J_i'(t_i)},$$

where for $2 \leq i \leq n$;

$$J_i(t_i) = (n-i+1) + \frac{1}{\Psi_i(t_i) - 1};$$

for $i = 1$

$$J_1(t_1) = \frac{1}{\Psi_1(t_1) - 1} + n - l + \sum_{i \in [n] \cap I_2} \frac{c_j}{\Psi_1(t_1) + c_j}$$

where $c_j = \frac{1}{y_j x_1}$ *for* $j \in [n] \cap I_2$. *Moreover, the curves* $C_1, \ldots, C_n$ *are disjoint cloud curves.*

Proof. For each $i \in [n]$, the parametric equation of C_i and the fact that C_i is a cloud curve of class D_i follow from the same arguments as in the proof of Theorem 7.10 in [11].

We need to show that $C_1, \ldots, C_n$ are disjoint. Note that C_i is characterized by the condition that the system (4.7) or (4.8) of equations have double roots in z. When N is an integer multiple of n, recall that the partition $\phi^{(i,\sigma)}(N) \in \mathbb{GT}_{\frac{N}{n}}$ was defined as in (2.6), and $\sigma_0 \in \Sigma_N$ was defined as in (4.1). When the bottom boundary conditions satisfy Assumptions 3.12(1) and 4.2, for each $i \in [n]$, the counting measures for $\phi^{(i,\sigma_0)}(N)$ converge weakly to a limit measure, denoted by $\mathbf{m}_i$, as $N \to \infty$. The measure $\mathbf{m}_i$ has density 1 in $\cup_{k=0}^{D_i} [\beta_{i,k}, \gamma_{i,k}]$, and density 0 elsewhere; see Lemma 4.8 of [11]. Note that the supports for $\mathbf{m}_i$, $i \in [n]$, are pairwise disjoint.

We make a change of variables in (4.7) as follows.

$$\begin{cases} \tilde{\chi}_1 = \chi + \frac{n-1}{n} \\ \tilde{\kappa}_1 = \kappa \end{cases},$$

Then after the change of variables, (4.7) becomes the same as (3.20) when $\gamma = \frac{n-1}{n}$, $D_1 = s - 1$ and $(\beta_{1,D_1}, \gamma_{1,D_1}, \ldots, \beta_{1,0}, \gamma_{1,0}) = (\tilde{a}_1, \tilde{b}_1, \ldots, \tilde{a}_s, \tilde{b}_s)$, and $x_1 = x$. Let $\tilde{C}_1 = C_1 + \left(\frac{n-1}{n}, 0\right)$. Then $\tilde{C}_1$ is the frozen boundary of the limit shape of perfect matchings a contracting bipartite graph, hence it is restricted in the region bounded by

$$\begin{aligned} &\tilde{\kappa}_1 = 0; \qquad \tilde{\kappa}_1 = 1; \\ &\tilde{\chi}_1 = \beta_{1,D_1}(1-\gamma) = \frac{\beta_{1,D_1}}{n} \\ &\tilde{\chi}_1 = \gamma_{1,0}(1-\gamma) + \gamma = \frac{\gamma_{1,0}}{n} + \frac{n-1}{n} \end{aligned}$$

Hence C_1 is restricted in the region bounded by

$$\begin{aligned} &\kappa = 0; \qquad \kappa = 1 \\ &\chi = \frac{\beta_{1,D_1}}{n} - \frac{n-1}{n} \\ &\chi = \frac{\gamma_{1,0}}{n} \end{aligned}$$

