$P = B \cdot \frac{A_x}{\ddot{a}_x}$

$\ddot{a}_x$ $i=0$

$\underbrace{E[K_x+1]}_{\text{average lifetime}}$

$K_x = K$

net premium

gross premium

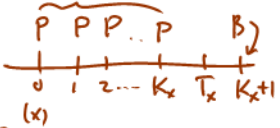
> Reserves

Recall: premiums

fully discrete - whole life
- premium discrete b.o.y.
- benefit discrete e.o.y.

# Policy Values

Lecture: Weeks 1-2



loss at issue $L_0^n$ $= \underbrace{B \cdot v^{K+1} - P\ddot{a}_{\overline{K+1}|}}_{\text{equivalence principle}}$

n = net

no expenses

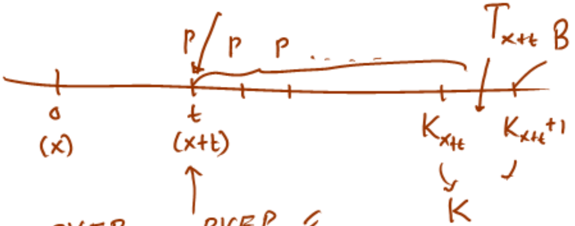
$E[L_0] = 0 = B\,E[v^{K+1}] - P\,E[\ddot{a}_{\overline{K+1}|}]$

$= \underbrace{BA_x}_{} - P\ddot{a}_x = 0 \Rightarrow APV(FP_0) = APV(FB_0)$

$APV(FB_0) - APV(FP_0)$

$P\ddot{a}_x = BA_x$

fully discrete whole life,



prospective loss r.v. at time t $= L_t = PVFB_t - PVFP_t$ ,

$$\downarrow$$

$$E\left[B v^{K+1} - P \ddot{a}_{\overline{K+1}|}\right] \quad , \quad K = K_{x+t} \, .$$

$t=0 \Rightarrow L_0$ = loss at issue

$K = K_{x+t}$

in force,

$$E[L_t \mid T_x > t] = \underbrace{E[L_t]} = \underbrace{E[PVFB_t]} - E[PVFP_t]$$

net premium reserve ${}_tV$ $= APV(FB_t) - APV(FP_t)$

$$= B \underbrace{E[v^{K+1}]}_{A_{x+t}} - P \underbrace{E[\ddot{a}_{\overline{K+1}|}]}_{\ddot{a}_{x+t}} = \underbrace{B A_{x+t}} - P \ddot{a}_{x+t}$$

n,g. $_tV$ or $_t\overset{n}{V} = B A_{x+t} - P\ddot{a}_{x+t} = \underbrace{APV(FB_t) - APV(FP_t)}$ .

$\boxed{P = \text{at issue}}$

B = fixed.

$$Var(L_t)^{*} = \ ?$$

$$L_t = B\underline{v}^{K+1} - P\ddot{a}_{\overline{K+1}|} \longrightarrow \frac{1-v^{K+1}}{d} = \ddot{a}_{\overline{K+1}|}$$

$$= Bv^{K+1} - P\left(\frac{1-v^{K+1}}{d}\right)$$

P, d constant

$$= \underbrace{\left(B+\frac{P}{d}\right)}v^{K+1} - \frac{P}{d}$$

PV of a whole life

$$Var(L_t) = \left(B+\frac{P}{d}\right)^2 \underbrace{Var\left(v^{K+1}\right)}$$

$$= \left(B+\frac{P}{d}\right)^2\left[{}^2A_{x+t} - \left(A_{x+t}\right)^2\right]$$

important.

↓

$= A_{x+t}$ but at $2\delta$

fully discrete whole life to $x=45$, $B=1000$

Mortality follows $\underset{\sim}{\text{SULT}}$ at $i=\underline{5\%}$.

Survival Ultimate Life Table

Premiums are calculated according to Equivalence Principle.

$E[L_0]=0$ ✓

No expenses

Calculate ${}_{10}V$ and $Var(L_{10})$

$$P = \frac{\overset{1000}{B}\overset{.15161}{A_{45}}}{\underset{17.8162}{\ddot{a}_{45}}} = 8.509671$$

$${}_{10}V = \underset{\sim}{APV(FB_{10})} - APV(FP_{10})$$

$$= BA_{55} - P\ddot{a}_{55}$$

$$= 1000(.23524) - P(16.0599) = 98.57554 \approx \underline{98.58}$$

$$Var(L_{10}) = \left(B+\frac{P}{d}\right)^2\left[\underset{.67483}{{}^2A_{55}} - \big(\underset{\underline{.23524}}{A_{55}}\big)^2\right] = \boxed{27{,}081.23}$$

$$d = \frac{.05}{1.05}$$

endowment policy issued to (x)
n-year

B= benefit

$$\underbrace{APV(FP_0)}_{P\ddot{a}_{x:\overline{n}|}} = \underbrace{APV(FB_0)}_{B\,A_{x:\overline{n}|}}$$

$$A_{x:\overline{n}|} = A^1_{x:\overline{n}|} + {}_nE_x$$

$$\downarrow$$

$${}_nE_x = v^n \cdot {}_np_x$$

$$P = B\,A_{x:\overline{n}|} / \ddot{a}_{x:\overline{n}|}$$

$$L_t = PVFB_t - PVFP_t$$

$$= \underbrace{B\,v^{\min(K+1,n)}} - P\,\ddot{a}_{\overline{\min(K+1,n)}|}$$

$$E[L_t] = B\,\underbrace{E[v^{\min(K+1,n)}]}_{A_{x:\overline{n}|}} - P\,\underbrace{E[\ddot{a}_{\overline{\min(K+1,n)}|}]}_{\ddot{a}_{x:\overline{n}|}}$$

$$= B\,A_{x:\overline{n}|} - P\,\ddot{a}_{x:\overline{n}|}, \quad \text{provide } t < n.$$

$$0, \quad t > n$$

$$B, \quad t = n$$

$$L_t = B v^{\min(K+1,n)} - P \underbrace{\ddot{a}_{\overline{\min(K+1,n)}|}}$$

$$\frac{1-v^{\min(K+1,n)}}{d}$$

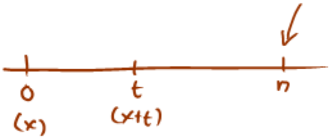
$$= \left(B+\frac{P}{d}\right) v^{\min(K+1,n)} - \frac{P}{d}$$

$$Var(L_t) = \left(B+\frac{P}{d}\right)^2 \underbrace{Var\left(v^{\min(K+1,n)}\right)}$$

$$x+t \qquad {}^2A_{x+t:\overline{n-t}|} - \left(\underbrace{A_{x+t:\overline{n-t}|}}\right)^2$$

$$\swarrow$$

$$2\delta$$



n-year term insurance

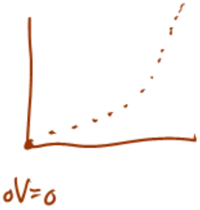
$$P = B A^{1}_{x:\overline{n}|} / \ddot{a}_{x:\overline{n}|}$$

$$ {}_tV = E[L_t] = \underbrace{APV(FB_t)} - APV(FP_t)$$

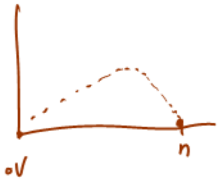
$$= \begin{cases} B A^{1}_{x+t:\overline{n-t}|} - P \ddot{a}_{x+t:\overline{n-t}|}, & t<n \\ 0, & t \geq n \end{cases}$$

---

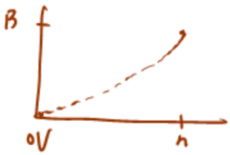
### whole life



### term insurance



### endowment



# Chapter summary

- Insurance reserves (policy values)
  - what are they? how do we calculate them? why are they important?
- Reserves or policy values
  - benefit reserves (no expenses considered)
  - gross premium reserves (expenses accounted for)
  - prospective calculation of reserves (based on the future loss random variable)
  - retrospective calculation of reserves (not emphasized)
- Other topics to be covered (in separate slides)
  - analysis of profit or loss and analysis by source (mortality, interest, expenses)
  - Thiele's differential equation for reserve calculation
  - policy alterations
- Chapters 7 (Dickson, et al.)

