MATH 3631
Actuarial Mathematics II
Final Examination
Tuesday, 3 May 2016
Time Allowed: 2 hours (1:00 - 3:00 pm)
Room: LH 305
Total Marks: 120 points

Please write your name and student number at the spaces provided:

Name: EMIL **Student ID:** Suggested Solutions

- **There are thirteen (13) written-answer questions here and you are to answer only the first twelve. Question 13 is a bonus problem. Each question is worth 10 points. Your final mark will be divided by 120 to convert to a unit of 100%.**
- **Please provide details of your workings in the appropriate spaces provided; partial points will be granted.**
- **Please write legibly.**
- **Anyone caught writing after time has expired will be given a mark of zero.**
- **Good luck to everyone. Congratulations to those graduating.**
- **Have a safe and enjoyable summer!**

Question	Worth	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
13	10	
Total	130	
%	÷ 120	

Question No. 1:

For a special fully discrete 10-year endowment insurance of $20,000 on (50), you are given:

- Mortality follows the `Illustrative Life Table`.
- $i = 0.06$
- Premium is determined according to the equivalence principle.

Now suppose, at issue, the policyholder was offered an optional 'return of premium' feature which provides an additional death benefit equal to the return of net annual premiums accumulated with interest at 6% to the end of the year of death.

Calculate the increase in the net annual premium because of this optional feature.

Premium without feature

$$P = 20{,}000 \frac{A_{50:\overline{10}|}}{\ddot{a}_{50:\overline{10}|}} \quad A_{50:\overline{10}|} = 1 - d\ddot{a}_{50:\overline{10}|}$$

$$\ddot{a}_{50:\overline{10}|} = \ddot{a}_{50} - {}_{10}E_{50}\,\ddot{a}_{60}$$

$$= 20{,}000\left(\frac{1}{\ddot{a}_{50:\overline{10}|}} - d\right) = 1508.67 \quad d = .06/1.06$$

$$= 13.2668 - .51081(11.1454)$$

$$= 7.573618$$

Premium with optional feature

$$L_0 = \begin{cases} 20000v^{K+1} + \cancel{P^*\ddot{s}_{\overline{K+1}|}v^{K+1} - P^*\ddot{a}_{\overline{K+1}|}}, & K < 10 \\ 20000v^{10} - P^*\ddot{a}_{\overline{10}|}, & K \geq 10 \end{cases}$$

Thus, $E[L_0] = 20000\,A_{50:\overline{10}|} - P^*\ddot{a}_{\overline{10}|}\,{}_{10}p_{50} = 0$

$$A_{50:\overline{10}|} = 1 - d\ddot{a}_{50:\overline{10}|} = 0.5713046$$

$$\Rightarrow \quad P^* = \frac{20000\,A_{50:\overline{10}|}}{\ddot{a}_{\overline{10}|}\,{}_{10}p_{50}}$$

$$\ddot{a}_{\overline{10}|} = \frac{1-v^{10}}{d} = 7.801692$$

$$= \frac{20000(0.5713046)}{7.801692(0.9147765)} = 1601.01$$

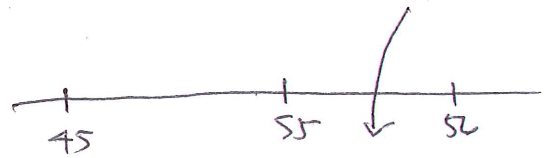
$${}_{10}p_{50} = \frac{l_{60}}{l_{50}} = \frac{8188074}{8950901} = 0.9147765$$

$$P^* - P = \text{increase} = 1601.01 - 1508.67 = \underline{\underline{92.34}}$$

Question No. 2:

For a fully discrete whole life insurance issued to (45), you are given:

- The death benefit is $1,500.
- $i = 5\%$
- $q_{55} = 0.0268$
- $q_{56} = 0.0288$
- ${}_{10.5}V = 343.29$
- Deaths are uniformly distributed over each year of age.



Calculate ${}_{10.7}V$.

$$343.29(1-0.5\,q_{55}) = \underbrace{({}_{10}V+P)}(1.05)^{0.5} - 1500 \times 0.5\,q_{55}\,v^{0.5}$$

$${}_{10}V+P = \underbrace{\frac{343.29(1-.5(0.0268)) + 1500(.5(.0268))\,v^{0.5}}{1.05^{0.5}}}_{349.6704}$$

Similarly,

$${}_{10.7}V = \underbrace{\frac{({}_{10}V+P)(1.05)^{0.7} - 1500 \times 0.7(.0268)\,v^{0.7}}{1-0.7(.0268)}}$$

$$= \frac{334.6238}{0.98124} = \underline{\underline{341.0214}}$$

Question No. 3:

For a fully discrete whole life policy of \$10,000 issued to (45), you are given:

- Expenses, incurred at the beginning of each year, are summarized below:

	% of premium	Per policy
First year	15%	1.5
Renewal years	5%	0.5

- Gross premium is determined according to the actuarial equivalence principle.
- $i = 4\%$
- $\ddot{a}_{45} = 17.8$
- $\ddot{a}_{50} = 17.0$

$$A_{45} = 1 - \frac{.04}{1.04}(17.8)$$
$$= \cancel{0.3461538}\ 0.3153846$$

Calculate ${}_5V^g$, the fifth-year gross premium reserve.

