Exercise 2.11

(a) It is not difficult to show that under Makeham's law, we have

$$S_0(x) = \exp\left\{\int_0^x (A + Bc^z)dz\right\} = \exp\left\{-\left[Ax + \frac{B}{\log(c)}(c^x - 1)\right]\right\}.$$

It follows therefore that

$$\begin{aligned} {}_tp_x &= S_x(t) = \frac{S_0(x+t)}{S_0(x)} \\ &= \frac{\exp\left\{-\left[A(x+t) + \frac{B}{\log(c)}(c^{x+t} - 1)\right]\right\}}{\exp\left\{-\left[Ax + \frac{B}{\log(c)}(c^x - 1)\right]\right\}} \\ &= e^{-At - \frac{B}{\log(c)}c^x(c^t - 1)} = s^t g^{c^x(c^t-1)} \end{aligned}$$

where clearly $s = e^{-A}$ and $g = e^{-B/\log(c)}$.

(b) We use result of part (a) by noting that

$$\log {}_tp_x = t\log(s) + c^x(c^t - 1)\log(g).$$

It therefore follows that

$$\begin{aligned} \frac{\log {}_{10}p_{70} - \log {}_{10}p_{60}}{\log {}_{10}p_{60} - \log {}_{10}p_{50}} &= \frac{10\log(s) + c^{70}(c^{10} - 1)\log(g) - 10\log(s) - c^{60}(c^{10} - 1)\log(g)}{10\log(s) + c^{60}(c^{10} - 1)\log(g) - 10\log(s) - c^{50}(c^{10} - 1)\log(g)} \\ &= \frac{c^{60}(c^{10} - 1)}{c^{50}(c^{10} - 1)} = c^{10}. \end{aligned}$$

The result follows immediately by raising both sides to the power of 0.10. Such property can indeed be generalized as follows: fix x and t, the following can be similarly verified:

$$c = \left[\frac{\log {}_tp_{x+2t} - \log {}_tp_{x+t}}{\log {}_tp_{x+t} - \log {}_tp_x}\right]^{1/t}$$