# Mortality assumptions

For illustration purposes, we may base our calculations on the following assumptions:

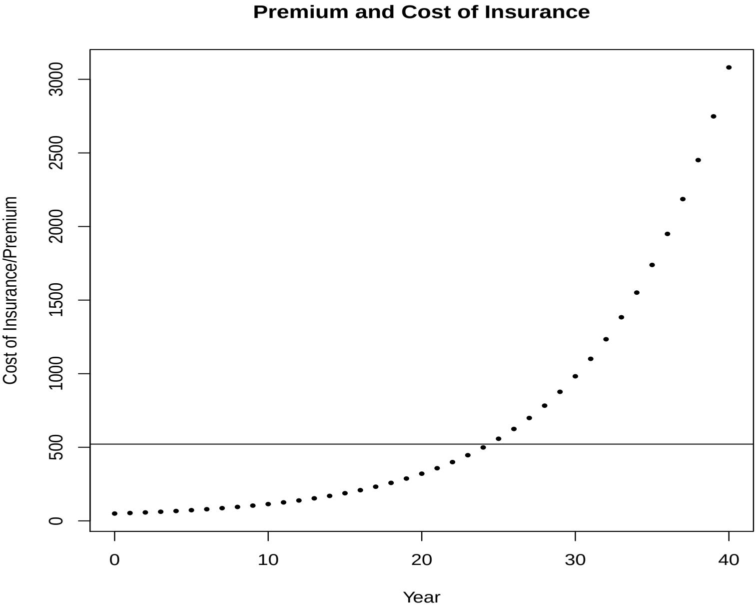
- Survival Ultimate Life Table (SULT)
  - the (official) Life Table used for Exam LTAM with $i = 0.05$
- Standard Ultimate Survival Model, pp. 583, 586-587
  - introduced in Section 4.3
  - Makeham's law $\mu_x = A + Bc^x$, with $A = 0.00022$, $B = 2.7 \times 10^{-6}$ and $c = 1.124$, and interest rate $i = 5\%$
- Standard Select Survival Model, pp. 583, 584-585
  - introduced in Example 3.13
  - the ultimate part follows the same Makeham's law as above; the select part follows
    $$\mu_{[x]+s} = 0.9^{2-s}\mu_{x+s}, \quad \text{for } 0 \leq s \leq 2,$$
    and interest rate $i = 5\%$

# Insurance reserves (policy values)

- Money set aside to be able to cover insurer's future financial obligations as promised through the insurance contract.
    - reserves show up as a liability item in the balance sheet;
    - increases in reserves are an expense item in the income statement.
- Reserve calculations may vary because of:
    - purpose of reserve valuation: statutory (solvency), GAAP (realistic, shareholders/investors), mergers/acquisitions
    - assumptions and basis (mortality, interest) - may be prescribed
- Actuary is responsible for preparing an Actuarial Opinion and Memorandum: that the company's assets are sufficient to back reserves.
- Reserves are more often called provisions in Europe.
    - another term used is policy values

# Why hold reserves?

- For several life insurance contracts:
    - the expected cost of paying the benefits generally increases over the contract term; but
    - the periodic premiums used to fund these benefits are level.
- The portion of the premiums not required to pay expected cost in the early years are therefore set aside (or provisioned) to fund the expected shortfall in the later years of the contract.
- Reserves also help reduce cost of benefits as they also earn interest while being set aside.
- Although reserves are usually held on a per-contract basis, it is still the overall responsibility of the actuary to ensure that in the aggregate, the company's assets are enough to back these reserves.


Premium and Cost of Insurance
Cost of Insurance/Premium
0
500
1000
1500
2000
2500
3000
0
10
20
30
40
Year

# The insurer's future loss random variable

- At any future time $t \geq 0$, define the insurer's (net) future loss random variable to be
$$L_t^n = \mathsf{PVFB}_t - \mathsf{PVFP}_t.$$
- For most types of policies, it is generally true that for $t \geq 0$, $L_t^n \geq 0$, i.e. $\mathsf{PVFB}_t \geq \mathsf{PVFP}_t$.
- If we include expenses, the insurer's (gross) future loss random variable is said to be
$$L_t^g = \mathsf{PVFB}_t + \mathsf{PVFE}_t - \mathsf{PVFP}_t.$$
- For our purposes, we define the expected value of this future loss random variable to be the reserve or policy value at time $t$:
$${}_tV^n = \mathsf{E}\big[L_t^n\big] = \mathsf{E}\big[\mathsf{PVFB}_t\big] - \mathsf{E}\big[\mathsf{PVFP}_t\big]$$
or in the case with expenses,
$${}_tV^g = \mathsf{E}\big[L_t^g\big] = \mathsf{E}\big[\mathsf{PVFB}_t\big] + \mathsf{E}\big[\mathsf{PVFE}_t\big] - \mathsf{E}\big[\mathsf{PVFP}_t\big]$$

# Some remarks I

- ${}_tV^n$ and ${}_tV^g$ are respectively called net premium reserve and gross premium reserve. The primary difference between the two is the consideration of expenses.
- For Exam MLC, the term benefit reserve is often the preferred terminology to refer to the net premium reserve (no expenses).
- So if no confusion arises, we will often drop $n$ and $g$ in the superscripts for either the future loss random variable $L_t$ or the reserve ${}_tV$.
- Note that $\mathsf{E}\big[L_t\big]$ is actually conditional on the survival of $(x)$ at time $t$. Because otherwise, there is no reason to hold reserves when policy has been paid out (or matured or voluntarily withdrawn).
- Reserves are indeed released from the balance sheet when policy is paid out (or matured or voluntarily withdrawn).

# Some remarks II

- Technically speaking, ${}_tV$ is to be the (smallest) amount for which the insurer is required to hold to be able to cover future obligations.
- We can see this from the following equations (here, we consider expenses, but if we ignore expenses, the term with expenses will simply be zero - same principle will hold):

  $${}_tV = \mathsf{APV}(\mathsf{FB}_t) + \mathsf{APV}(\mathsf{FE}_t) - \mathsf{APV}(\mathsf{FP}_t)$$

  Rewriting this, we get

  $$\mathsf{APV}(\mathsf{FB}_t) + \mathsf{APV}(\mathsf{FE}_t) = \mathsf{APV}(\mathsf{FP}_t) + {}_tV.$$

- This equation tells us that the reserve ${}_tV$ is the balancing term in the equation to cover the deficiency of future premiums that arises at time $t$ to cover future obligations (benefits plus expenses, if any).

# A numerical illustration

Consider a whole life policy issued to a select age [40] with:

- \$100 of death benefit payable at the moment of death;
- premiums are annual payable at the beginning of each year;
- mortality follows the `Standard Select Survival Model` with $i = 5\%$; and
- mortality between integral ages follows the Uniform Distribution of Death (UDD).

