MATH 3631
Actuarial Mathematics II
Final Examination
Monday, 1 May 2017
Time Allowed: 2 hours (6:00 - 8:00 pm)
Room: AUST 110
Total Marks: 120 points

Please write your name and student number at the spaces provided:

Name: EMIL **Student ID:** Suggested Solutions

- There are twelve (12) written-answer questions here and you are to answer all questions. Each question is worth 10 points. Your final mark will be divided by 120 to convert to a unit of 100%.
- Please provide details of your workings in the appropriate spaces provided; partial points will be granted.
- Please write legibly.
- Anyone caught writing after time has expired will be given a mark of zero.
- Good luck to everyone. Congratulations, and all the best, to those graduating.
- Have a safe and enjoyable summer!

Question	Worth	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
Total	120	
%	$\div$ 120	

Question No. 1:

$\to = 1.25(2)\dfrac{A_{40}}{\ddot{a}_{40}} = \cancel{2.5}\,2.5\left(\dfrac{.16132}{14.8166}\right) = \cancel{.03266936}\ .02721947 = P$

An insurer sells ~~a~~ portfolio of fully discrete whole life insurance, ~~with~~ death benefit of 2, each to the same age 40. You are given:

- All policies have independent future lifetimes.
- The annual contract premium is 125% of the annual net premium calculated based on the equivalence principle.
- Mortality follows the `Illustrative Life Table`.
- $i = 0.06$ $\quad d = .06/1.06$
- The 95th percentile of a standard normal distribution is 1.645.

Using the normal approximation, calculate the smallest number of policies to sell so that this insurer has at least 95% probability of a gain from this portfolio of policies.

Let the loss per policy be $L_{0,i} = 2v^{K+1} - P\ddot{a}_{\overline{K+1}|} = \left(2+\frac{P}{d}\right)v^{K+1} - P/d$

$E[L_{0,i}] = \left(2+\frac{P}{d}\right)A_{40} - \frac{P}{d} = \cancel{0.1613226}\ -.08066218$

$Var[L_{0,i}] = \left(2+\frac{P}{d}\right)^2\left({}^2A_{40} - A_{40}^2\right) = \cancel{0.1507942}\ 0.1391335$

${}^2A_{40} = .04863$

$L_{agg} = \sum_{i=1}^{N} L_{0,i} \Rightarrow E[L_{agg}] = \cancel{0.1613226}\ -.08066218\,N$

$Var[L_{agg}] = \cancel{0.1507942}\ 0.1391335\,N$

$Pr[L_{agg} < 0] \geq 0.95 \Rightarrow Pr\left[Z < \dfrac{\cancel{0.1613226}\ -.08066218\,N}{\sqrt{\cancel{0.1507942N}\ 0.1391335N}}\right] \geq 0.95$

prob of a gain

$\Rightarrow \cancel{+0.1613226}\ .08066218\sqrt{N} \geq 1.645\sqrt{\cancel{0.1507942}\ 0.1391335}$

$\Rightarrow \sqrt{N} \geq \dfrac{1.645\sqrt{\cancel{0.1507942}\ 0.1391335}}{\cancel{0.1613226}\ .08066218}$

$\Rightarrow N \geq \cancel{15.67927}\ 57.866$ sell at least ~~16~~ 58 policies

Question No. 2:

For a fully discrete whole life insurance of 25,000 issued to (50), you are given:

- The following expenses are payable at the beginning of the year:

	% of premium	Per policy
First year	25%	15
Renewal years	10%	5

- Gross annual premium is determined according to the actuarial equivalence principle.
- $i = 3\%$
- $A_{50} = 0.479$

$$\ddot{a}_{50} = \frac{1-A_{50}}{d} = \frac{1-.479}{.03/1.03} = 17.88767$$

- The tenth-year gross premium reserve is ${}_{10}V^g = 5810$.

