Exercise 9.6

(a) By the independence assumption, we can write

$$\begin{aligned}
{}_tp_{xy} &= {}_tp_x \times {}_tp_y \\
&= \exp\left\{-\left[\frac{Bc^x}{\log(c)}(c^t-1)\right]\right\} \times \exp\left\{-\left[\frac{Bc^y}{\log(c)}(c^t-1)\right]\right\} \\
&= \exp\left[-\frac{B}{\log(c)}\left(c^x+c^y\right)(c^t-1)\right] \\
&= \exp\left[-\frac{B}{\log(c)}c^w(c^t-1)\right] \\
&= {}_tp_w
\end{aligned}$$

where the joint life (xy) is replaced by a single life (w) where it satisfies

$$c^w = c^x + c^y$$

or equivalently we have

$$w = \frac{\log\left(c^x+c^y\right)}{\log(c)}.$$

(b) First we write

$$A_{\overset{1}{x}:y} = \sum_{k=0}^{\infty} v^{k+1}\, {}_{k|}q_{\overset{1}{x}:y}$$

where we note that

$$\begin{aligned}
{}_{k|}q_{\overset{1}{x}:y} &= \int_k^{k+1} {}_sp_{xy}\, \mu_{x+s}\, ds \\
&= \int_k^{k+1} {}_sp_w \times Bc^{x+s}\, \frac{c^w}{c^w}\, ds \\
&= \frac{c^x}{c^w}\int_k^{k+1} {}_sp_w \times Bc^{w+s}\, ds \\
&= \frac{c^x}{c^w}\int_k^{k+1} {}_sp_w\, \mu_{w+s}\, ds \\
&= \frac{c^x}{c^w}\, {}_{k|}q_w.
\end{aligned}$$

This leads us finally to

$$A_{\overset{1}{x}:y} = \sum_{k=0}^{\infty} v^{k+1}\, \frac{c^x}{c^w}\, {}_{k|}q_w = \frac{c^x}{c^w}\, \sum_{k=0}^{\infty} v^{k+1}\, {}_{k|}q_w = \frac{c^x}{c^w}\, A_w.$$