Exercise 7.15

In the subsequent development of the formulas, we shall denote by P the applicable single benefit premium to be paid at time of policy issue.

(a) Starting with ${}_0V = 0$, the recurrence relation between subsequent policy values are:

$$ {}_1V = \frac{({}_0V + P)(1+i) - q_{[60]}}{1 - q_{[60]}} = \frac{P(1+i) - q_{[60]}}{1 - q_{[60]}}, $$

and

$$ {}_{t+1}V = \frac{{}_tV(1+i) - q_{[60]+t}}{1 - q_{[60]+t}}, $$

for $t = 1, 2, \ldots, 18$, and for year 20, we have ${}_{20}V = 0$.

(b) Following the same principle as in (a), start with ${}_0V = 0$ and with steps of h years, we have

$$ {}_hV = \frac{P(1+i)^h - {}_hq_{[60]}}{1 - {}_hq_{[60]}}, $$

and

$$ {}_{t+h}V = \frac{{}_tV(1+i)^h - {}_hq_{[60]+t}}{1 - {}_hq_{[60]+t}}, $$

for $t = h, 2h, \ldots, 20 - 2h$, and for year 20, we have ${}_{20}V = 0$.

(c) From (b), we have

$$ {}_{t+h}V(1 - {}_hq_{[60]+t}) = {}_tV(1+i)^h - {}_hq_{[60]+t}, $$

so that

$$ \frac{{}_{t+h}V - {}_tV}{h} = \frac{1}{h} \times \left\{ {}_tV\left[(1+i)^h - 1\right] - {}_hq_{[60]+t}(1 - {}_{t+h}V) \right\}. $$

By noting that when we let $h \to 0$, we have

$$ \lim_{h\to 0} \left[\frac{{}_{t+h}V - {}_tV}{h} \right] = \frac{d}{dt}\,{}_tV, $$

$$ \lim_{h\to 0} \left[\frac{(1+i)^h - 1}{h} \right] = \delta, $$

$$ \lim_{h\to 0} \frac{{}_hq_{[60]+t}}{h} = \mu_{[60]+t}, $$

and finally, of course,

$$ \lim_{h\to 0}(1 - {}_{t+h}V) = 1 - {}_tV. $$

This leads us to the following desired result:

$$ \frac{d}{dt}\,{}_tV = {}_tV \cdot \delta - \mu_{[60]+t}(1 - {}_tV) = \left(\mu_{[60]+t} + \delta\right) {}_tV - \mu_{[60]+t}. $$

(d) Start with

$$\begin{aligned} {}_tV &= \bar{A}^{\;1}_{[60]+t:\overline{20-t}|} \\ &= \int_0^{20-t} e^{-\delta s}\ {}_sp_{[60]+t}\ \mu_{[60]+t+s} ds \\ &= \frac{1}{e^{-\delta t}\ {}_tp_{[60]}} \int_0^{20-t} e^{-\delta(s+t)}\ {}_{s+t}p_{[60]}\ \mu_{[60]+s+t} ds \end{aligned}$$

Applying a change of variable of integration $r = s + t$ so that $dr = ds$, we then have

$$ {}_tV = \frac{1}{e^{-\delta t}\ {}_tp_{[60]}} \int_t^{20} e^{-\delta r}\ {}_rp_{[60]}\ \mu_{[60]+r} dr. $$

Finally, the next step is to take the derivative of both sides with respect to t and show that this leads us to the differential equation derived in (c). First, we note the following results are easy to verify:

$$\begin{aligned} \frac{d}{dt}\left(e^{-\delta t}\ {}_tp_{[60]}\right)^{-1} &= \left(e^{-\delta t}\ {}_tp_{[60]}\right)^{-2} \times \left[-e^{-\delta t}\left(\ {}_tp_{[60]}\ \mu_{[60]+t}\right) - \delta e^{-\delta t}\ {}_tp_{[60]}\right] \\ &= \frac{1}{e^{-\delta t}\ {}_tp_{[60]}} \times (\delta + \mu_{[60]+t}) \end{aligned}$$

and

$$ \frac{d}{dt} \int_t^{20} e^{-\delta r}\ {}_rp_{[60]}\ \mu_{[60]+r} dr = -e^{-\delta t}\ {}_tp_{[60]}\ \mu_{[60]+t}. $$

Applying product rule of derivative, we then get

$$ \frac{d}{dt}\,{}_tV = \frac{\delta + \mu_{[60]+t}}{e^{-\delta t}\ {}_tp_{[60]}} \times \int_t^{20} e^{-\delta r}\ {}_rp_{[60]}\ \mu_{[60]+r} dr - \frac{1}{e^{-\delta t}\ {}_tp_{[60]}} \times e^{-\delta t}\ {}_tp_{[60]}\ \mu_{[60]+t} $$

Simplifying,

$$\begin{aligned} \frac{d}{dt}\,{}_tV &= (\delta + \mu_{[60]+t}) \times \bar{A}^{\;1}_{[60]+t:\overline{20-t}|} - \mu_{[60]+t} \\ &= (\delta + \mu_{[60]+t}) \times\ {}_tV - \mu_{[60]+t} \\ &= (\mu_{[60]+t} + \delta)\,{}_tV - \mu_{[60]+t}, \end{aligned}$$

which proves the result.