We also make a change of variables in (4.8). For $i \in \{2, 3, \ldots, n\}$, let

$$\begin{cases} \tilde{\chi}_i = n\chi + \kappa(n-i) \\ \tilde{\kappa}_i = \kappa \end{cases},$$

and let $\tilde{C}_i$ be the corresponding curve in the new coordinate system $\tilde{\chi}_i, \tilde{\kappa}_i$. Then $\tilde{C}_i$ is the frozen boundary of a uniform dimer model on contracting hexagon lattice with boundary condition given by $\mathbf{m}_i$. Then the fact that $\tilde{C}_i$ and C_i are cloud curves follows from Proposition 5.4 of [3]. Hence $\tilde{C}_i$ is restricted in the region bounded by

$$\tilde{\kappa}_1 = 0; \qquad \tilde{\kappa}_1 = 1$$
$$\tilde{\chi}_1 = \beta_{i,D_i}; \qquad \tilde{\chi}_1 = \gamma_{i,0}$$

Hence C_i is restricted in the region bounded by

$$\kappa = 0; \qquad \kappa = 1$$
$$\chi = \frac{\beta_{i,D_i} - \kappa(n-i)}{n};$$
$$\chi = \frac{\gamma_{i,0} - \kappa(n-i)}{n}$$

It is straightforward to check that when Assumption 4.2(4)(b) holds with constant L_1 sufficiently large, the regions described above containing different C_i are disjoint; therefore C_i are disjoint. □

Proof of Theorem 4.4. We consider a contracting square-hexagon lattice with edge weights satisfying Assumption 2.1, $\gamma = \frac{n-1}{n}$, and boundary partition given by

$$\omega := \left(\phi^{(1,\sigma_0)}(N), 0^{\frac{N(n-1)}{n}}\right) \in \mathbb{GT}_N^+.$$

i.e., ω is a partition with N components, the $\frac{N}{n}$ components at the beginning are those of $\phi^{(1,\sigma_0)}(N)$'s, and the remaining $\frac{N(n-1)}{n}$ components are 0. Let $\mathbf{m}_\omega$ be the limit counting measure for ω, as $N \to \infty$, and let $\tilde{\mathbf{m}}_\omega$ be the limit counting measure for $\phi^{(1,\sigma_0)}(N)$. Let $\kappa \in (0,1)$, and $\mathbf{m}_\omega^\kappa$ be the limit counting measure for the partitions on the $\lfloor \frac{2\kappa N}{n} \rfloor$th row, counting from the bottom. Let $\tilde{\mathbf{m}}_\omega^\kappa$ be the limit counting measure of the partitions obtained from the partitions on the $\lfloor \frac{2\kappa N}{n} \rfloor$th row by removing the $\frac{(1-\kappa)(n-1)N}{n}$ 0's in the end. By Proposition 3.11, we obtain

$$\sum_{j=0}^{\infty} \frac{\int_{\mathbb{R}} x^j \tilde{\mathbf{m}}_\omega^\kappa(dx)}{s^{j+1}} = -\frac{1}{2\pi\mathbf{i}} \oint_1 \frac{dz}{z} \log\left(1 - \frac{F_\kappa(z)}{s}\right).$$

Following similar computations as in Section 7 of [11], we obtain that

$$\mathrm{St}_{\tilde{\mathbf{m}}_\omega^\kappa}(x) = \log(z^\kappa(x)) \tag{4.9}$$

where $z^\kappa(x)$ is the solution of $F_\kappa(z) = x$ in a neighborhood of 1 when x is in a neighborhood of infinity. When x is not ina neighborhood of infinity but outside the support of $\tilde{\mathbf{m}}_\omega^\kappa$, the identity (4.9) is obtained by analytic extension.

We claim that if complex roots exists for (3.20) with $x = \frac{\chi - \gamma(1-\kappa)}{(1-\gamma)(1-\kappa)}$ and $\gamma = \frac{n-1}{n}$, then $z^\kappa(x)$ cannot be real.