Calculate G = gross annual premium

$$G\ddot{a}_{45} = 10000A_{45} + 0.10G + .05G\ddot{a}_{45} + 1.0 + 0.5\ddot{a}_{45}$$

$$G = \frac{10000A_{45} + 1.0 + 0.5\ddot{a}_{45}}{0.95\ddot{a}_{45} - 0.10} = \frac{10000(\cancel{0.3461538}\ 0.3153846) + 1.0 + 0.5(17.8)}{0.95(17.8) - 0.10}$$

$$= 188.2062 \checkmark$$

$${}_5V^g = 10000A_{50} - G\ddot{a}_{50} + .05G\ddot{a}_{50} + .5\ddot{a}_{50}$$

$$= 10000\underbrace{\left(1 - \frac{.04}{1.04}17.0\right)}_{0.3461538 \checkmark} - 188.2062 * 17.0 * 0.95 + .5(17.0)$$

$$= \cancel{262.0331}\ \ 430.5084$$

25

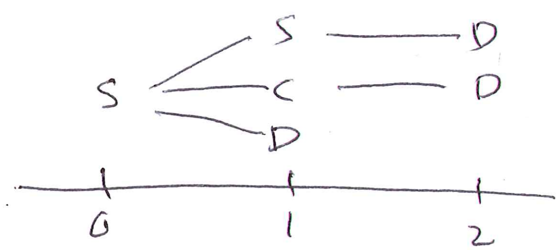
Question No. 4:

Hospital patients are classified as Sick (S), Critical (C), or Discharged (D). Transitions, in days, occur according to the following time-homogeneous probability matrix:

$$
\begin{array}{cc}
 & \begin{array}{ccc} \mathrm{S} & \mathrm{C} & \mathrm{D} \end{array} \\
\begin{array}{c} \mathrm{S} \\ \mathrm{C} \\ \mathrm{D} \end{array} &
\begin{pmatrix} 0.65 & 0.20 & 0.15 \\ 0.25 & 0.50 & 0.25 \\ 0.00 & 0.00 & 1.00 \end{pmatrix}
\end{array}
$$

Suppose today that there are exactly 10 sick patients in the hospital. The state of each patient is independent of the state of any other patient.

Calculate the probability that exactly 2 of these 10 sick patients will be discharged by the end of two days from today.



$S \to D$ 0.150

$S \to S \to D$ $0.65(0.15) = 0.0975$

$S \to C \to D$ $0.20(0.25) = 0.0500$

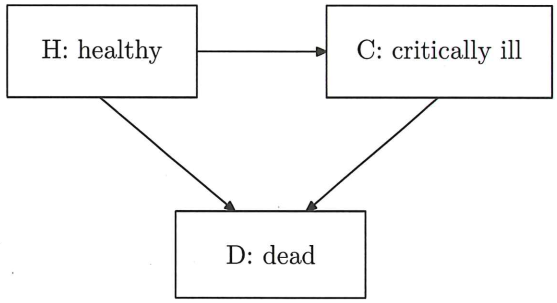
0.2975

probability 2 out of 10 $= \binom{10}{2}(.2975)^2(1-.2975)^8$

$= \underline{\underline{0.236242}}$

Question No. 5:

A life insurer uses the following three-state model to price a two-year critical illness policy issued to healthy policyholders at time $t = 0$:



You are given:

- The constant forces of transition are:

$$\mu^{HC} = 0.02 \qquad \mu^{HD} = 0.01 \qquad \mu^{CD} = 0.04$$

- The policy pays \$300 at the moment of death of the policyholder from the critically ill state. No other benefits are provided.
- $\delta = 5\%$

Calculate the actuarial present value of the benefits for this critical illness policy.

$$APV = \int_0^2 300 \int_t^2 e^{-.05s} \underbrace{{}_tp^{HH}}_{e^{-.03t}} \underbrace{\mu^{HC}}_{0.02} \cdot \underbrace{{}_{s-t}p^{CC}}_{e^{-.04(s-t)}} \underbrace{\mu^{CD}}_{0.04} \, ds \, dt$$

$$= 300\,(0.02) \int_0^2 e^{-.03t} e^{-.09t} \frac{.04}{.09} \underbrace{\int_t^2 0.09 e^{-.09(s-t)} \, ds}_{} \, dt$$

transform $u = s-t$

$$\int_0^{2-t} .09 e^{-.09u} \, du$$

$$= \frac{300\,(0.02)(0.04)}{0.09} \int_0^2 e^{-.12t} \left(1 - e^{-.09(2-t)}\right) dt$$

$$= \underline{\underline{0.4178386}}$$

Question No. 6:

Tom, who is exactly age 60 today, joins ABC Corporation. His annual salary is $500,000 and will remain constant each year. For simplicity, assume that retirement takes place on a birthday.