The first step in reserve calculation is to determine the annual premiums. Let $P$ be the annual premium in this case so that one can easily verify that

$$\begin{aligned} P &= 100 \times \frac{\bar{A}_{[40]}}{\ddot{a}_{[40]}} = 100 \times \frac{i}{\delta} \frac{A_{[40]}}{\ddot{a}_{[40]}} \\ &= 100 \left( \frac{0.05}{\log(1.05)} \right) \left( \frac{0.1209733}{18.45956} \right) = 0.6715928. \end{aligned}$$

# A numerical illustration - continued

The benefit reserve (or policy value) at the end of year 5 is given by

$$\begin{aligned} {}_5V &= \text{APV}(\text{FB}_5) - \text{APV}(\text{FP}_5) = 100 \times (i/\delta)A_{45} - P \times \ddot{a}_{45} \\ &= 100 \times \left(\frac{0.05}{\log(1.05)}\right)(0.151609) - 0.6715928 \times 17.81621 \\ &= 3.571607 \end{aligned}$$

Note that we have calculated the policy value above as the expectation of a future loss random variable. We can also view reserve in terms of the insurer's account value after policies have been in force after 5 years (retrospectively).

Suppose that insurer issues $N$ such similar but independent policies.

What happens to the insurer's account value after 5 years? [Done in lecture!]

# Fully discrete reserves - whole life insurance

Consider the case of a fully discrete whole life insurance issued to a life $(x)$ where premium of $P$ is paid at the beginning of each year and benefit of $\$B$ is paid at the e.o.y. of death.

- The insurer's future loss random variable at time $k$ (or at age $x+k$) is

  $$L_k = Bv^{K_{x+k}+1} - P\ddot{a}_{\overline{K_{x+k}+1}|},$$

  for $k = 0, 1, 2, \ldots$

- Applying the equivalence principle by solving $\mathsf{E}\big[L_0\big] = 0$, it can be verified that

  $$P = B \times \frac{A_x}{\ddot{a}_x} = B \times P_x.$$

- The benefit reserve (or policy value) at time $k$ can be expressed as

  $${}_kV = \mathsf{E}\big[L_k\big] = B \times \left(A_{x+k} - P_x\,\ddot{a}_{x+k}\right).$$

# - continued

The benefit reserve at time $k$ is indeed equal to the difference between

$$\text{APV}(\text{FB}_k) = B \times A_{x+k}$$

and

$$\text{APV}(\text{FP}_k) = B \times P_x \, \ddot{a}_{x+k}$$

Sometimes, the variance is a helpful statistic and one can easily derive the variance of $L_k$ with

$$\begin{aligned} \text{Var}\big[L_k\big] &= \text{Var}\left[B \cdot v^{K_{x+k}+1}\left(1+\frac{P_x}{d}\right) - B \cdot \frac{P_x}{d}\right] \\ &= B^2 \times \left(1+\frac{P_x}{d}\right)^2 \left[{}^2A_{x+k} - (A_{x+k})^2\right]. \end{aligned}$$

# Other special formulas

Note that it can be shown that other special formulas for the benefit premium reserves for the fully discrete whole life hold:

- $${}_kV = 1 - d\ddot{a}_{x+k} - \left(\frac{1}{\ddot{a}_x} - d\right)\ddot{a}_{x+k} = 1 - \frac{\ddot{a}_{x+k}}{\ddot{a}_x}$$
- $${}_kV = 1 - \frac{P_x + d}{P_{x+k} + d} = \frac{P_{x+k} - P_x}{P_{x+k} + d}$$
- $${}_kV = 1 - \frac{1 - A_{x+k}}{1 - A_x} = \frac{A_{x+k} - A_x}{1 - A_x}$$

Note that in these formulas we set $B = 1$. If the benefit amount $B$ is not \$1, then simply multiply these formulas with the corresponding benefit amount.

# A numerical illustration

Consider a fully discrete whole life policy of \$10,000 issued to a select age (40) with:

- mortality follows the `Standard Ultimate Survival Model` with $i = 5\%$; and

One can verify that $P = 65.58717$ and the following table of benefit reserves:

| $k$ | $\ddot{a}_{40+k}$ | ${}_kV$ | $k$ | $\ddot{a}_{40+k}$ | ${}_kV$ |
|---|---|---|---|---|---|
| 0 | 18.4578 | 0.000 | 13 | 16.4678 | 1078.103 |
| 1 | 18.3403 | 63.628 | 14 | 16.2676 | 1186.567 |
| 2 | 18.2176 | 130.096 | 15 | 16.0599 | 1299.123 |
| 3 | 18.0895 | 199.508 | 16 | 15.8444 | 1415.840 |
| 4 | 17.9558 | 271.966 | 17 | 15.6212 | 1536.774 |
| 5 | 17.8162 | 347.574 | 18 | 15.3901 | 1661.975 |
| 6 | 17.6706 | 426.437 | 19 | 15.1511 | 1791.478 |
| 7 | 17.5189 | 508.658 | 20 | 14.9041 | 1925.306 |
| 8 | 17.3607 | 594.340 | 21 | 14.6491 | 2063.467 |
| 9 | 17.1960 | 683.583 | 22 | 14.3861 | 2205.955 |
| 10 | 17.0245 | 776.487 | 23 | 14.1151 | 2352.744 |
| 11 | 16.8461 | 873.148 | 24 | 13.8363 | 2503.790 |
| 12 | 16.6606 | 973.658 | 25 | 13.5498 | 2659.027 |

# Endowment policy

To simplify the formula development, assume $B = 1$.

- The future loss random variable at time $k \leq n$ (or at age $x + k$) is

$$L_k = v^{\min(K_{x+k}+1, n-k)} - P_{x:\overline{n}|}\, \ddot{a}_{\overline{\min(K_{x+k}+1, n-k)}|},$$

  for $k = 0, 1, \ldots, n$. Loss is zero for $k > n$.

- The benefit reserve at time $k$ is

$${}_kV = A_{x+k:\overline{n-k}|} - P_{x:\overline{n}|}\, \ddot{a}_{x+k:\overline{n-k}|}.$$

- The variance of $L_k$ is

$$\mathsf{Var}\big[L_k\big] = \left(1 + \frac{P_{x:\overline{n}|}}{d}\right)^2 \left[{}^2A_{x+k:\overline{n-k}|} - \left(A_{x+k:\overline{n-k}|}\right)^2\right].$$

Without expenses, net premium reserves

$$L_t^n = \underbrace{PVFB_t - PVFP_t}$$

prospective loss at time t

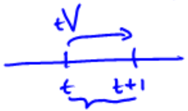
$$\big| T \geq t$$

$${}_tV^n = E[L_t^n] = \underbrace{E[PVFB_t]}_{APV(FB_t)} - E[PVFP_t]$$

$$\xrightarrow{\qquad} APV(FB_t) - APV(FP_t)$$

premiums are calculated using equivalence principle $E[L_0] = 0$

$t=0,\ {}_0V = 0$



Guojun → recursion formula

$${}_{t+1}V = \frac{({}_tV + P)(1+i) - B_{t+1} \cdot q_{x+t}}{1 - q_{x+t}}$$

$1 - q_{x+t}$ → % of survivors

because we do not reserve for those who died one policy -

$$ {}_{t+1}V = ({}_tV + P)(1+i) - \underbrace{(B_{t+1} - {}_{t+1}V)}_{\text{net amount at risk}} q_{x+t} $$

release reserve for those who died

---

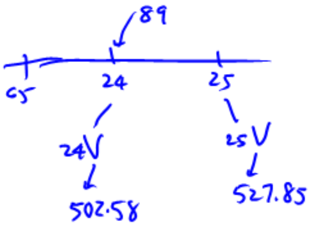
Review Problem 2 Q#5

B = 1000

(65)

