Exercise 4.21

(a) Denote by Z the present value random variable for this policy. The Actuarial Present Value of the benefits can be expressed as

$$\text{APV(benefits)} = \text{E}[Z] = 2000A_{50}^{(4)} - 1000v^{15}{}_{15}p_{50}A_{65}^{(4)}.$$

The following R code produces a table of $A_x^{(4)}$ starting at age $x = 50$:

```
A <- 0.00022
B <- 2.7*10^(-6)
c <- 1.124
x <- 50:147
y <- seq(50,147,0.25)
surv <- function(x){
exp(-A*x-(B*(c^x-1)/log(c)))}
px <- surv(y+0.25)/surv(y)
qx <- 1-px
int <- 0.05
v <- 1/(1+int)
vcum <- v^((1:length(y))/4)
Axqtr <- rep(0,length(y))
Axqtr[1] <- sum(vcum*cumprod(c(1,px[-length(px)]))*qx)
n <- 1
while (n<length(y)) {
n <- n+1
Axqtr[n] <- (Axqtr[n-1]*(1+int)^(1/4)-qx[n-1])/px[n-1]
}
x <- x[which(x==50):which(x==101)]
Axqtrz <- rep(0,length(x))
n<-0
while (n<length(x)) {
n <- n+1
Axqtrz[n] <- Axqtr[which(y==x[n])]
}
a <- matrix(1:length(x),nrow=length(x)/4,4)
output <- cbind(x[a[,1]],Axqtrz[a[,1]],x[a[,2]],Axqtrz[a[,2]],
  x[a[,3]],Axqtrz[a[,3]],x[a[,4]],Axqtrz[a[,4]])
colnames(output) <- c("x","Ax(4)","x","Ax(4)","x","Ax(4)","x","Ax(4)")
print(output)
```

This gives the output

```
> print(output)
        x     Ax(4)  x     Ax(4)  x     Ax(4)    x     Ax(4)
 [1,] 50 0.1927898 63 0.3338911 76 0.5350872  89 0.7524396
```

```
 [2,] 51 0.2014422 64 0.3474130 77 0.5522105  90 0.7673815
 [3,] 52 0.2104392 65 0.3613103 78 0.5694207  91 0.7818683
 [4,] 53 0.2197886 66 0.3755752 79 0.5866788  92 0.7958688
 [5,] 54 0.2294976 67 0.3901979 80 0.6039439  93 0.8093551
 [6,] 55 0.2395731 68 0.4051662 81 0.6211736  94 0.8223034
 [7,] 56 0.2500210 69 0.4204655 82 0.6383243  95 0.8346939
 [8,] 57 0.2608466 70 0.4360791 83 0.6553516  96 0.8465115
 [9,] 58 0.2720541 71 0.4519873 84 0.6722106  97 0.8577451
[10,] 59 0.2836468 72 0.4681683 85 0.6888565  98 0.8683882
[11,] 60 0.2956269 73 0.4845972 86 0.7052449  99 0.8784388
[12,] 61 0.3079950 74 0.5012470 87 0.7213326 100 0.8878989
[13,] 62 0.3207505 75 0.5180878 88 0.7370775 101 0.8967744
```

From this table, we find that

$$\text{APV(benefits)} = 2000(0.1927898) - 1000(1/1.05)^{15}(0.9594565)(0.3613103) = 218.8295.$$

(b) To evaluate the $\mathrm{E}[Z^2]$, we note that we write $Z = Z_1 - Z_2$ where Z_1 is the present value random variable associated with a whole life policy that pays 2000 at the end of the quarter in the year of death and Z_2 is the present value random variable associated with a 15-year deferred life policy that pays 1000 at the end of the quarter in the year of death. Thus,

$$\mathrm{E}[Z^2] = \mathrm{E}[Z_1^2] + \mathrm{E}[Z_2^2] - 2\mathrm{E}[Z_1 Z_2],$$

where clearly $Z_1 Z_2$ is the present value random variable associated with a 15-year deferred life policy that pays 2000×1000 at the end of the quarter in the year of death, but evaluated at 2δ. Thus, we see that

$$\begin{aligned}
\mathrm{E}[Z^2] &= (2000)^2\,{}^2A_{50}^{(4)} + (1000)^2 v^{30}\,{}_{15}p_{50}\,{}^2A_{50}^{(4)} - 2(2000)(1000)v^{30}\,{}_{15}p_{50}\,{}^2A_{50}^{(4)} \\
&= (2000)^2\,{}^2A_{50}^{(4)} + [(1000)^2 - 4(1000)^2]v^{30}\,{}_{15}p_{50}\,{}^2A_{50}^{(4)} \\
&= (2000)^2\,{}^2A_{50}^{(4)} - 3(1000)^2 v^{30}\,{}_{15}p_{50}\,{}^2A_{50}^{(4)}
\end{aligned}$$