ABC offers a pension plan to Tom with the following benefits:

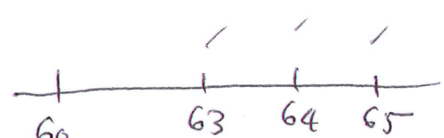
- a retirement benefit equal to $50,000 at time of retirement if he retires at age 63, or $75,000 at time of retirement if he retires at age 64, or $100,000 at time of retirement if he retires at age 65.
- no other benefits are provided.

To value the benefits, a force of interest $\delta = 4\%$ and the following multiple decrement table (w = withdrawal; r = retirement; and d = death) will be used:

age x	$\ell_x^{(\tau)}$	$d_x^{(w)}$	$d_x^{(r)}$	$d_x^{(d)}$
60	710	30	0	35
61	645	10	0	40
62	595	5	0	45
63	545	5	70	50
64	420	0	80	55
65	285	0	285	0

$V = e^{-.04}$

Calculate the actuarial present value of Tom's retirement benefits.



$$APV = 50000\,v^3 \frac{d_{63}^{(r)}}{\ell_{60}^{(\tau)}} + 75000\,v^4 \frac{d_{64}^{(r)}}{\ell_{60}^{(\tau)}} + 100000\,v^5 \frac{d_{65}^{(r)}}{\ell_{60}^{(\tau)}}$$

$$= 25000\,v^3 \left(2\,\frac{70}{710} + 3v\,\frac{80}{710} + 4v^2\,\frac{285}{710} \right)$$

$$= \underline{\underline{44{,}437.90}}$$

Question No. 7:

An insurer issued 4,000 fully discrete whole life insurance policies to lives all exactly age 50 on January 1, 2006. Each policy issued has a death benefit of \$100,000 with an annual gross premium of \$2,600.

You are given:

- The following values in Year 2015:

	anticipated	actual
Expenses as a percent of premium	0.05	0.06
Annual effective rate of interest	0.02	0.05
q_{59}	0.0085	0.0090

- The gross premium reserves per policy at the end of Year 2014 and Year 2015, respectively, are:

$$_9V = 17,033 \text{ and } _{10}V = 19,206$$

- A total of 3,851 remain in force at the beginning of Year 2015.
- Gains and losses are calculated in the following order: interest then expenses then mortality.

Calculate the gain (or loss) from each source (interest, expenses, mortality) for this portfolio of policies in Year 2015.

3851

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interest: $3851(17033+2600(.95))(.05-.02)$

$= 2,253,182$ gain

expenses: $3851(2600(.05-.06))(1.05) = -105,132.3$ loss

mortality: $3851(10000-19206)(.0085-0.0090) = -155,568.8$ loss

Question No. 8:

For two lives (x) and (y) with independent future lifetimes, you are given:

- ${}_nq_x = 0.0400$
- ${}_nq_{xy} = 0.1216$

Calculate ${}_nq_{\overline{xy}}$.

First solve for ${}_nq_y$ by noting that

$${}_nq_{xy} = 1 - {}_np_{xy} = 1 - {}_np_x \, {}_np_y$$

$$\Rightarrow \quad {}_np_y = \frac{1 - {}_nq_{xy}}{{}_np_x} = \frac{1 - 0.1216}{1 - 0.04} = .915$$

$${}_nq_y = 1 - 0.915 = 0.085$$

Thus, $${}_nq_{\overline{xy}} = {}_nq_x + {}_nq_y - {}_nq_{xy}$$

$$= 0.04 + 0.085 - 0.1216$$

$$= \underline{\underline{0.0034}}$$

Question No. 9:

For each of the following expressions, write down the standard actuarial symbol and explain in words what it means.