$A_{65} = .6135$

$i = .05$



$$ {}_{25}V = ({}_{24}V + P)(1.05) - (1000 - {}_{25}V)\, q_{89} $$

$$ q_{89} = \frac{{}_{25}V - ({}_{24}V + P)(1.05)}{{}_{25}V - 1000} $$

$$P = \frac{1000\,\overset{.6135}{A_{65}}}{\ddot{a}_{65}} \qquad \ddot{a}_{65} = \frac{1-\overset{.6135}{A_{65}}}{d}$$

$$A \leftrightarrow \ddot{a}$$

$$d = iv = 1-v = \frac{.05}{1.05}$$

$$= 1000\,\frac{.6135}{\frac{1-.6135}{.05/1.05}} = 75.58677 = ?$$

$$q_{89} = \underbrace{\frac{527.85 - (502.58 + 75.58677)(1.05)}{527.85 - 1000}}_{0.2207458} \approx \underline{22\%}$$

# Illustrative example 1

For a special fully discrete whole life insurance on (50), you are given:

- The death benefit is \$50,000 for the first 15 years and reduces to \$10,000 thereafter.
- The annual benefit premium is $5P$ for the first 15 years and reduces to $P$ thereafter.
- Mortality follows the `Survival Ultimate Life Table`.
- $i = 0.05$

Calculate the following:

1. the value of $P$; ✓
2. the benefit reserve at the end of 10 years; and ${}_{10}V$
3. the benefit reserve at the end of 20 years. ${}_{20}V$

$P = ?$ / $APV(FP_0) = APV(FB_0)$

$$5P\,\ddot{a}_{50} - 4P\cdot{}_{15}E_{50}\,\ddot{a}_{65}$$

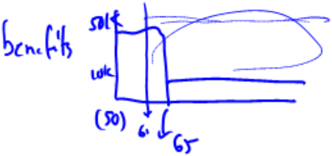
$\ddot{a}_{50}$: 17.0245

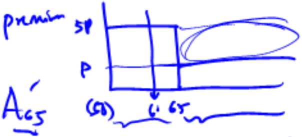
$\ddot{a}_{65}$: 13.5498

$$50000\,A_{50} - 40000\,{}_{15}E_{50}\,A_{65}$$

$A_{50}$: .18931

$A_{65}$: .35477





${}_nE_x$ is used for life insurance discounting -

${}_5E_x$

${}_{10}E_x$

${}_{20}E_x$

$${}_{15}E_{50} = \underbrace{{}_{10}E_{50}}_{.77772} * \underbrace{{}_{5}E_{60}}_{.59342}$$

$$\underbrace{.77772 \quad .59342}_{.4615146}$$

Solve for P :

$P = 48.51602 \approx \underline{48.52}$

(b) $t=10$ $\quad {}_{10}V = \underbrace{APV(FB_{10})} - APV(FP_{10})$

$$= \left(50000\,\overset{.29028}{A_{60}} - 40000\,\underbrace{{}_5E_{60}}\,\overset{.35477}{A_{65}}\right)$$

$$-\left(5P\,\ddot{a}_{60} - 4P\,{}_5E_{60}\,\ddot{a}_{65}\right)$$

48.51602

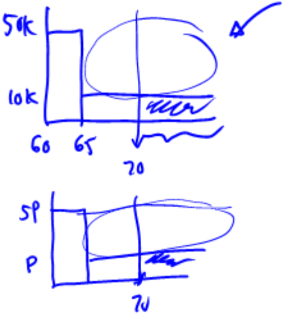
14.9041

14.9041

13.5498

$$= \underbrace{3631.501} - \underbrace{1598.931}$$

2032.57



(c) $t=20$ $\quad {}_{20}V = APV(FB_{20}) - APV(FP_{20})$

$$= 10000\,A_{70} - P\,\ddot{a}_{70} = 3699.205 \checkmark$$

With expenses, gross premium reserve

$G$ = gross premium

$$L_t^g = \underbrace{PVFB_t}_{} + \underbrace{PVFE_t}_{} - \underbrace{PVFG_t}_{}$$

$${}_tV^g = E[L_t^g] = \underbrace{APV(FB_t) + APV(FE_t)}_{} - APV(FG_t)$$

balancing item

$$\underbrace{APV(FB_t) + APV(FE_t)}_{} = \underline{APV(FG_t)} + {}_tV^g$$

balance item

$G_t$ is determined using equivalence principle

$$\underbrace{E[L_0^g] = {}_0V^g = 0}_{}$$

age = 40

SULT
$i = 5\%$

# Recursive formulas

To motivate development of recursive formulas, consider a fully discrete whole life insurance of $B to $(x)$. It can be shown (done in lecture) that:

$$ {}_{k+1}V = \frac{\left({}_kV + P\right)(1+i) - Bq_{x+k}}{1 - q_{x+k}}, $$

with $k = 1, 2, \ldots$ and ${}_0V = 0$ One can verify the following calculations of the successive reserves for $B = 10,000$. See slides page 13.

for a whole life

${}_\omega V = {}_\infty V = B$

${}_\omega V \Rightarrow 0$

| $k$ | $1000q_{40+k}$ | ${}_kV$ | $k$ | $1000q_{40+k}$ | ${}_kV$ |
|---|---|---|---|---|---|
| 0 | 0.52722 | 0.000 | 13 | 1.62346 | 1078.103 |
| 1 | 0.56531 | 63.628 | 14 | 1.79736 | 1186.567 |
| 2 | 0.60813 | 130.096 | 15 | 1.99278 | 1299.123 |
| 3 | 0.65625 | 199.508 | 16 | 2.21239 | 1415.840 |
| 4 | 0.71033 | 271.966 | 17 | 2.45917 | 1536.774 |
| 5 | 0.77112 | 347.574 | 18 | 2.73648 | 1661.975 |
| 6 | 0.83944 | 426.437 | 19 | 3.04808 | 1791.478 |
| 7 | 0.91622 | 508.658 | 20 | 3.39821 | 1925.306 |
| 8 | 1.00252 | 594.340 | 21 | 3.79161 | 2063.467 |
| 9 | 1.09952 | 683.583 | 22 | 4.23360 | 2205.955 |
| 10 | 1.20853 | 776.487 | 23 | 4.73017 | 2352.744 |
| 11 | 1.33104 | 873.148 | 24 | 5.28801 | 2503.790 |
| 12 | 1.46873 | 973.658 | 25 | 5.91465 | 2659.027 |

# Gross premium reserve calculation

B=10000

Consider a fully discrete whole life policy of $10,000 issued to (40) with:

- mortality follows the `Standard Ultimate Survival Model` with $i = 5\%$; and

SULT

50

Suppose expenses consist of: (a) $5 per 1,000 of death benefit in the first year and (b) $2 per 1,000 of death benefit in subsequent years.

20

It can be shown that the gross annual premium, $G$, is

$$
\begin{aligned}
G &= \frac{10000A_{40} + 30 + 20\ddot{a}_{40}}{\ddot{a}_{40}} \\
&= \frac{10000(0.1210592) + 30 + 20(18.45776)}{18.45776} \\
&= 87.21251.
\end{aligned}
$$

$B = 10000$

expenses 1st yr 50

renewal 20

$(x=40)$

SULT, $i=5\%$

${}_{10}V = ?$

recursive

$G = ?$

$$APV(FG_0) = APV(FB_0) + APV(FE_0)$$

$A_{40} = .121092$

$\ddot{a}_{40} = 18.4572$

$$G\,\cancel{\ddot{a}_{40}} = \frac{10000 A_{40} + 30 + 20\,\ddot{a}_{40}}{\ddot{a}_{40}}$$

$$G = 87.21251 \simeq 87.21$$

$$ {}_{10}V^{g} = APV(FB_{10}) + \underbrace{APV(FE_{10})} - APV(FG_{10})$$

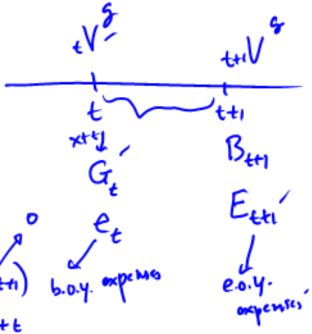
$A_{50} = ?$

$\ddot{a}_{50} = ?$

$$\underbrace{10000\,A_{50} + 20\,\ddot{a}_{50} - G\,\ddot{a}_{50}}$$

$$= 873.148$$

Recursive formula ${}_{t+1}V^g = ?$



$E_{t+1}$ = $\underbrace{\text{e.o.y. expenses}}$

death settlement expense

expenses associated with death

$${}_{t+1}V^g = \frac{({}_tV^g + G_t - e_t)(1+i) - (B_{t+1} + \overset{0}{\cancel{E_{t+1}}}) \cdot q_{x+t}}{1-q_{x+t}}$$

$${}_1V^g = \frac{(\overset{0}{{}_0V^g} + \overset{87.21}{G} - \overset{50}{e_0})(1+i) - \overset{10,000}{B_1} \cdot q_{40}}{1-q_{40}}$$

