Exercise 4.20

(a) For A_x, we start with $A_{50} = \sum_{k=0}^{\infty} v^k \,{}_kp_{50} q_{50+k}$, where ∞ refers to the end of the mortality table (in this case, when the p_x have reached zero), and use the recursive formula:

$$A_{x+1} = \frac{A_x - vq_x}{vp_x}.$$

We follow the same principle for $A_x^{(4)}$ except the increments are quarterly, so that

$$A_{x+1/4}^{(4)} = \frac{A_x^{(4)} - v^{1/4}\,{}_{1/4}q_x}{v^{1/4}\,{}_{1/4}p_x},$$

where the x's are of increments of $1/4$ and the starting value is

$$A_{50}^{(4)} = \sum_{k=0}^{\infty}\sum_{j=0}^{3} v^{k+\frac{j+1}{4}}\,{}_{k+\frac{j}{4}|\frac{1}{4}}q_{50+k} = \sum_{k=0}^{\infty}\sum_{j=0}^{3} v^{k+\frac{j+1}{4}}\,{}_{k+\frac{j}{4}}p_{50}\,{}_{\frac{1}{4}}q_{50+k+\frac{j}{4}}.$$

The following R code produces a table of A_x starting at age $x = 50$:

```
A <- 0.0001
B <- 0.00035
c <- 1.075
surv <- function(x){
exp(-A*x-(B*(c^x-1)/log(c)))}
# 165 is the last age when px=0
# payments increments of one year
x <- 50:165
px <- surv(x+1)/surv(x)
qx <- 1-px
int <- .06
v <- 1/(1+int)
vcum <- v^(1:length(x))
Ax <- rep(0,length(x))
Ax[1] <- sum(vcum*cumprod(c(1,px[-length(px)]))*qx)
n <- 1
while (n<length(x)) {
n <- n+1
Ax[n] <- (Ax[n-1]*(1+int)-qx[n-1])/px[n-1]
}

x <- x[which(x==50):which(x==101)]
Ax <- Ax[which(x==50):which(x==101)]
a1 <- matrix(1:length(x),nrow=length(x)/4,4)
output1 <- cbind(x[a1[,1]],Ax[a1[,1]],x[a1[,2]],Ax[a1[,2]],x[a1[,3]],Ax[a1[,3]],
   x[a1[,4]],Ax[a1[,4]])
colnames(output1) <- c("x","Ax","x","Ax","x","Ax","x","Ax")
print(output1)
```

This gives the output

```
> print(output1)
       x        Ax  x        Ax  x        Ax   x        Ax
 [1,] 50 0.3358681 63 0.4954272 76 0.6610838  89 0.7976610
 [2,] 51 0.3472031 64 0.5084051 77 0.6730273  90 0.8062168
 [3,] 52 0.3587338 65 0.5214078 78 0.6847750  91 0.8144673
 [4,] 53 0.3704515 66 0.5344183 79 0.6963128  92 0.8224111
 [5,] 54 0.3823466 67 0.5474191 80 0.7076271  93 0.8300482
 [6,] 55 0.3944087 68 0.5603926 81 0.7187056  94 0.8373791
 [7,] 56 0.4066267 69 0.5733212 82 0.7295365  95 0.8444054
 [8,] 57 0.4189887 70 0.5861869 83 0.7401094  96 0.8511296
 [9,] 58 0.4314821 71 0.5989720 84 0.7504145  97 0.8575548
[10,] 59 0.4440933 72 0.6116590 85 0.7604433  98 0.8636848
[11,] 60 0.4568085 73 0.6242306 86 0.7701882  99 0.8695242
[12,] 61 0.4696128 74 0.6366695 87 0.7796429 100 0.8750779
[13,] 62 0.4824910 75 0.6489593 88 0.7888019 101 0.8803515
```

