Exercise 4.17

(a) Z_1 is the present value random variable of an insurance that pays 20 at the end of the year of death of (x) if death occurs within the next 15 years, and 10 if death occurs thereafter. Z_2 is the present value random variable of an insurance issued to (x) that pays nothing if death occurs within the first 5 years, 10 at the end of the year of death if it occurs the following 10 years, and a pure endowment of 10 at the end of 15 years.

(b) For Z_1, we have

$$\mathrm{E}[Z_1] = 20 \int_0^{15} v^t \, {}_tp_x \mu_{x+t} dt + 10 \int_{15}^{\infty} v^t \, {}_tp_x \mu_{x+t} dt$$

and for Z_2, we have

$$\mathrm{E}[Z_2] = 10 \int_5^{15} v^t \, {}_tp_x \mu_{x+t} dt + 10 \int_{15}^{\infty} v^{15} \, {}_tp_x \mu_{x+t} dt \, .$$

(c) There are several ways to write the actuarial present values associated with Z_1 and Z_2. The following should not be difficult to verify:

$$\mathrm{E}[Z_1] = 20 \bar{A}^{\,1}_{x:\overline{15|}} + 10 \, {}_{15|}\bar{A}_x = 20 \bar{A}_x - 10 \, {}_{15|}\bar{A}_x = 10 \bar{A}_x + 10 \bar{A}^{\,1}_{x:\overline{15|}}$$

and

$$\mathrm{E}[Z_2] = 10 \, {}_5E_x \, \bar{A}^{\,1}_{x+5:\overline{10|}} + 10 \, {}_5E_x = 10 \bar{A}_{x:\overline{15|}} - 10 \bar{A}^{\,1}_{x:\overline{5|}} \, .$$

(d) To derive an expression for the covariance of Z_1 and Z_2, consider first its product:

$$Z_1 Z_2 = \begin{cases} 0, & \text{for } T \le 5, \\ 200 v^T, & \text{for } 5 < T \le 15, \\ 100 v^{15} v^T, & \text{for } T > 15. \end{cases}$$

Thus, we have

$$\mathrm{E}[Z_1 Z_2] = 200 \, {}_5E_x \, \bar{A}^{\,1}_{x+5:\overline{10|}} + 200 v^{15} \, {}_{15|}\bar{A}_x = 200 \, {}_5E_x \, \bar{A}^{\,1}_{x+5:\overline{10|}} + 200 v^{15} \, {}_5E_x \, \bar{A}_{x+15},$$

so that

$$\mathrm{Cov}[Z_1, Z_2] = \mathrm{E}[Z_1 Z_2] - \mathrm{E}[Z_1]\mathrm{E}[Z_2],$$

where the expectations of Z_1 and Z_2, respectively, are in part (c).