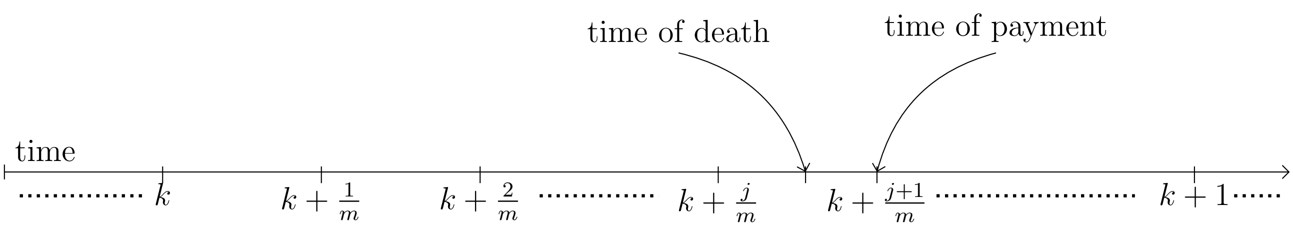
Exercise 4.3

For this whole life insurance $A_x^{(m)}$, the benefit is paid at the end of the m-th period in the year of death. This timing of payment is important to visualize to calculate the Actuarial Present Value. The following picture provides this timing if the year of death is within k to $k+1$:



It is therefore clear to see that

$$A_x^{(m)} = \sum_{k=0}^{\infty}\sum_{j=0}^{m-1} v^{k+\frac{j+1}{m}} \; {}_{k+\frac{j}{m}|\frac{1}{m}}q_x = \sum_{k=0}^{\infty}\sum_{j=0}^{m-1} v^{k+\frac{j+1}{m}} \, {}_kp_x \; {}_{\frac{j}{m}|\frac{1}{m}}q_{x+k}.$$

* corrected on March 3, 2012

Applying the UDD assumption between integral ages, we have ${}_{\frac{j}{m}|\frac{1}{m}}q_{x+k} = \dfrac{1}{m}\cdot q_{x+k}$ so that we have

$$\begin{aligned}
A_x^{(m)} &= \sum_{k=0}^{\infty} v^k \, {}_kp_x \sum_{j=0}^{m-1} \frac{1}{m} v^{\frac{j+1}{m}} \, q_{x+k} \\
&= \sum_{k=0}^{\infty} v^k \, {}_kp_x \, q_{x+k} \sum_{j=0}^{m-1} \frac{1}{m} v^{\frac{j+1}{m}} \\
&= \sum_{k=0}^{\infty} v^k \, {}_{k|}q_x \sum_{j=0}^{m-1} \frac{1}{m} v^{\frac{j+1}{m}} \\
&= \sum_{k=0}^{\infty} v^{k+1} \, {}_{k|}q_x \cdot \frac{1}{v} \sum_{j=0}^{m-1} \frac{1}{m} v^{\frac{j+1}{m}} \\
&= \frac{i}{i^{(m)}} A_x
\end{aligned}$$

The last term holds because clearly, we have

$$A_x = \sum_{k=0}^{\infty} v^{k+1} \, {}_{k|}q_x$$

and that

$$\frac{1}{v}\sum_{j=0}^{m-1} \frac{1}{m} v^{\frac{j+1}{m}} = \frac{1}{v}\frac{1}{m}\frac{v^{1/m}(1-v)}{1-v^{1/m}} = \frac{1-v}{v}\frac{1}{m\left[(1+i)^{1/m}-1\right]} = \frac{i}{i^{(m)}}.$$