The following R code produces a table of $A_x^{(4)}$ starting at age $x = 50$:

```
# payments increments of quarter of a year
y <- seq(50,165,0.25)
px <- surv(y+0.25)/surv(y)
qx <- 1-px
int <- .06
v <- 1/(1+int)
vcum <- v^((1:length(y))/4)
Axqtr <- rep(0,length(y))
Axqtr[1] <- sum(vcum*cumprod(c(1,px[-length(px)]))*qx)
n <- 1
while (n<length(y)) {
n <- n+1
Axqtr[n] <- (Axqtr[n-1]*(1+int)^(1/4)-qx[n-1])/px[n-1]
}

y <- y[which(y==50):which(y==100.75)]
Axqtr <- Axqtr[which(y==50):which(y==100.75)]
a2 <- matrix(1:length(y),nrow=length(y)/4,4)
output2 <- cbind(y[a2[,1]],Axqtr[a2[,1]],y[a2[,2]],Axqtr[a2[,2]],
   y[a2[,3]],Axqtr[a2[,3]],y[a2[,4]],Axqtr[a2[,4]])
colnames(output2) <- c("x","Ax(4)","x","Ax(4)","x","Ax(4)","x","Ax(4)")
print(output2)
```

This gives the output

```
> print(output2)
          x     Ax(4)      x     Ax(4)      x     Ax(4)       x     Ax(4)
```

[1,]	50.00	0.3433016	62.75	0.5031434	75.50	0.6697975	88.25	0.8093480
[2,]	50.25	0.3461794	63.00	0.5064558	75.75	0.6728966	88.50	0.8116381
[3,]	50.50	0.3490701	63.25	0.5097709	76.00	0.6759844	88.75	0.8139092
[4,]	50.75	0.3519737	63.50	0.5130885	76.25	0.6790608	89.00	0.8161613
[5,]	51.00	0.3548899	63.75	0.5164082	76.50	0.6821256	89.25	0.8183944
[6,]	51.25	0.3578186	64.00	0.5197299	76.75	0.6851784	89.50	0.8206084
[7,]	51.50	0.3607597	64.25	0.5230532	77.00	0.6882191	89.75	0.8228032
[8,]	51.75	0.3637130	64.50	0.5263778	77.25	0.6912474	90.00	0.8249789
[9,]	52.00	0.3666785	64.75	0.5297036	77.50	0.6942631	90.25	0.8271355
[10,]	52.25	0.3696559	65.00	0.5330302	77.75	0.6972660	90.50	0.8292728
[11,]	52.50	0.3726452	65.25	0.5363574	78.00	0.7002558	90.75	0.8313910
[12,]	52.75	0.3756461	65.50	0.5396848	78.25	0.7032324	91.00	0.8334899
[13,]	53.00	0.3786586	65.75	0.5430123	78.50	0.7061955	91.25	0.8355696
[14,]	53.25	0.3816824	66.00	0.5463394	78.75	0.7091448	91.50	0.8376301
[15,]	53.50	0.3847174	66.25	0.5496660	79.00	0.7120803	91.75	0.8396714
[16,]	53.75	0.3877634	66.50	0.5529918	79.25	0.7150016	92.00	0.8416935
[17,]	54.00	0.3908203	66.75	0.5563164	79.50	0.7179087	92.25	0.8436963
[18,]	54.25	0.3938879	67.00	0.5596397	79.75	0.7208012	92.50	0.8456800
[19,]	54.50	0.3969660	67.25	0.5629613	80.00	0.7236790	92.75	0.8476445
[20,]	54.75	0.4000545	67.50	0.5662809	80.25	0.7265419	93.00	0.8495898
[21,]	55.00	0.4031532	67.75	0.5695983	80.50	0.7293897	93.25	0.8515160
[22,]	55.25	0.4062618	68.00	0.5729132	80.75	0.7322222	93.50	0.8534231
[23,]	55.50	0.4093802	68.25	0.5762252	81.00	0.7350393	93.75	0.8553112
[24,]	55.75	0.4125083	68.50	0.5795342	81.25	0.7378407	94.00	0.8571802
[25,]	56.00	0.4156458	68.75	0.5828398	81.50	0.7406264	94.25	0.8590301
[26,]	56.25	0.4187925	69.00	0.5861417	81.75	0.7433960	94.50	0.8608612
[27,]	56.50	0.4219482	69.25	0.5894398	82.00	0.7461495	94.75	0.8626733
[28,]	56.75	0.4251128	69.50	0.5927336	82.25	0.7488868	95.00	0.8644665
[29,]	57.00	0.4282860	69.75	0.5960228	82.50	0.7516075	95.25	0.8662410
[30,]	57.25	0.4314676	70.00	0.5993074	82.75	0.7543117	95.50	0.8679967
[31,]	57.50	0.4346575	70.25	0.6025868	83.00	0.7569991	95.75	0.8697336
[32,]	57.75	0.4378553	70.50	0.6058609	83.25	0.7596696	96.00	0.8714520
[33,]	58.00	0.4410609	70.75	0.6091294	83.50	0.7623231	96.25	0.8731517
[34,]	58.25	0.4442741	71.00	0.6123920	83.75	0.7649594	96.50	0.8748330
[35,]	58.50	0.4474947	71.25	0.6156484	84.00	0.7675784	96.75	0.8764958
[36,]	58.75	0.4507224	71.50	0.6188983	84.25	0.7701800	97.00	0.8781402
[37,]	59.00	0.4539570	71.75	0.6221415	84.50	0.7727640	97.25	0.8797664
[38,]	59.25	0.4571982	72.00	0.6253777	84.75	0.7753304	97.50	0.8813743
[39,]	59.50	0.4604460	72.25	0.6286066	85.00	0.7778789	97.75	0.8829641
[40,]	59.75	0.4636999	72.50	0.6318279	85.25	0.7804096	98.00	0.8845358
[41,]	60.00	0.4669598	72.75	0.6350414	85.50	0.7829223	98.25	0.8860896
[42,]	60.25	0.4702254	73.00	0.6382469	85.75	0.7854168	98.50	0.8876255
[43,]	60.50	0.4734965	73.25	0.6414439	86.00	0.7878932	98.75	0.8891436
[44,]	60.75	0.4767729	73.50	0.6446323	86.25	0.7903513	99.00	0.8906440
[45,]	61.00	0.4800543	73.75	0.6478119	86.50	0.7927910	99.25	0.8921268
[46,]	61.25	0.4833405	74.00	0.6509822	86.75	0.7952123	99.50	0.8935921

```
[47,] 61.50 0.4866312 74.25 0.6541432 87.00 0.7976150  99.75 0.8950400
[48,] 61.75 0.4899261 74.50 0.6572944 87.25 0.7999991 100.00 0.8964706
[49,] 62.00 0.4932250 74.75 0.6604357 87.50 0.8023645 100.25 0.8978840
[50,] 62.25 0.4965277 75.00 0.6635669 87.75 0.8047112 100.50 0.8992803
[51,] 62.50 0.4998339 75.25 0.6666875 88.00 0.8070390 100.75 0.9006596
```