Calculate A_{60}.

$$G\ddot{a}_{50} = 25000A_{50} + .15G + .10G\ddot{a}_{50} + 10 + 5\ddot{a}_{50}$$

$$G = \frac{25000A_{50} + 10 + 5\ddot{a}_{50}}{.90\ddot{a}_{50} - .15} = 757.0703$$

$${}_{10}V^g = 25000A_{60} + .10G\ddot{a}_{60} + 5\ddot{a}_{60} - G\ddot{a}_{60}$$

$$= 25000A_{60} - .90G\ddot{a}_{60} + 5\ddot{a}_{60} \qquad \ddot{a}_{60} = \frac{1-A_{60}}{d}$$

$$5810 = \left(25000 + \frac{.90G - 5}{d}\right)A_{60} + \frac{.90G - 5}{d}$$

Solving for A_{60}, we get

$$A_{60} = \frac{5810 + \frac{.90G-5}{d}}{25000 + \frac{.90G-5}{d}} = \frac{29031.81}{48221.81} = \underline{\underline{0.6020473}}$$

Question No. 3:

For a fully discrete whole life insurance policy of 100 on (40), you are given:

(i) The gross annual premium is 3.2.

(ii) For calculating gross premium reserves in year 10, the following assumptions are made:

- The 9th-year gross premium reserve is ${}_9V^g = 8.5$.
- $q_{49} = 0.001$
- Annual expenses of 2, payable at the beginning of the year
- $i = 0.05$

(iii) Actual experience during year 10 for this policy is:

- The policy is still inforce and active at the end of year 10.
- The annual expenses are 3, paid at the beginning of the year.
- The actual interest earned is 4.5%.

(iv) Gain (or loss) for this single policy is calculated in the following order: mortality, then expenses, then interest.

Calculate the gain (or loss) from each source (mortality, expense, interest) for this policy in year 10.

$${}_{10}V^g = ? \Rightarrow \frac{({}_9V^g + G - 2)(1.05) - 100\, q_{49}}{1 - q_{49}} = \underbrace{\frac{(8.5+3.2-2)(1.05) - 100(.001)}{1-.001}}_{10.0951}$$

gain from mortality: $(100 - {}_{10}V^g)(q_{49} - 0) = .0899049$

gain from expenses: $(2-3)(1+.05) = -1.05$

gain from interest: $({}_{10}V^g + G - 3)(.045 - .05) = -.05147548$

total gain/loss = $\underline{\underline{-1.011571}}$ loss!

Question No. 4:

Hospital patients are classified as Sick (S), Critical (C), or Discharged (D). Transitions, in days, occur according to the following time-homogeneous probability matrix:

$$\begin{array}{c} \\ S \\ C \\ D \end{array} \begin{array}{c} \begin{array}{ccc} S & C & D \end{array} \\ \begin{pmatrix} 0.60 & 0.20 & 0.20 \\ 0.20 & 0.50 & 0.30 \\ 0.00 & 0.00 & 1.00 \end{pmatrix} \end{array}$$

It costs 2,000 for each day a 'Sick' patient is in the hospital. It costs 10,000 for each day a 'Critical' patient is in the hospital. It costs nothing to be in 'Discharged' state. Assume payment for each stay in the hospital is made at the end of each day.

You are given: $i = 0.0$

For a 'Sick' patient admitted to the hospital today, calculate the actuarial present value of the cost of being in the hospital for the next 3 days.

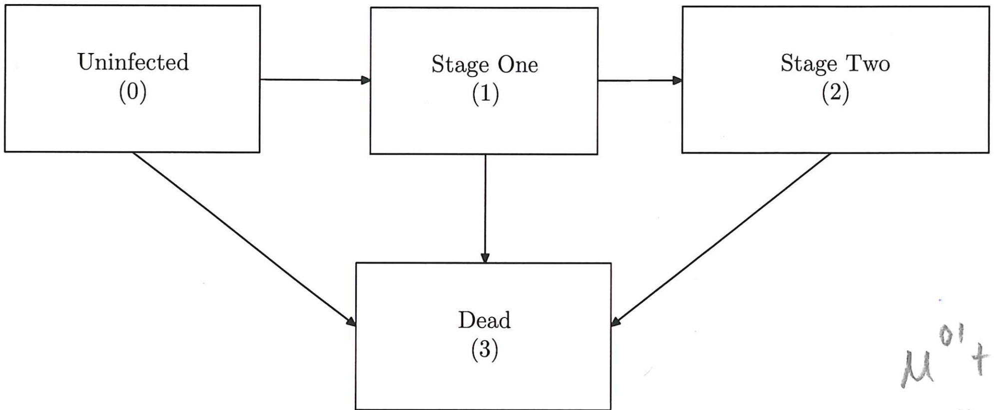
Possible transitions	probability	CF	product
$S \to S \to S$	$(.6)^2 = .36$	$2000 \times 3 = 6000$	2160
$S \to S \to C$	$.6(.2) = .12$	$2000 \times 2 + 10000 = 14000$	2240
$S \to C \to S$	$.2(.2) = .04$	$2000 \times 2 + 10000 = 14000$	
$S \to C \to C$	$.2(.5) = .10$	$2000 + 10000 \times 2 = 22000$	2200
$S \to D$	$.2$	2000	400
$S \to S \to D$	$.6(.2) = .12$	$2000 \times 2 = 4000$	480
$S \to C \to D$	$.2(.3) = .06$	$2000 + 10000 = 12000$	720