# - continued

To calculate gross premium reserves, use recursive formulas with ${}_0V = 0$:

$$ {}_1V = \frac{(\overset{0}{\boxed{{}_0V}} + G - \boxed{50})(1.05) - 10000q_{40}}{1 - q_{40}} \checkmark, \quad \text{and} $$

$$ {}_{k+1}V = \frac{({}_kV + G - \boxed{20})(1.05) - 10000q_{40+k}}{1 - q_{40+k}} \checkmark, \quad \text{for } k = 1, 2, \ldots $$

| $k$ | $1000q_{40+k}$ | ${}_kV$ | $k$ | $1000q_{40+k}$ | ${}_kV$ |
|---|---|---|---|---|---|
| 0 | 0.52722 | 0.000 | 13 | 1.62346 | 1051.338 |
| 1 | 0.56531 | 33.819 | 14 | 1.79736 | 1160.127 |
| 2 | 0.60813 | 100.487 | 15 | 1.99278 | 1273.021 |
| 3 | 0.65625 | 170.106 | 16 | 2.21239 | 1390.087 |
| 4 | 0.71033 | 242.781 | 17 | 2.45917 | 1511.384 |
| 5 | 0.77112 | 318.617 | 18 | 2.73648 | 1636.961 |
| 6 | 0.83944 | 397.716 | 19 | 3.04808 | 1766.852 |
| 7 | 0.91622 | 480.184 | 20 | 3.39821 | 1901.082 |
| 8 | 1.00252 | 566.123 | 21 | 3.79161 | 2039.658 |
| 9 | 1.09952 | 655.634 | 22 | 4.23360 | 2182.573 |
| 10 | 1.20853 | 748.817 | 23 | 4.73017 | 2329.802 |
| 11 | 1.33104 | 845.768 | 24 | 5.28801 | 2481.301 |
| 12 | 1.46873 | 946.579 | 25 | 5.91465 | 2637.004 |

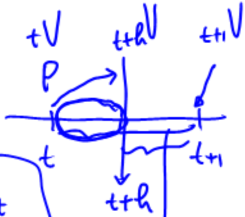
Compare these values with the benefit reserves. What do you observe?

interim reserves

$$\begin{tikzcd} {}_tV \arrow[r] \arrow[dr, dash] & {}_{t+1}V \arrow[d, dash] \\ & {}_{t+h}V \end{tikzcd}$$

${}_{t+h}V$, $0<h\leq 1$,

net premium reserve, without expenses,



$$\boxed{{}_{t+h}V = \frac{({}_tV + P)(1+i)^{h} - B\cdot v^{1-h}\,{}_{h}q_{x+t}}{1-{}_{h}q_{x+t}}}$$

UDD

pay benefit at e.o.y. so discounted

UDD $\Rightarrow$ ${}_hq_{x+t} = h\cdot q_{x+t}$

${}_{1-h}q_{x+t+h}$

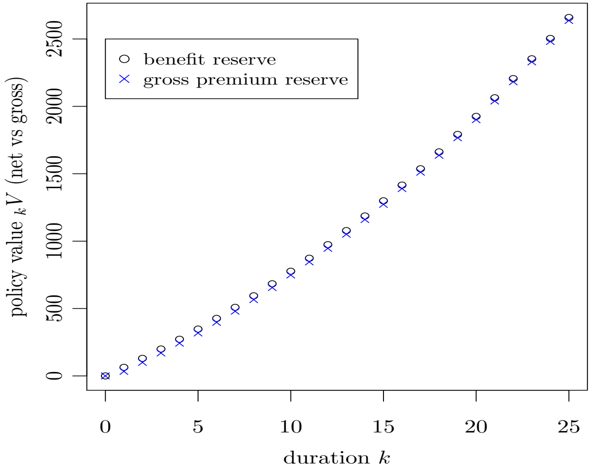


Figure: Comparison between benefit reserve and gross premium reserve

# A generalization of recursive relations

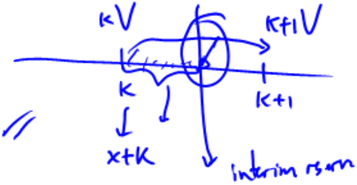
The reserve in the next period $t+1$ can be shown to be

$$_{t+1}V = \frac{\big({}_tV + G_t - e_t\big)(1+i_t) - \big(B_{t+1} + E_{t+1}\big)q_{x+t}}{1 - q_{x+t}}.$$

Intuitively, we have:

- accumulate previous reserves plus premium (less expenses) with interest;
- deduct death benefits (plus any claims-related expenses) to be paid at the end of the year; and
- divide the reserves by the proportion of survivors.

K = discrt



$$_{k+1}V = \frac{(_kV + P)(1+i) - B\cdot q_{x+k}}{1-q_{x+k}}$$

$_{k+h}V$, $0<h<1$

hold reserve for those who survive

$$_{k+1}V = (_kV+P)(1+i) - (\underbrace{B - {}_{k+1}V}_{\Downarrow})\, q_{x+k}$$

reserve release

$$_{k+h}V = \frac{(_kV+P)(1+i)^h - B\cdot {}_hq_{x+k}\, v^{1-h}}{1-{}_hq_{x+k}}$$

# Published SOA question #'s 274-277

For a special fully discrete life insurance on $(x)$, you are given:

- Deaths are uniformly distributed over each year of age. UDD
- The following extracted table:

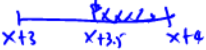
| $k$ | Benefit premium at beginning of year $k$ | Death benefit at end of year $k$ | Interest rate used during year $k$ | $q_{x+k-1}$ | Benefit reserve at at end of year $k$ ✓ |
|---|---|---|---|---|---|
| 2 | – | – | – | – | 84 $_2V$ |
| 3 | 18 ✓ | 240 ✓ | 0.07 ✓ | – | 96 $_3V$ ✓ |
| 4 | 24 ✓ | 360 ✓ | 0.06 ✓ | 0.101 | – $_4V$ |

(274) Calculate $q_{x+2}$. $= .09125$

(275) Calculate the benefit reserve at the end of year 4. $_4V$

$$_3V = \frac{(_2V + P)(1.07) - 240 \cdot q_{x+2}}{1 - q_{x+2}}$$

(276) Calculate $_{0.5}q_{x+3.5}$. ✓



(277) Calculate the benefit reserve at the end of 3.5 years.

