Exercise 5.10

First, we make the following observations:

- For $(IA)_x$, this provides a death benefit of \$1 in the first year, increasing by \$1 each year thereafter, where benefit is payable at the end of the year of death. The present value random variable associated with this increasing insurance is

$$Z = (K+1)v^{K+1}, \quad \text{for } K = 0, 1, 2, \cdots$$

- For $\ddot{a}_x$, this provides a whole life annuity-due of \$1 with present value random variable

$$Y_1 = \ddot{a}_{\overline{K+1}|}, \quad \text{for } K = 0, 1, 2, \cdots$$

- For $(I\ddot{a})_x$, this provides a whole life annuity-due of \$1 payable in the first year increasing by \$1 each year thereafter. The present value random variable can be expressed as

$$Y_2 = (I\ddot{a})_{\overline{K+1}|}, \quad \text{for } K = 0, 1, 2, \cdots$$

Therefore, we see that

$$\begin{aligned} Y_1 - dY_2 &= \ddot{a}_{\overline{K+1}|} - d\,(I\ddot{a})_{\overline{K+1}|} \\ & \text{use the hint} \\ &= \ddot{a}_{\overline{K+1}|} - d\left[\frac{\ddot{a}_{\overline{K+1}|} - (K+1)v^{K+1}}{d}\right] \\ &= \ddot{a}_{\overline{K+1}|} - \ddot{a}_{\overline{K+1}|} + (K+1)v^{K+1} \\ &= (K+1)v^{K+1} = Z \end{aligned}$$

Taking the expectations of both sides, we get

$$(IA)_x = \mathrm{E}[Z] = \mathrm{E}[Y_1] - d\mathrm{E}[Y_2] = \ddot{a}_x - d(I\ddot{a})_x,$$

which proves the desired result.