Exercise 8.6

(a) Since you cannot return to state 0 once you leave the state, we must have:

$$ {}_tp_{30}^{00} = {}_tp_{30}^{\overline{00}} = \exp\left[-\int_0^t \left(\mu_{30+s}^{01} + \mu_{30+s}^{02}\right) ds\right] = \mathrm{e}^{-10^{-5}t} \times \mathrm{e}^{-At - \frac{Bc^{30}}{\log(c)}(c^t-1)}. $$

Plug $t = 10$ together with the given values of A, B and c, we get

$$ {}_{10}p_{30}^{00} = 0.9791219. $$

For the probability of being in state 1 at the end of 10 years, we have

$$ {}_{10}p_{30}^{01} = \int_0^{10} {}_tp_{30}^{00}\, \mu_{30+t}^{01}\, dt = \int_0^{10} \mathrm{e}^{-10^{-5}t} \times \mathrm{e}^{-At - \frac{Bc^{30}}{\log(c)}(c^t-1)} \times 10^{-5} dt. $$

and for the probability of being in state 2 at the end of 10 years, we have

$$ {}_{10}p_{30}^{02} = \int_0^{10} {}_tp_{30}^{00}\, \mu_{30+t}^{02}\, dt = \int_0^{10} \mathrm{e}^{-10^{-5}t} \times \mathrm{e}^{-At - \frac{Bc^{30}}{\log(c)}(c^t-1)} \times \left(A + Bc^{30+t}\right) dt. $$

Because states 1 and 2 are both absorbing states, both probabilities ${}_{10}p_{30}^{01}$ and ${}_{10}p_{30}^{02}$ can also be interpreted as the probabilities of being in that (respective) state by the end of 10 years. Neither integrals can be explicitly calculated but R codes show that

$$ {}_{10}p_{30}^{01} = 0.0000991 $$

and

$$ {}_{10}p_{30}^{02} = 0.0207791. $$

Note that the solutions printed in the textbook should be flip-flopped.

```
x <- 30
A <- 5*10^(-4)
B <- 7.6*10^(-5)
c <- 1.09
muxt01 <- 10^(-5)
muxt02 <- function(t){
out <- A + B*c^(x+t)
out}
tpx00 <- function(t){
temp1 <- exp(-muxt01*t)
temp2 <- exp(-A*t -(B*c^x)*(c^t-1)/log(c))
temp1*temp2}
integ01 <- function(s){
tpx00(s)*muxt01}
tpx01 <- function(t){
integrate(integ01, lower=0, upper=t)
```

```
}
integ02 <- function(s){
tpx00(s)*muxt02(s)}
tpx02 <- function(t){
integrate(integ02, lower=0, upper=t)
}
```

This produces the following result:

```
> tpx00(10)
[1] 0.9791219
> tpx01(10)
9.906477e-05 with absolute error < 1.1e-18
> tpx02(10)
0.02077908 with absolute error < 2.3e-16
```

(b) Let P denote the annual premium rate payable continuously. Then, the APV of future benefits at issue is given by

$$\begin{aligned}\text{APV}(\text{FB}_0) &= 20000 \times \int_0^{10} v^t \, {}_tp_{30}^{00} \, \mu_{30+t}^{01} \, dt + 10000 \times \int_0^{10} v^t \, {}_tp_{30}^{00} \, \mu_{30+t}^{02} \, dt \\ &= 20000\,(0.00007846) + 10000\,(0.01602768) = 1618.46\end{aligned}$$

and the APV of future premiums at issue is

$$\text{APV}(\text{FP}_0) = P \times \int_0^{10} v^t \, {}_tp_{30}^{00} \, dt = P \times 7.845802.$$

This leads us to

$$P = \frac{1618.46}{7.845802} = 206.2836.$$

The policy value at the end of 5 years can be expressed as

$${}_5V = \text{APV}(\text{FB}_5) - \text{APV}(\text{FP}_5)$$

where

$$\begin{aligned}\text{APV}(\text{FB}_5) &= 10000 \times \left[2 \int_0^{10} v^t \, {}_tp_{35}^{00} \, \mu_{35+t}^{01} \, dt + \int_0^{10} v^t \, {}_tp_{35}^{00} \, \mu_{35+t}^{02} \, dt \right] \\ &= 10000 \times \left[2\,(0.00004412) + 0.01068526 \right] = 1077.351\end{aligned}$$

and

$$\text{APV}(\text{FP}_5) = P \times \int_0^{10} v^t \, {}_tp_{35}^{00} \, dt = P \times 4.412399 = 910.2054.$$

Thus, we have

$${}_5V = 1077.35 - 910.2054 = 167.1451.$$

The R code for the numerical computation required above is provided below for your convenience:

```
x <- 35
muxt01 <- 10^(-5)
muxt02 <- function(t){
out <- A + B*c^(x+t)
out}
tpx00 <- function(t){
temp1 <- exp(-muxt01*t)
temp2 <- exp(-A*t -(B*c^x)*(c^t-1)/log(c))
temp1*temp2}
V5 <- APVFB(5) - P*APVFP(5)

> V5
[1] 167.1451
```