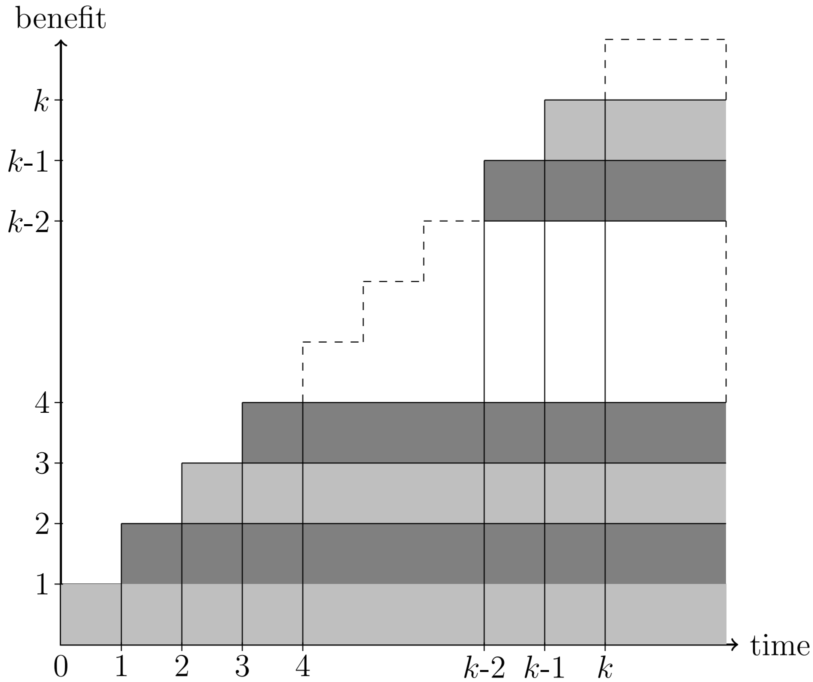
Exercise 4.6

For $(IA^{(m)})_x$, the benefit starts with \$1 if death occurs in the first year and increasing by \$1 each year for deaths thereafter. Obviously, the benefit is payable at the end of the m-th interval in the year of death. The benefit payment pattern is best visualize in the following picture:



This visualization allows us to interpret the Actuarial Present Value as the sum of deferred insurances each with \$1 of death benefits with detailed proof as follows:

$$
\begin{aligned}
(IA^{(m)})_x &= \sum_{k=0}^{\infty}\sum_{j=0}^{m-1}(k+1)v^{k+\frac{j+1}{m}}\ {}_{k+\frac{j}{m}|\frac{1}{m}}q_{x+k} \\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{m-1}\sum_{s=0}^{k} v^{k+\frac{j+1}{m}}\ {}_{k+\frac{j}{m}|\frac{1}{m}}q_{x+k} \\
&= \sum_{k=0}^{\infty}\sum_{s=0}^{k}\sum_{j=0}^{m-1} v^{k+\frac{j+1}{m}}\ {}_{k+\frac{j}{m}|\frac{1}{m}}q_{x+k} \\
&\quad \text{change the order of the first two summations} \\
&= \sum_{s=0}^{\infty}\left[\sum_{k=s}^{\infty}\sum_{j=0}^{m-1} v^{k+\frac{j+1}{m}}\ {}_{k+\frac{j}{m}|\frac{1}{m}}q_{x+k}\right] \\
&= \sum_{s=0}^{\infty} {}_{s|}A_x^{(m)},
\end{aligned}
$$

where in the second line, we write $(k+1) = \sum_{s=0}^{k} 1$, that is, the sum of 1's $k+1$ times.

The result immediately follows because we can write

$$\begin{aligned}
(IA^{(m)})_x &= \sum_{s=0}^{\infty} {}_{s|}A_x^{(m)} \\
&= \sum_{s=0}^{\infty} v^s \, {}_sp_x \, A_{x+s}^{(m)} \\
&= A_x^{(m)} + vp_x \, A_{x+1}^{(m)} + v^2 \, {}_2p_x \, A_{x+2}^{(m)} + \cdots
\end{aligned}$$

Thus, we see that equivalently, if death occurs in the first year, benefit is \$1, increasing by \$1 each year of death thereafter. The extra \$1 comes from each term whenever death is prolonged for another year.