Rewrite $$3V = (2V + G_2)(1+i_2) - (B_3 - 3V)\, q_{x+2}$$

$$q_{x+2} = \frac{3V - (2V + G_2)(1+i_2)}{3V - B_3}$$

$$= \frac{96 - (84+18)(1.07)}{96 - 240} = \boxed{.09125}$$

0

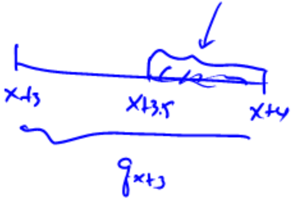
---

$$3V = \frac{(2V + P)(1+i) - B \cdot q_{x+2}}{1 - q_{x+2}}$$

$$4V = \frac{(3V + P)(1+i) - B\, q_{x+3}}{1 - q_{x+3}} = \underbrace{\frac{(96 + 24)(1.06) - 360 \cdot (.101)}{1 - .101}}$$

$$= 101.0456$$

$$.5q_{x+3.5} \neq \underline{\underline{.5 \cdot q_{x+3}}}$$



$$\frac{.5 \times q_{x+3}}{1-.5q_{x+3}} \checkmark$$

$\checkmark$

$$1-.5p_{x+3.5}$$

$$= 1-\frac{p_{x+3}}{.5p_{x+3}} \;(= .5q_{x+3.5}$$

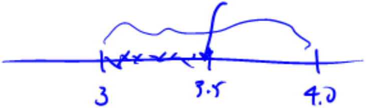
$$\frac{p_{x+3} = .5p_{x+3} \times .5p_{x+3.5}}{.5p_{x+3}}$$

$$= 1-\frac{1-q_{x+3}}{1-.5q_{x+3}} = \frac{1-.5q_{x+3}-1+q_{x+3}}{1-.5q_{x+3}} = \frac{.5 \times q_{x+3}}{1-.5 \times q_{x+3}}$$

0.101

0.101

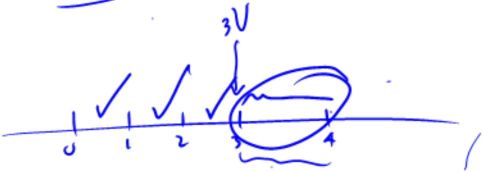
$${}_{3.5}V = \frac{({}_3V + P)(1+i)^{1/2} - B \cdot {}_{1/2}q_{x+3}}{1 - {}_{1/2}q_{x+3}}$$



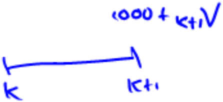
$$= \frac{(\overset{{}_3V}{96} + 24)(1.06)^{1/2} - 360 * \frac{1}{2}(.101) \cdot \left(\frac{1}{1.06}\right)^{1/2}}{1 - \frac{1}{2}(.101)}$$

$${}_{1/2}q_{x+3} = \frac{1}{2} \times \underset{.101}{q_{x+3}}$$

$= ??$ 111.5214



# Illustrative example 2



20 year term

For a special single premium 20-year term insurance on (70):

- The death benefit, payable at the end of the year of death, is equal to 1000 plus the benefit reserve.
- $q_{70+k} = 0.03$, for $k = 0, 1, 2, \dots$
- $i = 0.07$

Calculate the single benefit premium for this insurance.

Solve for P = same each year

$$_{k+1}V = \frac{(_kV + P)(1+i) - B \cdot q_{x+k}}{1-q_{x+k}}$$

or

$$_{k+1}V = (_kV+P)(1+i) - \underbrace{(B - \cancel{_{k+1}V})}_{1000 + \cancel{_{k+1}V}} q_{x+k}$$

$$_0V = 0$$

$$_1V = (0 + P)(1.07) - (1000 + \cancel{_1V} - \cancel{_1V}) \cdot 0.03$$

$$= P(1.07) - 1000(.03)$$

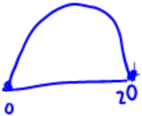
$$_2V = (P(1.07) - 1000(.03))(1.07) - (1000 + \cancel{_2V} - \cancel{_2V})\, 0.03$$

$$= P(1.07)^2 - 1000(.03)[1.07 + 1]$$

⋮

$$_{20}V = P(1.07)^{20} - 1000(.03)\Big[\underbrace{1.07^{19} + 1.07^{18} + \dots + 1.07^{0}}_{\frac{1-1.07^{20}}{1-1.07}}\Big] = 0$$

$$P = 317.8204$$



$${}_{k+1}V = \frac{({}_kV + P)(1+i) - B\, q_{x+k}}{1-q_{x+k}}$$

net premium reserves

With expenses

$G_k$ = gross premium in year $k$

$i_k$, $e_k$, $B_{k+1}$, $E_{k+1}$

$${}_{k+1}V^g = \frac{({}_kV^g + \overset{\checkmark}{G_k} - \overset{\checkmark}{e_k})(1+i_k) - (B_{k+1} + E_{k+1}')\cdot q_{x+k}}{1-q_{x+k}}$$

# Net amount at risk

- The difference $B_{t+1} + E_{t+1} - {}_{t+1}V$ is called the net amount at risk.
- Sometimes called death strain at risk (DSAR) or sum at risk.
- The recursive formula can then alternatively be written as

$$\big({}_tV + G_t - e_t\big)(1 + i_t) = {}_{t+1}V + \big(B_{t+1} + E_{t+1} - {}_{t+1}V\big)q_{x+t}$$

where the term $\big(B_{t+1} + E_{t+1} - {}_{t+1}V\big)q_{x+t}$ can then be called the expected net amount at risk.

# SOA MLC question #13 Fall 2014

gross premium

For a fully discrete whole life insurance of $\overset{B}{100,000}$ on (45), you are given:

- The gross premium reserve at duration 5 is 5500 and at duration 6 is 7100.

$_5V = 5500$ $_6V = 7100$

- $q_{50} = 0.009$
- $i = 0.05$
- Renewal expenses at the start of each year are 50 plus 4% of the gross premium.
- Claim expenses are 200.

Calculate the annual gross premium.

G

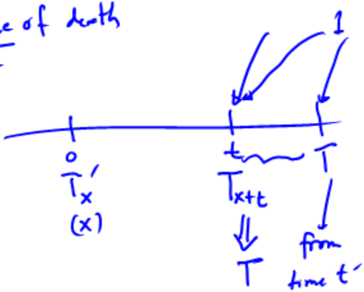
$$\underset{7100}{_6V} = \frac{(\underset{5500}{_5V} + G - 50 - .04G)(1.05) - \overset{+200)}{\{100000}(.009)}{1-.009}$$

$G = 2197.817 \approx 2200$

fully continuous whole life

- premiums are continuous
- benefit are at the $\underbrace{\text{exact time}}_{\text{moment}}$ of death

$_tV^n$, $_tV^g$



$$L_t = v^{T} - \bar{P} * \bar{a}_{\overline{T|}}$$

$$E[L_t] = \underbrace{E[v^T]}_{} - \bar{P}\, E[\bar{a}_{\overline{T|}}]$$

$T = T_{x+t}$

$$\bar{A}_{x+t} - \underbrace{\bar{P}\, \bar{a}_{x+t}}_{}$$

$$APV(FB_t) - APV(FP_t)$$

$$- APV(FE_t)$$

# Fully continuous reserves - whole life

Consider now the case of a fully continuous whole life insurance with an annual premium rate of $\bar{P}(\bar{A}_x)$.