APV = sum

= 8200

Question No. 5:

A disease progresses according to the following Markov model:



$\mu^{01} + \mu^{03} = .003$

$\mu^{13} + \mu^{12} = .15$

All forces of transitions are constant and independent of age:

$$\mu^{01} = 0.001, \quad \mu^{12} = 0.100, \quad \mu^{03} = 0.002, \quad \mu^{13} = 0.050, \quad \text{and} \quad \mu^{23} = 0.400$$

Calculate the probability that a person 'Uninfected' today will be in 'Stage Two' at the end of 10 years.

0 2

$0 \longrightarrow 0,1 \longrightarrow 1,2 \longrightarrow$

0 s t 10

$$\text{Probability} = \int_0^{10}\int_0^t {}_sp^{\overline{00}}\,\mu_{01} \cdot {}_{t-s}p^{\overline{11}}\,\mu_{12}\,ds\,dt$$

$$e^{-.003s} \quad 0.001 \quad e^{-.15(t-s)} \quad 0.10$$

$$= (.0001)\int_0^{10} e^{-.15t}\int_0^t e^{+.147s}\,ds\,dt$$

$$= \frac{.0001}{.147}\int_0^{10} e^{-.15t}\left(e^{.147t}-1\right)dt$$

$$= \underbrace{\frac{1}{147}\left[\frac{1}{.003}\left(1-e^{-.03}\right) - \frac{1}{.15}\left(1-e^{-1.5}\right)\right]}_{0.03178474}$$

Question No. 6:

An employee, now age 50, receives the following one-year term benefits, payable at the end of the year of decrement:

- 100 if decrement results from cause 1;
- 200 if decrement results from cause 2; and
- 500 if decrement results from cause 3.

Only three possible decrements exist. In their associated single-decrement tables, all three decrements follow de Moivre's Law with $\omega = 100$. In addition, assume UDD in each single decrement table. You are given $i = 0.05$.

Find the actuarial present value at age 50 of the benefits.

$$\text{APV(benefits)} = 100\, v\, q_{50}^{(1)} + 200\, v\, q_{50}^{(2)} + 500\, v\, q_{50}^{(3)}$$

$$q_{50}^{(1)} = q_{50}^{\prime(1)} \int_0^1 (1 - t/50)(1 - t/50)\, dt$$

$$\frac{1}{50}\left(\frac{50}{3} - \frac{1}{3}\frac{49^3}{50^2}\right) = \frac{1}{3} - \frac{1}{3}\frac{49^3}{50^3} = \underbrace{\frac{1}{3}\left(1 - \left(\frac{49}{50}\right)^3\right)}_{.0196026\,7}$$

$$= q_{50}^{(2)} = q_{50}^{(3)}$$ same for all 3 decrements

$$= 800 \frac{1}{1.05}(.0196026\,7)$$

$$= \underline{\underline{14.93537}}$$

Question No. 7:

You are given the following extract from a triple decrement model: .05

x	$\ell_x^{(\tau)}$	$q_x^{(1)}$	$q_x^{(2)}$	$q_x^{(3)}$
55	-	0.02	0.04	0.06
56	~~8800~~ 8700	0.05	0.10	0.15

Suppose that after preparing this table, you corrected that $q_{55}^{(2)}$ should have been 0.05 while all other probability values in the table above remain the same.

