Exercise 5.9

We can write

$$
\begin{aligned}
{}_kp_x &= \exp\left[-\int_0^k \mu_{x+s} ds\right] = \exp\left[-\int_0^k \frac{1}{2}\left(\mu^A_{x+s} + \mu^B_{x+s}\right) ds\right] \\
&= \exp\left[-\frac{1}{2}\int_0^k \mu^A_{x+s} ds\right] \times \exp\left[-\frac{1}{2}\int_0^k \mu^A_{x+s} ds\right] \\
&= \left({}_kp^A_x\right)^{1/2} \times \left({}_kp^B_x\right)^{1/2},
\end{aligned}
$$

the geometric average of the survival probabilities from mortality tables A and B. Therefore, the true value of a_x satisfies

$$
\begin{aligned}
a_x &= \sum_{k=1}^{\infty} v^k \, {}_kp_x \\
&= \sum_{k=1}^{\infty} v^k \left({}_kp^A_x\right)^{1/2} \cdot \left({}_kp^B_x\right)^{1/2} \\
&\leq \sum_{k=1}^{\infty} v^k \left[\frac{1}{2}\left({}_kp^A_x + {}_kp^B_x\right)\right] \\
&= \frac{1}{2}\left(a^A_x + a^B_x\right)
\end{aligned}
$$

The third line follows from the fact that geometric averages always produce smaller values than arithmetic averages. Hence, the approximation suggested by the student will always overestimate the true value of the annuity. Equality will hold if and only if

$$
{}_kp^A_x = {}_kp^B_x, \quad \text{for all } k.
$$

* corrected on Nov 11, 2011