- The future loss random variable at time $t$ (or at age $x+t$):

$$L_t = v^{T_{x+t}} - \bar{P}(\bar{A}_x)\,\bar{a}_{\overline{T_{x+t}}|} = v^{T_{x+t}}\left[1 + \frac{\bar{P}(\bar{A}_x)}{\delta}\right] - \frac{\bar{P}(\bar{A}_x)}{\delta}.$$

- The benefit reserve at time $t$ is

$$_tV = \mathsf{E}[L_t] = \overset{\checkmark}{\bar{A}_{x+t}} - \bar{P}(\cancel{\bar{A}_x})\,\overset{\checkmark}{\bar{a}_{x+t}}.$$

$\bar{P} = \bar{P}(\bar{A}_x)$

- The variance of $L_t$ is

$$\mathsf{Var}[L_t] = \left[1 + \frac{\bar{P}(\bar{A}_x)}{\delta}\right]^2 \left[{}^2\bar{A}_{x+t} - (\bar{A}_{x+t})^2\right].$$

$$L_t = v^T - \bar{P}\,\bar{a}_{\overline{T}|}\checkmark = v^T - \bar{P} \times \frac{1-v^T}{\delta}$$

$$= \underbrace{\left(1+\frac{\bar{P}}{\delta}\right)v^T - \frac{\bar{P}}{\delta}}$$

$Var(L_t)$

$$= \left(1+\frac{\bar{P}}{\delta}\right)^2 \left[\, \underset{E@25}{{}^2\bar{A}_{x+t}} - \underset{E^2}{\left(\bar{A}_{x+t}\right)^2} \right]$$

# Other formulas

$$ {}_kV = 1 - \frac{\ddot{a}_{x+k}}{\ddot{a}_x} $$

Some continuous analogues of the discrete case:

- $\boxed{{}_tV = 1 - \dfrac{\bar{a}_{x+t}}{\bar{a}_x}}$ ✓

  in terms only of life annuity ✓

- $${}_tV = \frac{\bar{P}(\bar{A}_{x+t}) - \bar{P}(\bar{A}_x)}{\bar{P}(\bar{A}_{x+t}) + \delta}$$

  in terms of premium

- $${}_tV = \frac{\bar{A}_{x+t} - \bar{A}_x}{1 - \bar{A}_x}$$

  in terms of insurance

$A \to \ddot{a}$

$A_x = 1 - d\ddot{a}_x$

$P = \frac{A_x}{\ddot{a}_x} = \frac{1 - d\ddot{a}_x}{\ddot{a}_x}$

$= \boxed{\frac{1}{\ddot{a}_x} - d}$

fully discrt.
whole life

$_kV = A_{x+k} - P\,\ddot{a}_{x+k}$

$= 1 - d\ddot{a}_{x+k} - \underline{\left(\frac{1}{\ddot{a}_x} - d\right)\ddot{a}_{x+k}}$

$= 1 - \cancel{d\ddot{a}_{x+k}} - \frac{\ddot{a}_{x+k}}{\ddot{a}_x} + \cancel{d\ddot{a}_{x+k}}$

$= 1 - \frac{\ddot{a}_{x+k}}{\ddot{a}_x}$

---

fully continuous
whole life

$\bar{A}_x = 1 - \delta\bar{a}_x$

$\bar{P} = \frac{\bar{A}_x}{\bar{a}_x} = \frac{1 - \delta\bar{a}_x}{\bar{a}_x}$

$= \frac{1}{\bar{a}_x} - \delta$

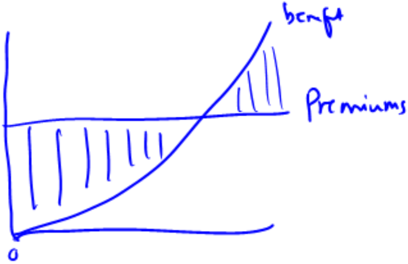
$_tV = \bar{A}_{x+t} - \bar{P} \cdot \bar{a}_{x+t}$

$1 - \delta\bar{a}_{x+t}$

$\left(\frac{1}{\bar{a}_x} - \delta\right)\bar{a}_{x+t}$

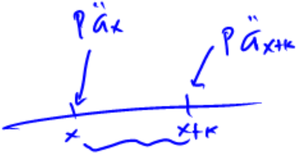
$\vdots$

$= 1 - \frac{\bar{a}_{x+t}}{\bar{a}_x}$ $\quad P \longrightarrow$



$$ {}_kV = 1 - \frac{\ddot{a}_{x+k}}{\ddot{a}_x} \cdot \frac{P}{P} $$

$$ = \frac{P\ddot{a}_x - P\ddot{a}_{x+k}}{P\ddot{a}_x} $$



$P\ddot{a}_x$ at issue

$$ \underbrace{P\ddot{a}_x - P\ddot{a}_{x+k}} $$

paid between x to x+k

should be enough to cover benefits yet to be paid

# Illustrative example 3 (study if you have time)

For a 10-year deferred whole life annuity of 1 on (35) payable continuously, you are given:

- Mortality follows deMoivre's law with $\omega = 85$. Recall Uniform
- Level benefit premiums are payable continuously for 10 years.
- $i = 0$ $\Rightarrow v = 1$

Calculate the benefit reserve at the end of five years.

step 1 Calculate premium P

APV(FP$_0$) = APV(FB$_0$)

$$P\,\bar{a}_{35:\overline{10}|} = 1 \cdot {}_{10}E_{35}\,\bar{a}_{45}$$

$$P = \frac{4/5\,\overset{4}{(20)}}{5} = 16/5$$

$${}_{10}E_{35} = \cancel{v^{10}}\,{}_{10}p_{55} = 1 - \frac{10}{50} = 4/5$$

$$\bar{a}_{45} = \int_0^{40} v^t\,{}_tp_{45}\,dt = 40 - \frac{1}{40}\frac{40^2}{2} = 20$$

$$\frac{t}{1}\left(1 - \frac{t}{40}\right)$$

$$\bar{a}_{35:\overline{10}|} = \int_0^{10} v^t\,{}_tp_{35}\,dt = 10 - \frac{1}{50}\frac{10^2}{2} = 9$$

$$\overset{\prime}{1}\left(1 - \frac{t}{51}\right)$$

Reserve at end of year 5 = APV$(FB_5)$ − APV$(FP_5)$

$$= {}_5E_{40}\,\bar{a}_{45} - P\,\bar{a}_{40:\overline{5}|}$$

$$= \frac{8}{9}\cdot 20 - \frac{16}{9}\cdot 5\left(\frac{17}{18}\right)$$

$$= \underline{\underline{9.382716}}$$

$$\approx 9.4$$

2/11/2010

$P = 16/9$

$${}_5E_{40} = v^5\,{}_5p_{40} = \frac{8}{9}$$

$$1-\frac{5}{45}$$

$$\bar{a}_{40:\overline{5}|} = \int_0^5\left(1-\frac{t}{40}\right)dt$$

$$= 5-\frac{1}{45}\cdot\frac{5^2}{2}$$

$$= 5\left(1-\frac{1}{18}\right) = 5\left(\frac{17}{18}\right)$$

# Illustrative example 4 - modified SOA MLC Spring 2012

A special fully discrete 3-year endowment insurance on $(x)$ pays death benefits as follows:

| Year of Death | Death Benefit |
|---|---|
| 1 | \$ 10,000 |
| 2 | \$ 20,000 |
| 3 | \$ 30,000 |

premium

$P = ?$

You are given:

- The endowment benefit amount is \$ 50,000.
- Annual benefit premiums increase at 10% per year, compounded annually.
- $i = 0.05$ ✓
- $q_x = 0.08 \qquad q_{x+1} = 0.10 \qquad q_{x+2} = 0.12$

Calculate the benefit reserve at the end of year 2.

at $t=0$, $APV(FP_0) = APV(FB_0)$

✓

$$\boxed{P = 36477.10}$$

$$\Leftarrow \quad P + \frac{P(1.1)}{1.05}.92 + P\frac{(1.1)^2}{1.05^2}.92(.9) = 10{,}000\frac{(.08)}{1.05} + \frac{20{,}000(.92)(0.1)}{1.05^2} + \frac{30{,}000(.92)(.9)(.12)}{1.05^3}$$

$$+ \frac{50000(.92)(.9)(.88)}{1.05^3}$$

$P = ?$

use recursive $\quad {}_0V = 0 \rightarrow {}_1V \rightarrow {}_2V \rightarrow {}_3V = 50{,}000$

$${}_1V = \frac{(\cancel{0}+P)(1.05) - 10000(.08)}{.92} = P\frac{(1.05)}{.92} - 10{,}000\frac{(.08)}{.92} = X$$

$$✓\ {}_2V = \frac{(X + P(1.1))(1.05) - 20000(.10)}{.90} = Y$$

$${}_3V = \frac{(Y + P(1.1)^2)(1.05) - 30000(.12)}{.88} = 50{,}000 \rightarrow P$$

$$\Rightarrow \underline{{}_2V = 29{,}968.11}$$

$${}_1V = 13{,}623.33$$

# SOA MLC question #15 Fall 2015 - modified

$G = ?$

$1000 A_{35}$

For a fully discrete whole life insurance of 1000 on $(35)$, you are given:

- First year expenses are 30% of the gross premium plus 300.
- Renewal expenses are 4% of the gross premium plus 30.
- All expenses are incurred at the beginning of the policy year.
- Gross premiums are calculated using the equivalence principle.
- Mortality follows the `Survival Ultimate Life Table` with $i = 0.05$.

