Exercise 4.10

One way to approach this problem is to use random variables. Recall that we can write

$$T = K + S$$

where $S \sim U(0,1)$ and K, S are independent in the case where UDD is assumed. Thus, we have

$$\begin{aligned}
(\bar{I}\bar{A})_x &= \mathrm{E}[Tv^T] = \mathrm{E}[(K+S)v^{K+S}] \\
&= \mathrm{E}[(K+1+S-1)v^{K+1+S-1}] \\
&= \mathrm{E}[(K+1)v^{K+1}v^{S-1}] + \mathrm{E}[v^{K+1}(S-1)v^{S-1}] \\
&= \mathrm{E}[v^{S-1}]\mathrm{E}[(K+1)v^{K+1}] + \mathrm{E}[(S-1)v^{S-1}]\mathrm{E}[v^{K+1}] \\
&= \mathrm{E}[v^{S-1}](IA)_x + \mathrm{E}[(S-1)v^{S-1}]A_x
\end{aligned}$$

When S is uniformly distributed on $(0,1)$, it can be shown that

$$\mathrm{E}[v^{S-1}] = \int_0^1 v^{s-1}ds = \frac{i}{\delta}$$

and that

$$\mathrm{E}[(S-1)v^{S-1}] = \int_0^1 (s-1)v^{s-1}ds = \frac{i-\delta e^{\delta}}{\delta^2}$$

so that

$$(\bar{I}\bar{A})_x = \frac{i}{\delta}(IA)_x + \frac{i-\delta e^{\delta}}{\delta^2}A_x$$