The following R code produces a table of ${}^2A_x^{(4)}$ starting at age $x = 50$:

```
# to calculate the variance, we evaluate APV at interest 2*delta
v <- 1/(1+int)^2
vcum <- v^((1:length(y))/4)
Ax2qtr <- rep(0,length(y))
Ax2qtr[1] <- sum(vcum*cumprod(c(1,px[-length(px)]))*qx)
n <- 1
while (n<length(y)) {
n <- n+1
Ax2qtr[n] <- (Ax2qtr[n-1]*(1+int)^(2/4)-qx[n-1])/px[n-1]
}
x <- x[which(x==50):which(x==101)]
Ax2qtrz <- rep(0,length(x))
n<-0
```

```
while (n<length(x)) {
n <- n+1
Ax2qtrz[n] <- Ax2qtr[which(y==x[n])]
}
a <- matrix(1:length(x),nrow=length(x)/4,4)
output <- cbind(x[a[,1]],Ax2qtrz[a[,1]],x[a[,2]],Ax2qtrz[a[,2]],
  x[a[,3]],Ax2qtrz[a[,3]],x[a[,4]],Ax2qtrz[a[,4]])
colnames(output) <- c("x","2Ax(4)","x","2Ax(4)","x","2Ax(4)","x","2Ax(4)")
print(output)
```

This gives the output

```
> print(output)
       x     2Ax(4)  x    2Ax(4)  x    2Ax(4)   x    2Ax(4)
 [1,] 50 0.05296794 63 0.1391988 76 0.3193349  89 0.5863475
 [2,] 51 0.05721316 64 0.1492685 77 0.3376135  90 0.6076205
 [3,] 52 0.06177959 65 0.1599316 78 0.3564491  91 0.6286322
 [4,] 53 0.06668705 66 0.1712048 79 0.3758061  92 0.6493072
 [5,] 54 0.07195597 67 0.1831030 80 0.3956429  93 0.6695728
 [6,] 55 0.07760734 68 0.1956386 81 0.4159125  94 0.6893603
 [7,] 56 0.08366261 69 0.2088217 82 0.4365622  95 0.7086055
 [8,] 57 0.09014356 70 0.2226592 83 0.4575342  96 0.7272499
 [9,] 58 0.09707224 71 0.2371544 84 0.4787656  97 0.7452413
[10,] 59 0.10447077 72 0.2523072 85 0.5001893  98 0.7625345
[11,] 60 0.11236119 73 0.2681130 86 0.5217343  99 0.7790917
[12,] 61 0.12076527 74 0.2845630 87 0.5433267 100 0.7948826
[13,] 62 0.12970430 75 0.3016433 88 0.5648903 101 0.8098848
```

From the table above, we find that

$$\mathrm{E}[Z^2] = (2000)^2(0.05296794) - 3(1000)^2(1/1.05)^{30}(0.9594565)(0.1599316) = 105359.0,$$

so that

$$\begin{aligned} \mathrm{SD}[Z] &= \sqrt{\mathrm{Var}[Z]} = \sqrt{\mathrm{E}[Z^2] - (\mathrm{E}[Z])^2} \\ &= \sqrt{105359.0 - (218.8295)^2} = \sqrt{57472.61} = 239.7345. \end{aligned}$$

(c) The benefit is 2000 in the first 15 years and reduces to 1000 thereafter. Note that after 15 years, the accumulated value of the single premium of 500 will only be $500(1.05)^{15} = 1039.464$, which means that the policy benefit will have a greater value than this accumulation only if the insured dies within the first 15 years. This is equivalent to

$${}_{15}q_{50} = 1 - {}_{15}p_{50} = 1 - 0.9594565 = 0.0405435.$$