Michigan State University
STT 455 - Actuarial Models I
Fall 2014 semester
Homework No. 1
due Friday, 5:00 pm, September 19, 2014

**Please follow the instructions below:**

Return this page with your signature.

Submit your work to our graduate assistant, Ed Cruz, at C505 Wells.

Write your name and section number at the spaces provided:

**Name:** SUGGESTED SOLUTION **Section:** ____________________

I certify that this is my own work, and that I have not copied the work of another student.

**Signature:** ____________________________ **Date:** ____________________

1. (35 points) Suppose that the future lifetime of a newborn follows the survival function

$$S_0(t) = \left(\frac{105 - t}{105}\right)^{1/3}, \quad \text{for } 0 < t \leq 105.$$

   (a) [15 points] Explain why this is a valid survival function.

   (b) [10 points] Calculate $\mathrm{E}(T_0) = \mathring{e}_0$

   (c) [5 points] Calculate ${}_{10}q_{30}$ and interpret this value.

   (d) [5 points] Calculate the probability that (40) will die between ages 65 and 75.

(a) Valid since $S_0(0) = 1$, $S_0(\infty) = S_0(105) = 0$,

and $\frac{d}{dt} S_0(t) = -\frac{1}{3}\frac{1}{105}\left(\frac{105-t}{105}\right)^{-2/3} \leq 0 \Rightarrow$ nonincreasing

(b) $\mathring{e}_0 = \int_0^{105} \left(\frac{105-t}{105}\right)^{1/3} dt = \int_0^{105} \left(1 - t/105\right)^{1/3} dt = -105 \int_1^0 u^{1/3} du$

use substitution $u = 1 - t/105$

$du = -\frac{1}{105} dt$

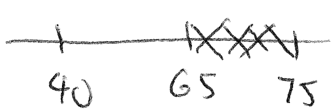
$= -105 \frac{u^{4/3}}{4/3} = 105\left(\frac{3}{4}\right)$

$= \underline{78.75}$

(c) ${}_{10}q_{30} = 1 - {}_{10}p_{30} = 1 - \frac{S_0(40)}{S_0(30)} = 1 - \left(\frac{65}{75}\right)^{1/3} = .0465805$

This gives the probability that (30) will die within the next 10 years, or between ages 30 and 40.

(d)




${}_{25|10}q_{40} = {}_{25}p_{40} - {}_{35}p_{40}$

$= \frac{S_0(65) - S_0(75)}{S_0(40)} = \frac{40^{1/3} - 30^{1/3}}{65^{1/3}}$

$= \underline{.07777685}$

2. (25 points) Suppose that $T_0$ follows a constant force with density

$$f_0(t) = \frac{1}{100}e^{-t/100}, \quad \text{for } t > 0. \Rightarrow S_0(t) = e^{-t/100}$$

(a) [8 points] Explain why $T_x$, for any age $x > 0$, follows a constant force with similar density as

$$f_x(t) = \frac{1}{100}e^{-t/100}, \quad \text{for } t > 0.$$

(b) [5 points] Calculate $\mathrm{E}(T_x) = \mathring{e}_x$.

(c) [7 points] Calculate $e_x$.

(d) [5 points] Explain briefly why (b) is different from (c).

(a) $$f_x(t) = \frac{f_0(x+t)}{S_0(x)} = \frac{\frac{1}{100}e^{-(\cancel{x}+t)/100}}{\cancel{e^{-x/100}}} = \frac{1}{100}e^{-t/100}, \; t > 0$$

(b) $$E(T_x) = \int_0^\infty t \cdot \frac{1}{100}e^{-t/100}\,dt \quad \text{OR} \quad \underbrace{\int_0^\infty e^{-t/100}\,dt}_{100}$$

(c) $$e_x = \sum_{k=1}^{\infty} {}_kp_x = \underbrace{\sum_{k=1}^{\infty}\left(e^{-1/100}\right)^k}_{\text{geometric sum}} \qquad {}_kp_x = \frac{S_0(x+k)}{S_0(x)} = \frac{e^{-(\cancel{x}+k)/100}}{\cancel{e^{-x/100}}}$$

$$= \frac{e^{-1/100}}{1-e^{-1/100}} = \underline{\underline{99.50083}}$$

(d) $e_x = E[K_x]$ where $K_x$ ignores the fractional part of the year ~~at~~ death Therefore, we would expect $e_x$ to be lower than $\mathring{e}_x$. And by about ½ year on the average, which approximately holds true in this case!

3. (40 points) You are given the force of mortality:

$$\mu_x = a + e^{bx}$$

where $a$ and $b$ are positive constants. In addition, you are given the following values:

$$p_0 = 0.30068 \qquad p_1 = 0.26920 \qquad p_2 = 0.23822$$

(a) [15 points] Show that the following expression is true:

$$_tp_x = e^{-at} \exp\left[-\frac{e^{bx}}{b}(e^{bt} - 1)\right]$$

(b) [20 points] Calculate the constants $a$ and $b$. HINT: Calculate the expression:

$$\frac{\log(p_2) - \log(p_1)}{\log(p_1) - \log(p_0)}$$

See also DHW, Exercise 2.11.

(c) [5 points] Calculate $\mu_{45}$.

(a) $$_tp_x = e^{-\int_0^t \mu_{x+s} ds} = \exp\left[-\int_0^t \left(a + e^{b(x+s)}\right) ds\right]$$

$$= \exp\left[-\left(\int_0^t a\, ds + e^{bx} \int_0^t e^{bs} ds\right)\right]$$

$$= \exp\left[-at + e^{bx} \cdot \frac{1}{b}(e^{bt} - 1)\right]$$

(b) $$\frac{\log(p_2) - \log(p_1)}{\log(p_1) - \log(p_0)} = \frac{\left[\cancel{-a} + e^{2b} \cancel{\tfrac{1}{b}} \cancel{(e^b - 1)}\right] - \left[\cancel{-a} + e^{b} \cdot \cancel{\tfrac{1}{b}} \cancel{(e^b - 1)}\right]}{\left[\cancel{-a} + e^{b} \cancel{\tfrac{1}{b}} \cancel{(e^b - 1)}\right] - \left[\cancel{-a} + \cancel{\tfrac{1}{b}} \cancel{(e^b - 1)}\right]}$$

$$= \frac{e^{2b} - e^{b}}{e^b - 1} = \frac{e^b(e^b - 1)}{e^b - 1} = e^b = \frac{\log(.23822) - \log(.26920)}{\log(.26920) - \log(.30068)}$$

$$= 1.003022$$

$$\Rightarrow b = .1003022$$

Since $P_0 = e^{-a} \exp\left[-\frac{1}{b}(e^{b}-1)\right]$

$$= e^{-a} \underbrace{\exp\left[\frac{-1}{.1003022}\left(e^{.1003022}-1\right)\right]}_{.3492837} = .30068$$

$$\Rightarrow e^{-a} = \frac{.30068}{.3492837} = .8608475$$

$$\Rightarrow a = -\log(.8608475) = .1498379$$

(c) Plug $a = .1498379$ and $b = .1003022$

$$\mu_{45} = .1498379 + e^{.1003022}$$

$$= \underline{\underline{91.39952}}$$