Exercise 2.14

(a) Starting with

$$\begin{aligned} \mathring{e}_x &= \int_0^\infty {}_tp_x dt = \int_0^1 {}_tp_x dt + \int_1^\infty {}_tp_x dt \\ &\leq 1 + \int_1^\infty {}_tp_x dt = 1 + \int_1^\infty p_x \cdot {}_{t-1}p_{x+1} dt \\ &\leq 1 + \int_1^\infty {}_{t-1}p_{x+1} dt = 1 + \int_0^\infty {}_sp_{x+1} ds \\ &= 1 + \mathring{e}_{x+1} \end{aligned}$$

The inequalities hold because we know that ${}_tp_x \leq 1$ for all x and t.

A heuristic approach is to use the inequality $T_x \leq T_{x+1} + 1$ and by taking the expectation of both sides, you get the desired result. Here, intuitively, a person age x who reaches to live another year will live a longer life.

(b) Because we know that $T_x \geq \lfloor T_x \rfloor = K_x$, taking the expectation of both sides gives us $\mathrm{E}[T_x] \geq \mathrm{E}[K_x]$ and the result immediately follows.

(c) The difference between $\mathring{e}_x$ and e_x is the additional average lifetime a person has in the year of death. If we assume deaths uniformly occur between integral ages, an extra half-year would be expected to be lived.

(d) From Exercise 2.9, we found that if the force of mortality μ_x is non-increasing with age x, then $\dfrac{d}{dx} {}_tp_x \leq 0$. This leads us to

$$\frac{d}{dx} \mathring{e}_x = \frac{d}{dx} \int_0^\infty {}_tp_x dt = \int_0^\infty \frac{d}{dx} {}_tp_x dt \leq 0.$$

Thus for forces of mortality that are non-increasing, then the average future lifetime will also be non-increasing. However, we know that for human mortality pattern, the force of mortality generally decreases at infancy so that this does not generally hold for all ages x.