Exercise 7.18

First, we calculate the single premium, and let this be P. Then we have

$$\begin{aligned} P &= \text{APV}(\text{FB}_0) = \int_0^{20} P(1+i)^t v^t {}_tp_{x+t}\mu_{x+t}dt + 50000{}_{20}E_x\bar{a}_{x+20} \\ &= P{}_{20}q_x + 50000{}_{20}E_x\bar{a}_{x+20} \end{aligned}$$

Solving for P, we get

$$P = 50000\frac{v^{20}{}_{20}p_x\bar{a}_{x+20}}{1-{}_{20}q_x} = 50000v^{20}\bar{a}_{x+20}$$

(a) During the deferred period, we have $t \leq 20$. Prospectively, the reserve is the actuarial present value of future benefits minus the actuarial present value of future premiums; because it is a single premium, there are no premiums more to be collected. Thus, we have the prospective reserve formula:

$$\begin{aligned} {}_tV &= \text{APV}(\text{FB}_t) \\ &= \int_0^{20-t} P(1+i)^s v^{s-t}{}_sp_{x+t}\mu_{x+t+s}ds + 50000{}_{20-t}E_{x+t}\bar{a}_{x+20} \\ &= P(1+i)^t{}_{20-t}q_{x+t} + 50000{}_{20-t}E_{x+t}\bar{a}_{x+20} \end{aligned}$$

After the deferred period, $t > 20$, the prospective reserve formula is clearly

$${}_tV = 50000\bar{a}_{x+t}$$

since only the annuity payments are being disbursed during this period.

(b) During the deferred period, we have $t \leq 20$. Retrospectively, the reserve is the actuarial accumulated value of past premiums minus the actuarial accumulated value of past benefits. Thus, we have the retrospective reserve formula:

$$\begin{aligned} {}_tV &= \frac{P}{{}_tE_x} - \frac{\int_0^t P(1+i)^s v^s{}_sp_x\mu_{x+s}ds}{{}_tE_x} \\ &= \frac{P(1-{}_tq_x)}{v^t{}_tp_x} = P(1+i)^t \end{aligned}$$

After the deferred period, $t > 20$, the retrospective reserve formula is

$${}_tV = \frac{P}{{}_tE_x} - \frac{P{}_{20}q_x + 50000{}_{20}E_x\bar{a}_{x+20:\overline{t-20}|}}{{}_tE_x}$$

(c) For $t \leq 20$, starting with the prospective formula, we have

$$\begin{aligned} \text{Prospective} &= P(1+i)^t{}_{20-t}q_{x+t} + 50000{}_{20-t}E_{x+t}\bar{a}_{x+20} \\ &= P(1+i)^t{}_{20-t}q_{x+t} + P(1+i)^{20}v^{20-t}{}_{20-t}p_{x+t} \\ &= P(1+i)^t\left({}_{20-t}q_{x+t} + {}_{20-t}p_{x+t}\right) = P(1+i)^t \\ &= \text{Retrospective} \end{aligned}$$

The second line follows because

$$P = 50000 v^{20} \bar{a}_{x+20} \quad \text{or equivalently} \quad P(1+i)^{20} = 50000 \bar{a}_{x+20}$$

For $t > 20$, starting with the retrospective formula, by substituting

$$\bar{a}_{x+20:\overline{t-20}|} = \bar{a}_{x+20} - {}_{20-t}E_{x+t}\,\bar{a}_{x+t}$$

we have

$$\begin{aligned} \text{Retrospective} &= \frac{P}{{}_tE_x} - \frac{P\,{}_{20}q_x + 50000\,{}_{20}E_x\,\bar{a}_{x+20} - 50000\,{}_{20}E_x\,{}_{20-t}E_{x+t}\,\bar{a}_{x+t}}{{}_tE_x} \\ &= \frac{50000\,{}_tE_x\,\bar{a}_{x+t}}{{}_tE_x} = 50000\bar{a}_{x+t} \\ &= \text{Prospective} \end{aligned}$$

The last line follows because

$$P = P\,{}_{20}q_x + 50000\,{}_{20}E_x\,\bar{a}_{x+20}$$

and that

$${}_{20}E_x\,{}_{20-t}E_{x+t} = {}_tE_x$$