Exercise 7.14

(a) In general, with no expenses, the Thiele's differential equation can be expressed as

$$\frac{d}{dt}\,{}_tV = \delta_t\,{}_tV + P - (S_t - {}_tV)\mu_{[x]+t}.$$

For this problem, premiums are level payable continuously at the rate P. And since $S_t = 20000$ for $0 < t < 10$ and $S_t = {}_tV$ for $10 \leq t < 20$, we can then express the differential equation as

$$\frac{d}{dt}\,{}_tV = (0.06 - 0.001t)\,{}_tV + P - (20000 - {}_tV)\mu_{[40]+t},$$

for $0 < t < 10$ and as

$$\frac{d}{dt}\,{}_tV = (0.06 - 0.001t)\,{}_tV + P,$$

for $10 \leq t < 20$. The boundary conditions are ${}_{20}V = 60000$ for the endowment payment upon reaching age 60, and the initial value of ${}_0V = 0$.

(b) Based on the Euler's method for a fixed step size of h, we can approximate the derivative as

$$\frac{d}{dt}\,{}_tV \approx \frac{1}{h}\left({}_{t+h}V - {}_tV\right).$$

Starting with the final boundary condition of ${}_{20}V = 60000$ and a step size of h, we can then solve for P using backward recursion based on the following approximation:

$${}_tV = \frac{{}_{t+h}V - hP}{h(0.06 - 0.001t) + 1},$$

for $10 \leq t < 20$ and

$${}_tV = \frac{{}_{t+h}V - hP + 20000h\mu_{[40]+t}}{h(0.06 - 0.001t) + h\mu_{[40]+t} + 1},$$

for $0 < t < 10$. We can solve for P using these approximate formulas by iteratively calculating for P that will yield us an initial reserve of ${}_0V = 0$, working backwards with the reserve value of ${}_{20}V = 60000$. The following R code solves for this premium P:

```
S <- 20000
deltak <- function(t){
.06 - .001*t}
h <- 0.05
k <- seq(0,20,h)
n <- length(k)
Vt <- rep(1,n)
Vt[n] <- 60000
```

```
# need to write the mu function next
mu <- function(x,t){
A <- .00022
B <- 2.7*10^(-6)
c <- 1.124
mutemp <- A + B*c^(x+t)
out <- ifelse(t<=2, 0.9^(2-t)*mutemp,mutemp)
out}
# express the initial reserve as a function of premium rate P
V0 <- function(P) {
m <- n
while (m>which(k==10)) {
m <- m-1
num <- Vt[m+1] - h*P
den <- h*deltak(k[m]) + 1
Vt[m] <- num/den
}
m <- which(k==10)
while (m>1) {
m <- m-1
num <- Vt[m+1] - h*P + h*S*mu(40,k[m])
den <- h*deltak(k[m]) + h*mu(40,k[m]) + 1
Vt[m] <- num/den
}
Vt[1]}
# the following solves for P such that initial reserve V0=0
P <- uniroot(V0,c(0,60000))$root
```

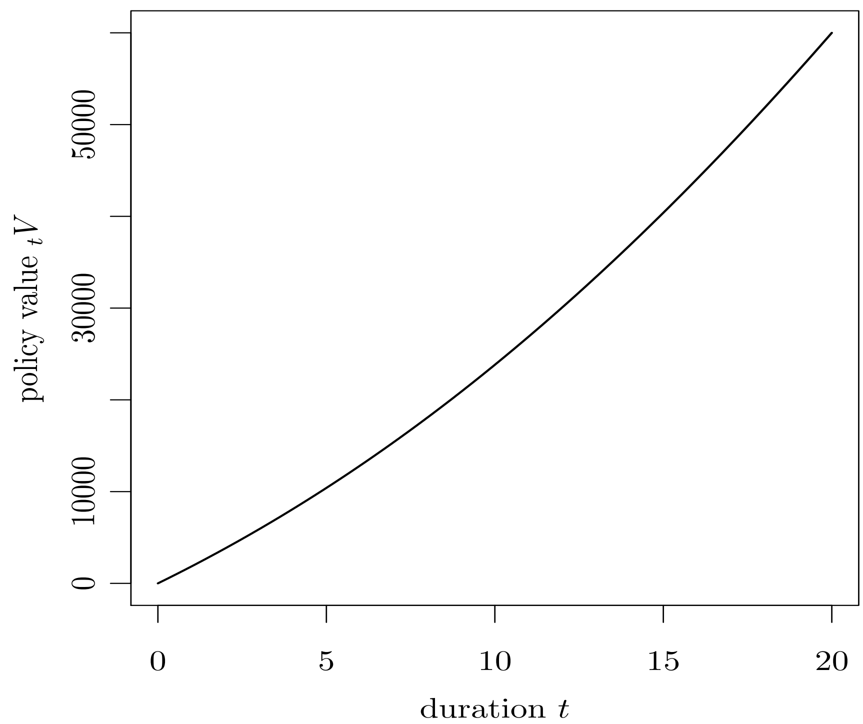
This produces the following result:

```
> P
[1] 1810.726
```

(c) The following table displays the policy values for integral years:

t	${}_tV$	t	${}_tV$
0	0.0000		
1	1853.8638	11	26882.9244
2	3817.7860	12	30070.8515
3	5894.9409	13	33384.7327
4	8088.4838	14	36823.9198
5	10400.9021	15	40387.3491
6	12834.5166	16	44073.5260
7	15391.4745	17	47880.5110
8	18073.7445	18	51805.9074
9	20883.1160	19	55846.8507
10	23821.2011	20	60000.0000

The following figure displays the representation of the policy values over the 20-year policy period:



The increasing pattern in the policy values observed in this case should not at all be surprising. In the first 10 years, we expect the increase because of the increasing probabilities of death with age, and for periods from 10 to 20 years, on top of these increasing probabilities, the benefits are also increasing with age because of the addition of the policy value included as benefit. Hence, we observe an even slightly sharper increase in policy values beyond 10 years.