Calculate the correct value of $d_{56}^{(3)}$.

$$p_{55}^{(\tau)} = 1-(.02+.04+.06) = 0.88 \Rightarrow \ell_{56}^{(\tau)} = \frac{\ell_{55}^{(\tau)}}{p_{55}^{(\tau)}} = \frac{8800}{.88} = 10000$$

Thus,

$$\ell_{56}^{(\tau)} = 10000(1-(.02+.05+.06)) = 8700$$

and

$$d_{56}^{(3)} = 8700\, q_{56}^{(3)} = 8700(0.15) = \underline{\underline{1305}}$$

Question No. 8:

For a Type A Universal Life policy with a total death benefit of 200,000, you are given:

policy year	annual premium deposit	percent of premium charge	annual fixed expense charge	annual cost of insurance rate per 1,000	interest credited
1	3,500	5%	50	2.0	10%
2	5,000	5%	50	3.0	10%

If you surrender your policy, there is a penalty of 5% of the account value.

Calculate the surrender value for this policy at the end of the second year.

$$AV_1 = \underbrace{(3500(.95) - 50)(1.10)}_{3602.5} - (200000 - AV_1)\left(\frac{2}{1000}\right)$$

$$- 400 + \frac{2}{1000} AV_1$$

$$AV_1 = \frac{3202.5}{1 - 2/1000} = 3208.918$$

$$AV_2 = \underbrace{(3208.918 + 5000(.95) - 50)(1.10)}_{8699.81} - (200000 - AV_2)\frac{3}{1000}$$

$$- 600 + \frac{3}{1000} AV_2$$

$$AV_2 = \frac{8099.81}{1 - 3/1000} = 8124.183$$

With 5% penalty, the

surrender value $= AV_2 * .95 = \underline{\underline{7717.973}}$

Question No. 9:

Suppose you are now age 59 and you buy a Type B universal life policy with an additional death benefit of 10,000. You pay an annual premium of 4,000. You are given:

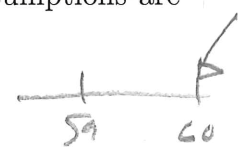
policy year	percent of premium charge	annual expense charge	annual COI rate per 1 of insurance	annual discount rate for COI	annual credited interest rate
1	10%	5	0.030	4.0%	6.0%

At the end of the first year, you decided to surrender your policy (zero surrender charge) and to use your account value to pay for a special life annuity due with a net single premium that provides the following benefit payments at the beginning of each year:

- For the first 10 years, a payment of k.
- After the first 10 years, a payment of $2k$.

In calculating this net single premium for this special annuity, the following assumptions are made:

- Mortality follows the `Illustrative Life Table`.
- $i = 0.06$

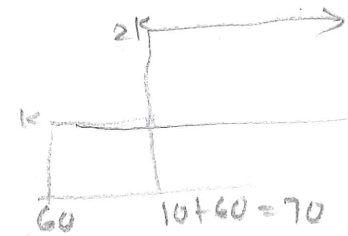


Calculate the value of k.

$$AV_1 = \left(4000(.90) - 5 - \frac{4000}{1.04}(.03)\right)(1.06) = 3688.392$$

$$APV(\text{annuity}) = k\,\ddot{a}_{60} + k\,{}_{10}E_{60}\,\ddot{a}_{70}$$

equal



$$= k\left(\ddot{a}_{60} + {}_{10}E_{60}\,\ddot{a}_{70}\right)$$

$$k = \frac{3688.392}{\ddot{a}_{60} + {}_{10}E_{60}\,\ddot{a}_{70}} = \frac{3688.392}{15.01187} = \underline{\underline{245.6984}}$$

$\ddot{a}_{60}$: 11.1454

${}_{10}E_{60}$: .45120

$\ddot{a}_{70}$: 8.5693

Question No. 10:

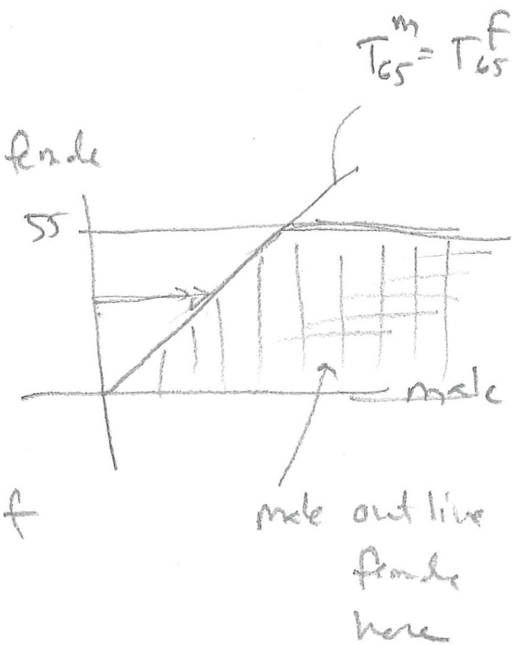
You are given:

- Male mortality is based on a constant force of mortality with $\mu = 0.05$.
- Female mortality follows deMoivre's law with $\omega = 120$.
- Assume for any pairs of male and female, their future lifetimes are independent.

Calculate the probability that a male age 65 dies after a female age 65.

$T_{65}^{m} \sim$ exponential $\mu = .05$

$T_{65}^{f} \sim$ uniform on $(0, 55)$



$P[T_{65}^{m} > T_{65}^{f}]$ take complement

$$= 1 - \int_0^{55}\int_0^{f} .05e^{-.05m}\frac{1}{55}\,dm\,df$$

$$= 1 - \int_0^{55} \frac{1}{55}(1 - e^{-.05f})\,df$$

$$= \cancel{1 - \frac{1}{55}(55)} + \frac{1}{.05}(1 - e^{-.05(55)})\frac{1}{55}$$

$$= \underline{\underline{0.3403899}}$$

Question No. 11:

Kanye and Kim are both age 50 and both have independent future lifetimes. They purchase a special 15-year deferred joint and last survivor annuity policy. This policy has the following features:

- An annuity of 50,000 is payable at the beginning of each year starting at age 65 if both Kanye and Kim are alive, increasing to 75,000 per year after the first death continuing to this amount until the last death.
- The annual premium of π is payable at the beginning of each year that both Kanye and Kim are alive during the deferred period (i.e. the first 15 years).
- Mortality follows the `Illustrative Life Table`.
- $i = 0.06$

Calculate π.

benefit 50K → 75K

premium π

50 65

$$APV(\text{premium}) = \underbrace{\ddot{a}_{50:50:\overline{15}|}}_{9.329183} \times \pi$$

$$\ddot{a}_{50:50} - {}_{15}E_{50:50}\, \ddot{a}_{65:65}$$

$\ddot{a}_{50:50} = 11.6513$, $\ddot{a}_{65:65} = 7.8552$

$${}_{15}E_{50:50} = {}_{15}E_{50} \times \frac{l_{65}}{l_{50}}, \quad l_{65} = 7533964, \quad l_{50} = 8950901$$

$${}_{15}E_{50} = {}_{10}E_{50} \times {}_5E_{60} = .51081 \times .68756$$

$$APV(\text{benefits}) = \left(50000\, \ddot{a}_{65:65} + 75000\, \underbrace{\ddot{a}_{\overline{65|65}}}_{\ddot{a}_{65} - \ddot{a}_{65:65}}\right) \times {}_{15}E_{50:50}$$

same above

$$= 75000\, \underset{9.8969}{\ddot{a}_{65}} - 25000\, \underset{7.8552}{\ddot{a}_{65:65}}$$

$$= 161372.6$$

$$\pi = \frac{161372.6}{9.329183} = 17297.62$$

Question No. 12:

For a life insurance policy issued to (40) and (50), you are given:

- Both lives have independent future lifetimes.
- Death benefit of 10,000 is payable at the end of the year of the first death.
- Death benefit of 25,000 is payable at the end of the year of the second death.
- Mortality follows the `Illustrative Life Table`.
- $i = 0.06$

Calculate the actuarial present value of this insurance.

$$APV(\text{insurance}) = 10000\, A_{40:50} + 25000\, \underbrace{A_{\overline{40:50}}}_{A_{40} + A_{50} - A_{40:50}}$$

$$= 25000(A_{40} + A_{50}) - 15000\, A_{40:50}$$

$$= 25000(.16132 + .24905) - 15000(.29368)$$

$$= \underline{\underline{5854.05}}$$

EXTRA PAGE FOR ADDITIONAL OR SCRATCH WORK