To see why that is true, from (4.9) we obtain

$$z^\kappa(x) = \exp\left(\int_{\mathbb{R}} \frac{\tilde{\mathbf{m}}_\omega^\kappa[ds]}{x-s}\right);$$

and

$$z^\kappa(x) = \exp\left(\int_{\mathbb{R}} \frac{(x-s-\mathbf{i}\epsilon)\tilde{\mathbf{m}}_\omega^\kappa[ds]}{(x-s)^2+\epsilon^2}\right)$$

Therefore $\Im[z^{\kappa}(x+\mathbf{i}\epsilon)]<0$ when ϵ is a small positive number. However, when complex roots exist for (3.20), for real root $s(x)$, Lemma 7.9 of [11] implies that $\Im[s(x+\mathbf{i}\epsilon)]\geq 0$ when ϵ is a small positive number. This implies that when complex roots exist for (4.9), $z^{\kappa}(x+\mathbf{i}\epsilon)$ cannot be real.

From expression (7.2) of [11], we obtain

$$\mathrm{St}_{\mathbf{m}^{\kappa}}(x)=\sum_{i=1}^{n}\log(z_i^{\kappa}(x))$$

where $\mathbf{m}^{\kappa}$ is the limit counting measure for partitions corresponding to dimer configurations at level κ on a contracting square-hexagon lattice with edge weights and bottom boundary conditions satisfying Assumptions 4.1, 4.2, 4.3 and 3.12(1). Note that

- When x is in a neighborhood of ∞, $z_1(x)$ is the root of (4.7) in a neighborhood of (4.7) when $x=n\chi$. Therefore

$$z_1^{\kappa}(x)=z^{\kappa}\left(nx+\frac{\kappa(n-1)}{1-\kappa}\right);$$

  and
- when x is in a neighborhood of ∞ and $2\leq i\leq n$, $z_i(x)$ is the root of (4.8) in a neighborhood of 1. It follows from Section 7.1 of [11], that when (4.8) has a pair of complex conjugate roots (non-real) in z, $z_i(x)$ cannot be real; and
- By Lemma 4.5, the regions when each one of (4.7) or (4.8) has a pair of complex conjugate roots are disjoint. That is, at each point of the trapezoid domain occupied by the square-hexagon lattice, at most one of the z_i's $(1\leq i\leq n)$ is not real.

Therefore, under the assumptions of the theorem, the frozen boundary is given by the condition that one of the n equation systems in (4.7), (4.8) has double roots in z. Then the theorem follows from Lemma 4.5. □

Example 4.6. *Consider a contracting square-hexagon lattice with period* 1×2. *Let* $x_1=1$, *and* $\frac{x_2}{x_1}\leq e^{-\alpha N}$; $y_1=4$ *and* $y_2=1/4$. *Assume* N *is an integer multiple of* 6. *Assume the boundary partition* $\lambda(N)$ *satisfies*

$$\begin{aligned}
&\lambda_1(N)=\lambda_2(N)=\ldots=\lambda_{\frac{N}{4}}(N)=\mu_1(N)\\
&\lambda_{\frac{N}{4}+1}(N)=\lambda_{\frac{N}{4}+2}(N)=\ldots=\lambda_{\frac{N}{2}}(N)=\mu_2(N)\\
&\lambda_{\frac{N}{2}+1}(N)=\lambda_{\frac{N}{2}+2}(N)=\ldots=\lambda_{\frac{2N}{3}}(N)=\mu_3(N)\\
&\lambda_{\frac{2N}{3}+1}(N)=\lambda_{\frac{N}{2}+2}(N)=\ldots=\lambda_{\frac{5N}{6}}(N)=\mu_4(N)\\
&\lambda_{\frac{5N}{6}+1}(N)=\lambda_{\frac{N}{2}+2}(N)=\ldots=\lambda_{N}(N)=\mu_5(N)=0.
\end{aligned}$$