(b) From the tables above, we find that

$$A_{50} = 0.3358681 \text{ and } A_{100} = 0.8750779$$

and

$$A_{50}^{(4)} = 0.3433016 \text{ and } A_{100}^{(4)} = 0.8964706$$

which when we assume UDD between integral ages gives us

$$A_{50}^{(4)} = \frac{i}{i^{(4)}} A_{50} = 0.3433334 \text{ and } A_{100}^{(4)} = \frac{i}{i^{(4)}} A_{100} = 0.8945281,$$

where $i^{(4)} = 4[(1 + .06)^{1/4} - 1] = 0.05869538$.

The R code to generate these values are given below:

```
# comparing values to UDD
Ax50 <- Ax[which(x==50)]
Ax100 <- Ax[which(x==100)]
Ax50qtr <- Axqtr[which(y==50)]
Ax100qtr <- Axqtr[which(y==100)]
i4 <- 4*((1+int)^(1/4)-1)
Ax50qtrUDD <- (int/i4)*Ax50
Ax100qtrUDD <- (int/i4)*Ax100
Ax50qtr
Ax50qtrUDD
Ax100qtr
Ax100qtrUDD
```

This gives the output

```
> Ax50qtr
[1] 0.3433016
> Ax50qtrUDD
[1] 0.3433334
> Ax100qtr
[1] 0.8964706
> Ax100qtrUDD
[1] 0.8945281
```

(c) UDD assumes that deaths are uniformly distributed over integral ages. What we find when we compare the true values of $A^{(4)}$ versus that produced by the UDD assumption is that there appears to be a larger discrepancy for age 100 than that for age 50. This may not be surprising since generally for later ages, the mortality rate is much more rapidly increasing and higher than for earlier ages. Hence, there is possible greater discrepancy in the linearity of the number of survivors between integral ages at later ages. Indeed, UDD tends to slightly overestimate $A^{(4)}$ for lower ages while it underestimates $A^{(4)}$ for older ages.

The following figure also confirms this. The plot below provides the difference between the true value of $A^{(4)}$ versus that produced by the UDD assumption for ages 50 to 100.