Exercise 8.8

(a) We ignore expenses, or we can treat expenses as addition to benefits. Similarly, we can treat premium as a negative benefit. For an entity in state i then, the policy value at time t is equal to the APV of future benefits (including premiums as negative benefits).

Thus, for benefits (including the setting up of reserves) associated with transitions from i to some $j \neq i$, this is equal to

$$\int_0^\infty \frac{v(t+s)}{v(t)} \left[S_{t+s}^{(ij)} + {}_{t+s}V^{(j)} \right] {}_sp_{x+t}^{\overline{ii}} \, \mu_{x+t+s}^{ij} ds.$$

We then sum this for all possible $j \neq i$. For benefits associated with being in state i continuously, this APV is equal to

$$\int_0^\infty \frac{v(t+s)}{v(t)} B_{t+s}^{(i)} \, {}_sp_{x+t}^{\overline{ii}} ds.$$

If we add all these terms and possibilities, we get the desired result.

(b) Substitute $r = t + s$ so that we have

$${}_tV^{(i)} = \sum_{j \neq i} \int_t^\infty \frac{v(r)}{v(t)} \left[S_r^{(ij)} + {}_rV^{(j)} \right] {}_{r-t}p_{x+t}^{\overline{ii}} \, \mu_{x+r}^{ij} dr + \int_t^\infty \frac{v(r)}{v(t)} B_r^{(i)} \, {}_{r-t}p_{x+t}^{\overline{ii}} dr.$$

It is clear that ${}_rp_x^{\overline{ii}} = {}_tp_x^{\overline{ii}} \times {}_{r-t}p_{x+t}^{\overline{ii}}$ so that

$${}_{r-t}p_{x+t}^{\overline{ii}} = \frac{{}_rp_x^{\overline{ii}}}{{}_tp_x^{\overline{ii}}}.$$

Therefore, we have

$$v(t) \, {}_tp_x^{\overline{ii}} \, {}_tV^{(i)} = \sum_{j \neq i} \int_t^\infty v(r) \left(S_r^{(ij)} + {}_rV^{(j)} \right) {}_rp_x^{\overline{ii}} \, \mu_{x+r}^{ij} dr + \int_t^\infty v(r) B_r^{(i)} \, {}_rp_x^{\overline{ii}} dr.$$

The final step is to differentiate both sides of the above equation. Differentiating the RHS gives us

$$-\sum_{j \neq i} v(t) \left(S_t^{(ij)} + {}_tV^{(j)} \right) {}_tp_x^{\overline{ii}} \, \mu_{x+t}^{ij} - v(t) B_t^{(i)} \, {}_tp_x^{\overline{ii}}.$$

For the LHS, we first note the following:

$$\frac{d}{dt} v(t) = -\delta_t v(t)$$

Since we can write ${}_tp_x^{\overline{ii}} = \exp\left(-\int_0^t \sum_{j \neq i} \mu_{x+s}^{ij} ds \right)$, then

$$\frac{d}{dt} {}_tp_x^{\overline{ii}} = -\sum_{j \neq i} {}_tp_x^{\overline{ii}} \, \mu_{x+t}^{ij}.$$

Furthermore, we have

$$\begin{aligned} \frac{d}{dt}\left(v(t)\,{}_tp_x^{\overline{ii}}\right) &= -v(t)\sum_{j\neq i}{}_tp_x^{\overline{ii}}\,\mu_{x+t}^{ij} - v(t)\,{}_tp_x^{\overline{ii}}\cdot\delta_t \\ &= -v(t)\,{}_tp_x^{\overline{ii}}\left(\delta_t + \sum_{j\neq i}\mu_{x+t}^{ij}\right). \end{aligned}$$

Continuing to differentiate the LHS then, we have

$$\begin{aligned} \frac{d}{dt}\left(v(t)\,{}_tp_x^{\overline{ii}}\,{}_tV^{(i)}\right) &= v(t)\,{}_tp_x^{\overline{ii}}\frac{d}{dt}\,{}_tV^{(i)} + {}_tV^{(i)}\frac{d}{dt}\left(v(t)\,{}_tp_x^{\overline{ii}}\right) \\ &= v(t)\,{}_tp_x^{\overline{ii}}\left[\frac{d}{dt}\,{}_tV^{(i)} - {}_tV^{(i)}\left(\delta_t + \sum_{j\neq i}\mu_{x+t}^{ij}\right)\right]. \end{aligned}$$

Equating the two derivatives, we get

$$-v(t)\,{}_tp_x^{\overline{ii}}\left[\sum_{j\neq i}\left(S_t^{(ij)} + {}_tV^{(j)}\right)\mu_{x+t}^{ij} + B_t^{(i)}\right] = v(t)\,{}_tp_x^{\overline{ii}}\left[\frac{d}{dt}\,{}_tV^{(i)} - {}_tV^{(i)}\left(\delta_t + \sum_{j\neq i}\mu_{x+t}^{ij}\right)\right]$$

and solving for the derivative of the policy value at time t (with some re-arrangement) leads us to

$$\frac{d}{dt}\,{}_tV^{(i)} = \delta_t\,{}_tV^{(i)} - B_t^{(i)} - \sum_{j\neq i}\mu_{x+t}^{ij}\left(S_t^{(ij)} + {}_tV^{(j)} - {}_tV^{(i)}\right),$$

which gives us the desired result.