Exercise 9.8

(a) Consider the distribution function of T_{xy} when we have independence:

$$F_{T_{xy}}(t) = 1 - {}_tp_{xy} = 1 - {}_tp_x {}_tp_y$$

Differentiating, we get

$$\begin{aligned} f_{T_{xy}}(t) &= \frac{dF_{T_{xy}}(t)}{dt} \\ &= -{}_tp_x \frac{{}_tp_y}{dt} - -{}_tp_y \frac{{}_tp_x}{dt} \\ &= -{}_tp_x \left(-{}_tp_y \mu_{y+t}\right) - {}_tp_y \left(-{}_tp_x \mu_{x+t}\right) \\ &= {}_tp_x {}_tp_y \left(\mu_{y+t} + \mu_{x+t}\right), \end{aligned}$$

which proves the result. Notice also that because

$$f_{T_{xy}}(t) = {}_tp_{xy} \mu_{x+t:y+t} = {}_tp_x {}_tp_y \mu_{x+t:y+t},$$

when we have independence, the following holds:

$$\mu_{x+t:y+t} = \mu_{x+t} + \mu_{y+t}$$

(b) Because of independence, we have

$$f_{T_x T_y}(t,s) = f_{T_x}(t) f_{T_y}(s) = {}_tp_x \mu_{x+t} \cdot {}_sp_y \mu_{y+s}.$$

Thus, since the insurance pays at the moment of death of (x) provided before the death of (y), the actuarial present value can then be expressed as

$$\begin{aligned} \bar{A}^{1}_{xy} &= \mathrm{E}\left[v^{T_x}\left(T_x \leq T_y\right)\right] \\ &= \int_0^\infty \int_t^\infty v^t {}_tp_x \mu_{x+t} \cdot {}_sp_y \mu_{y+s} ds dt \\ &= \int_0^\infty v^t {}_tp_x \mu_{x+t} \int_0^t {}_sp_y \mu_{y+s} ds dt \\ &= \int_0^\infty v^t {}_tp_x \mu_{x+t} \cdot {}_tp_y dt \\ &= \int_0^\infty v^t {}_tp_{xy} \mu_{x+t} dt \end{aligned}$$

which give (9.13).

(c) Here, the insurance pays at the moment of death of (x) provided (y) is dead. Thus, the actuarial present value can be expressed as

$$\begin{aligned}
\bar{A}^{2}_{xy} &= \mathrm{E}\left[v^{T_x}\left(T_x > T_y\right)\right] \\
&= \int_0^\infty \int_0^t v^t\, {}_tp_x \mu_{x+t} \cdot {}_sp_y \mu_{y+s} ds dt \\
&= \int_0^\infty v^t\, {}_tp_x \mu_{x+t} \int_0^t {}_sp_y \mu_{y+s} ds dt \\
&= \int_0^\infty v^t\, {}_tp_x \mu_{x+t} \left(1 - {}_tp_y\right) dt \\
&= \int_0^\infty v^t\, {}_tp_x \mu_{x+t} dt - \int_0^\infty v^t\, {}_tp_{xy} \mu_{x+t} dt \\
&= \bar{A}_x - \bar{A}^{1}_{xy}
\end{aligned}$$

Since we know that the present value random variables satisfies

$$v^{T_x} = v^{T_x}\left(T_x \leq T_y\right) + v^{T_x}\left(T_x > T_y\right),$$

taking expectations of both sides lead us to:

$$\bar{A}_x = \bar{A}^{1}_{xy} + \bar{A}^{2}_{xy}$$

The insurance payable to (x) is paid either at the first death or the second death of (x).