(a) $\int_0^\infty \bar{a}_{\overline{t}|} \cdot {}_tp_{xy} \cdot \mu_{x+t:y+t} dt$

(b) $\int_0^\infty v^t \bar{A}_{x+t}\, {}_tp_{xy}\, \mu_{y+t} dt$

(c) $\mathrm{E}[Z]$ where $Z = \begin{cases} \bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_x}|}, & T_x \leq T_y \\ 0, & T_x > T_y \end{cases}$

(a) $\Rightarrow$ $\bar{a}_{xy}$ = APV of a continuous annuity of 1 per year payable so long as both (x) and (y) are alive

(b) $\Rightarrow$ $\bar{A}_{\overset{2}{x}y}$ = APV of an insurance of 1 paid at the moment of death of (x) provided (y) predeceases (x)

(c) $\Rightarrow$ $\bar{a}_{x|y}$ = APV of a reversionary annuity of 1 per year payable continuously to (y) commencing at the moment of death of (x)

Question No. 10:

For a Type B Universal Life policy with additional death benefit of $5,000 issued to (50), you are given:

- Expense charges in each year are 1.5% of premium plus $20.
- The cost of insurance rate is equal to 125% of the mortality rate at the attained age based on the `Illustrative Life Table`.
- $i^c = 6\%$ for all years
- $i^q = 5\%$ for all years
- The account value at the end of the first year is equal to $1,194.37.
- The corridor factor requirement is a minimum of 2.0 each year.

Calculate the largest amount of premium this policyholder can pay at the beginning of the second year.

$$AV_2 = \underbrace{\left(1194.37 + \pi_1(1-.015) - 20 - 1.25 * \frac{6.42}{1000}\frac{1}{1.05}5000\right)(1.06)}_{\leq 5000}$$

$$cf = \frac{AV_2 + 5000}{AV_2} \geq 2.0 \Rightarrow AV_2 \leq 5000$$

Solving for π_1, we get

$$\pi_1 \leq \underbrace{\frac{5000 - \left(1194.37 - 20 - 1.25 * \frac{6.42}{1000}\frac{1}{1.05}5000\right)(1.06)}{(1-.015)(1.06)}}$$

$$\underline{\underline{3635.356}}$$

Question No. 11:

For a Type A Universal Life policy with a total death benefit of $100,000, you are given:

policy year	annual premium deposit	percent of premium charge	annual fixed expense charge	annual cost of insurance rate per 1,000	interest credited
1	$2,500	10%	$25	2.5	5%
2	$2,750	5%	$25	3.0	5%

Calculate the account value at the end of the second year.

$$AV_1 = \frac{\left(2500(.90) - 25 - 100000\left(\frac{2.5}{1000}\right)\left(\frac{1}{1.05}\right)\right)(1.05)}{1 - 2.5/1000}$$

$$= 2091.479$$

$$AV_2 = \frac{\left(2091.479 + 2750(.95) - 25 - 100000\left(\frac{3}{1000}\right)\left(\frac{1}{1.05}\right)\right)(1.05)}{1 - 3/1000}$$

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$$= 4626.808$$

Question No. 12:

Suppose you are currently age 45 and you buy a Type B Universal Life policy with an additional death benefit of $250,000.

You made an initial premium of $5,400.

The charges in the policy consist of:

- Expense charges payable at the beginning of each year:
  - (i) Fixed expenses: $60
  - (ii) Variable expenses: 3% of premium
- Cost of insurance rate: 125% of the mortality rates from the `Illustrative Life Table`

You are also given that: $i^c = 5\%$ and $i^q = 4\%$.

Calculate your account value at the end of the first year.

$$q_{45} = 4/1000$$

$$AV_1 = \underbrace{\left(5400(1-.03) - 60 - 250000 * 1.25 * \frac{4}{100} * \frac{1}{1.04}\right)(1.05)}$$

$$= \underline{\underline{4174.881}}$$

Question No. 13: (BONUS)

You are given:

- (40) and (50) have independent future lifetimes.
- Mortality for either life follows deMoivre's law with $\omega = 100$.
- $\delta = 5\%$

Calculate $\bar{a}_{\overline{40:50}}$.

Hint: $\int te^{-kt}dt = -\frac{1}{k}te^{-kt} - \frac{1}{k^2}e^{-kt}$, for any constant $k > 0$.

$$F_{T_{\overline{xy}}}(t) = \begin{cases} \frac{t}{60}\frac{t}{50}, & 0 < t \leq 50 \\ t/60, & 50 < t \leq 60 \\ 0, & \text{else} \end{cases} \Rightarrow f_{T_{\overline{xy}}}(t) = \begin{cases} \frac{t}{1500}, & 0 < t \leq 50 \\ 1/60, & 50 < t \leq 60 \\ 0, & \text{else} \end{cases}$$

$$\bar{A}_{\overline{40:50}} = \underbrace{\int_0^{50} e^{-.05t}\frac{t}{1500}\,dt}_{0.190054} + \underbrace{\int_{50}^{60} e^{-.05t}\frac{1}{60}\,dt}_{0.0107659 8}$$

$$= 0.20082$$

$$\bar{a}_{\overline{40:50}} = \frac{1 - 0.20082}{.05} = \underline{\underline{15.9836}}$$

EXTRA PAGE FOR ADDITIONAL OR SCRATCH WORK