Calculate the gross premium reserve at the end of the first policy year.

$G = ?$ $\quad APV(FP_0) = APV(FB_0) + APV(FE_0)$ /

$$G\ddot{a}_{35} = 1000A_{35} + .04G\ddot{a}_{35} + 30\ddot{a}_{35} + .26G + 270$$

look up SULT

$A_{35} = .09653$

$\ddot{a}_{35} = 18.9728$

$$\cancel{(.96\ddot{a}_{35} - .26)}G = \frac{1000A_{35} + 30\ddot{a}_{35} + 270}{.96\ddot{a}_{35} - .26}$$

${}_1V = ?$ $\qquad G = ?\ 52.11762$

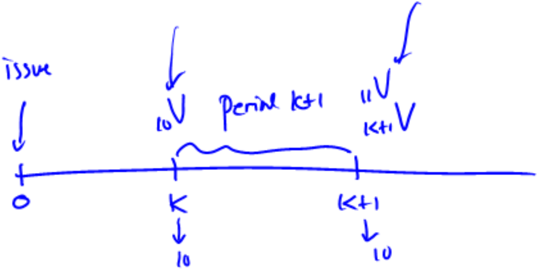
$${}_0V = 0$$

$${}_1V = \frac{(0 + G - .30G - 300)(1.05) - 1000q_{35}}{\boxed{1 - q_{35}}} = ???$$

$q_{35} = .000391$

# Other terminologies and notations used

| Expression | Other terms/symbols used |
|---|---|
| ✓ reserves net<br>gross | policy values ✓ |
| ✓ future loss random variable | prospective loss ✓ |
| $(B - {}_tV)$<br>✓ net amount at risk | death strain at risk (DSAR)<br>sum at risk |
| reserve at end of the year | terminal reserve ✓ |
| reserve at beginning of year plus applicable premium | initial reserve ✓ |


Issue
10V
Period k+1
11V
k+1V
0
K
K+1
10
10

Is reserve suffiient? **Study this!** – $_{K}V = 100$

fully discrete whole life

no expenses

$B=100$

issue (45)

SULT at $i = 5\%$

$A_{45} = .15161$ ✓ $A_{55} = .23524$

$\ddot{a}_{45} = 17.8162$ $\ddot{a}_{55} = 160599$

$_{10}V = ?$ $P = 100 \dfrac{A_{45}}{\ddot{a}_{45}} = .850961$

$K = K_{55}$

$_{10}V = 100 A_{55} - P\ddot{a}_{55} = 9.857554$ ✓

Future loss $= L_{10} = \underbrace{100 v^{K+1} - P\ddot{a}_{\overline{K+1}|}}$

Big question: ~~Is this enough?~~ Is reserves not sufficient?

$Pr[L_{10}^{*} > 9.857554]$

$$\downarrow$$

$$100v^{K+1} - P\underbrace{\ddot{a}_{\overline{K+1}|}}_{\frac{1-v^{K+1}}{d}} = \left(100 + \frac{P}{d}\right)v^{K+1} - \frac{P}{d} > 9.857554$$

$$\Pr[L_{10} > 9.857554] \Rightarrow \Pr\left[ v^{K+1} > \overbrace{\frac{9.857554 + P/d}{100 + P/d}}^{a} \right]$$

$v = e^{-\delta}$

$\log v = -\delta$

$$\log v^{K+1} > \log a$$

$$(K+1)\cancel{\log v} < \frac{\log a}{\log v}$$

$$K < \underbrace{\frac{\log a}{-\delta} - 1}_{28.66063}$$

$P = 9.857554$

$d = .05/1.05$

$\log v = -\delta$

$\frac{1}{1.05} = \frac{1}{1+i} = v = e^{-\delta}$

$x = 45$

$$\Pr[K < 28.66063] = \Pr[K \leq 28]$$

$$= {}_{29}q_{45} = 1 - {}_{29}p_{45} = 1 - \frac{l_{74}}{l_{45}} = 1 - \frac{86627.6}{99033.9}$$

$$= 0.1252733$$

$$\approx \boxed{0.13}$$

net premium reserve
no expenses

$${}_tV^n = APV(\cancel{F}B_t) - APV(FP_t)$$

-) Gross premium reserve
with expenses

$${}_tV^g = APV(\cancel{F}B_t) - APV(FG_t) + APV(FE_t)$$

---

$${}_tV^g - {}_tV^n = \underbrace{APV(FE_t)} - \big(\underbrace{APV(FG_t) - APV(FP_t)}\big)$$

fully discrete whole life

$$G\ddot{a}_{x+t} - P\ddot{a}_{x+t}$$

$$= \text{all expenses} - (G-P)\ddot{a}_{x+t}$$

most often negative

G>>P
↓
comm expens

define $G-P$ = expense loading -

$${}_tV^e = APV(FE_t) - APV(FEL_t)$$

expense reserve $\quad {}_tV^e = {}_tV^\delta - {}_tV^n$

$$= APV(FE_t) - APV(FE\ell_t)$$

fully discrete whole life to (x)

expenses: 1st yr 10
subsequent 9

recuperate 1st yr expense over time

$B = 100$

(x)

$P = 100\, A_x / \ddot{a}_x$

$G = P + 9 + \frac{1}{\ddot{a}_x}$

expense loading $= G - P = 9 + \frac{1}{\ddot{a}_x}$

10 years $\quad {}_{10}V^\delta$ ✓ $\quad {}_{10}V^n$ ✓ $\quad \underline{{}_{10}V^e}$ ✓

$${}_{10}V^e = APV(FE_{10}) - APV(FE\ell_{10})$$

$$= (\cancel{9}\,\ddot{a}_{x+10}) - \left(\cancel{9} + \frac{1}{\ddot{a}_x}\right)\ddot{a}_{x+10} = \boxed{-1 * \frac{\ddot{a}_{x+10}}{\ddot{a}_x}}$$

Some clarification : FEL = Future Expense loading

difference between gross and net premium $G - P$ = expense loading

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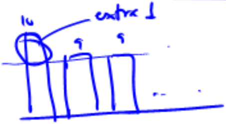
Consider age $(x)$

benefit = 100

fully discrete

expenses are: 10 in 1st yr

9 in renewal yr



net premium $P = 100\, A_x / \ddot{a}_x$

$APV(FG_0) = APV(FB_0) + APV(FE_{10})$

gross premium $G\ddot{a}_x = 100\, A_x + 9\, \ddot{a}_x + 1$

$$G = \underbrace{100\, \frac{A_x}{\ddot{a}_x}} + 9\, \frac{\cancel{\ddot{a}_x}}{\cancel{\ddot{a}_x}} + \frac{1}{\ddot{a}_x}$$

$$G = P + 9 + \frac{1}{\ddot{a}_x}$$ ✓