For $1\leq j\leq 5$, *let*

$$r_j=\lim_{N\to\infty}\frac{\mu_j(N)}{N}.$$

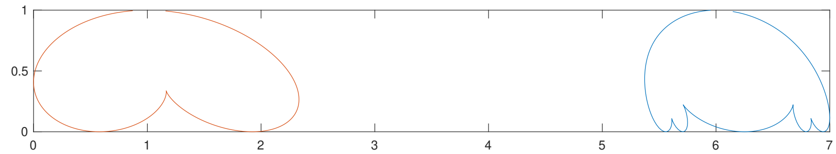

FIGURE 4.1. Frozen boundary of Example 4.6

Note that $r_5 = 0$. Then

$$\begin{aligned}\Psi_1(t_1) &= \frac{\left(t_1 - 2r_1 - \frac{3}{2}\right)(t_1 - 2r_2 - 1)}{(t_1 - 2r_1 - 2)\left(t_1 - 2r_2 - \frac{3}{2}\right)};\\ \Psi_2(t_2) &= \frac{t_2\left(t_2 - 2r_4 - \frac{1}{3}\right)\left(t_2 - 2r_3 - \frac{2}{3}\right)}{\left(t_2 - \frac{1}{3}\right)\left(t_2 - 2r_4 - \frac{2}{3}\right)(t_2 - 2r_3 - 1)};\\ J_1(t_1) &= \frac{1}{\Psi_1(t_1) - 1} + \frac{1}{\Psi_1(t_1) + 1} + \frac{1}{2\Psi_1(t_1) + 1};\\ J_2(t_2) &= \frac{1}{\Psi_2(t_2) - 1} + 1.\end{aligned}$$

Then the frozen boundary is a union of the following two parametric curves.

$$\begin{cases}\chi_1(t_1) = \frac{1}{2}\left[t_1 - \frac{J_1(t_1)}{J_1'(t_1)}\right]\\ \kappa_1(t_1) = \frac{1}{J_1'(t_1)}\end{cases}$$

and

$$\begin{cases}\chi_2(t_2) = \frac{1}{2}\left[t_2 - \frac{J_2(t_2)}{J_2'(t_2)}\right]\\ \kappa_2(t_2) = \frac{1}{J_2'(t_2)}\end{cases}$$

For $(r_1, r_2, r_3, r_4) = (6, 5, 2, 1)$, see Figure 4.1 for a picture of the frozen boundary.

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REFERENCES

[1] A. Borodin and P. Ferrari, *Random tilings and markov chains for interlacing particles*, Markov Process. Related Fields **24** (2018), 419–451.

[2] C. Boutillier, J. Bouttier, G. Chapuy, S. Corteel, and S. Ramassamy, *Dimers on rail yard graphs*, Annales de l'Institut Henri Poincaré D **4** (2017), 479–539.

[3] C. Boutillier and Z. Li, *Limit shape and height fluctuations of random perfect matchings on square-hexagon lattices*, Annales de Institut Fourier, arXiv:1709.09801.

[4] A. Bufetov and V. Gorin, *Representations of classical lie groups and quantized free convolution*, Geom.Funct.Anal. **25** (2015), 763–814.

[5] A. Bufetov and A. Knizel, *Asymptotics of random domino tilings of rectangular Aztec diamond*, Ann. Inst. H. Poincaré Probab. Statist. **54** (2018), 1250–1290.

[6] H. Cohn, R. Kenyon, and Propp. J., *A variational principle for domino tilings*, J. Amer. Math. Soc. **14** (2001), 297–346.

[7] V. Gorin and G. Panova, *Asymptotics of symmetric polynomials with applications to statistical mechanics and representation theory*, Ann. Probab. **43** (2015), 3052–3132.

[8] R. Kenyon and A. Okounkov, *Limit shapes and the complex Burgers equation*, Acta Math. **199** (2007), 263–302.
[9] R. Kenyon, A. Okounkov, and S. Sheffield, *Dimers and amoebae*, Ann. Math. **163** (2006), 1019–1056.
[10] Z. Li, *Fluctuations of dimer heights on contracting square-hexagon lattices*, arXiv:1809.08727.
[11] ———, *Schur function at generic points and limit shape of perfect matchings on contracting square hexagon lattices with piecewise boundary conditions*, arXiv:1807.06175.
[12] ———, *Asymptotics of schur functions on almost staircase partitions*, Electronic Communications in Probability **25** (2020), 13pp.
[13] S. Sheffield, *Random surfaces*, Astérisque **304** (2005), 175pp.

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