Exercise 5.8

(a)
$$\begin{aligned}\ddot{a}_{[40]:\overline{4}|} &= 1 + v\frac{\ell_{[40]+1}}{\ell_{[40]}} + v^2\frac{\ell_{42}}{\ell_{[40]}} + v^3\frac{\ell_{43}}{\ell_{[40]}} \\ &= 1 + \frac{(1.06)^{-1}33485 + (1.06)^{-2}33440 + (1.06)^{-3}33378}{33519} \\ &= 3.666425\end{aligned}$$

* corrected on 7 December 2012; Thanks to K. Panther

(b)
$$\begin{aligned}a_{[40]+1:\overline{4}|} &= v\frac{\ell_{42}}{\ell_{[40]+1}} + v^2\frac{\ell_{43}}{\ell_{[40]+1}} + v^3\frac{\ell_{44}}{\ell_{[40]+1}} + v^4\frac{\ell_{45}}{\ell_{[40]+1}} \\ &= \frac{1}{33485}\left[(1.06)^{-1}33440 + (1.06)^{-2}33378 + (1.06)^{-3}33309 + (1.06)^{-4}33231\right] \\ &= 3.450572\end{aligned}$$

(c)
$$\begin{aligned}(Ia)_{[40]:\overline{4}|} &= v\frac{\ell_{[40]+1}}{\ell_{[40]}} + 2v^2\frac{\ell_{42}}{\ell_{[40]}} + 3v^3\frac{\ell_{43}}{\ell_{[40]}} + 4v^4\frac{\ell_{44}}{\ell_{[40]}} \\ &= \frac{1}{33519}\left[(1.06)^{-1}33485 + 2(1.06)^{-2}33440 + 3(1.06)^{-3}33378 + 4(1.06)^{-4}33309\right] \\ &= 8.375024\end{aligned}$$

(d)
$$\begin{aligned}(IA)_{[40]:\overline{4}|} &= v\frac{d_{[40]}}{\ell_{[40]}} + 2v^2\frac{d_{[40]+1}}{\ell_{[40]}} + 3v^3\frac{d_{42}}{\ell_{[40]}} + 4v^4\frac{d_{43}}{\ell_{[40]}} + 4v^4\frac{\ell_{44}}{\ell_{[40]}} \\ &= \frac{1}{33519}\left[(1.06)^{-1}(33519 - 33485) + 2(1.06)^{-2}(33485 - 33440)\right. \\ &\quad \left. +3(1.06)^{-3}(33440 - 33378) + 4(1.06)^{-4}(33378 - 33309) + 4(1.06)^{-4}33309\right] \\ &= 3.163052\end{aligned}$$

(e) Denote by $Y = \ddot{a}_{\overline{K+1}|}I(K \leq 3) + \ddot{a}_{\overline{4}|}I(K > 3)$ the present value random variable for a four-year life annuity-due of \$1 per year on [41]. The following table provides necessary details of the calculations:

k	$\Pr[K_{[41]} = k]$	$y = \ddot{a}_{\overline{k+1}|}$	$y \cdot \Pr[K_{[41]} = k]$	y^2	$y^2 \cdot \Pr[K_{[41]} = k]$
0	0.00117	1.00000	0.00117	1.00000	0.00117
1	0.00149	1.94340	0.00290	3.77679	0.00564
2	0.00206	2.83339	0.00584	8.02811	0.01655
≥ 3	0.99528	3.67301	3.65567	13.49102	13.42732
sum	1.00000		3.665582		13.45068

Thus, we find from this table that

$$\mathrm{E}[Y] = 3.665582 \quad \text{and} \quad \mathrm{E}[Y^2] = 13.45068$$

so that the required standard deviation is given by

$$\mathrm{SD} = 1000 \times \sqrt{\mathrm{E}[Y^2] - (\mathrm{E}[Y])^2} = 1000 \times \sqrt{13.45068 - (3.665582)^2} = 119.1382.$$

(f) Examining the column of $\ddot{a}_{\overline{k+1}|}$ from the above table, we note that the present value of a life annuity-due of 1 per year is less than 3.0 for $k \leq 2$. Thus, the required probability is

$$\begin{aligned}
\Pr[K_{[40]} \leq 2] &= \frac{d_{[40]} + d_{[40]+1} + d_{42}}{\ell_{[40]}} \\
&= \frac{(33519 - 33485) + (33485 - 33440) + (33440 - 33378)}{33519} \\
&= 0.004206569
\end{aligned}$$