MATH 3630
Actuarial Mathematics I
Final Examination

Tuesday, 12 December 2017
Time Allowed: 2 hours (1:00 - 3:00 pm)
Room: MONT 104
Total Marks: 120 points

Please write your name and student number at the spaces provided:

Name: Emil

Student ID: Suggested Solutions

- There are twelve (12) written-answer questions here and you are to answer all twelve. Each question is worth 10 points. Your final mark will be divided by 120 to convert to a unit of 100%.
- Please provide details of your workings in the appropriate spaces provided; partial points will be granted.
- Please write legibly.
- Anyone caught cheating will be subject to university's disciplinary action.
- Best of luck.
- Have a Happy and Healthy Christmas and New Year!

Question	Worth	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
Total	120	
%	÷ 120	

Question No. 1:

You are given the following survival function of a newborn:

$$S_0(x) = \frac{1}{1+x}, \quad \text{for } x \geq 0.$$

Calculate the force of mortality at age 65, μ_{65}.

$$\mu_x = -\frac{d}{dx} \log S_0(x) = \cancel{-}\frac{d}{dx}\left[\cancel{-} \log(1+x)\right]$$

$$= \frac{1}{1+x}$$

$$\mu_{65} = \frac{1}{1+65} = \frac{1}{66} = \underline{\underline{0.01515152}}$$

Question No. 2:

Mortality for a population consisting of females and males follow a select-and-ultimate table, an extract of which is given below. Females have a 3-year select period while males have a 2-year select period. Assume mortality follows the Uniform Distribution of Death (UDD) between integral ages.

Females						Males				
x	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	$x+3$	x	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
50	80960	79827	78522	77025	53	50	70764	69124	67224	52
51	79530	78334	76958	75382	54	51	68823	67118	65146	53
52	78021	76760	75312	73655	55	52	66805	65036	62993	54
53	76430	75103	73581	71842	56	53	64711	62879	60768	55
54	74756	73362	71765	69944	57	54	62544	60651	58475	56
55	72998	71535	69863	67958	58	55	60305	58354	56117	57

At select age 50, the population consists of 65% female and 35% male.

Calculate the probability that a randomly chosen person from this population, at select age 50, will survive the next 3.5 years.

Female: $${}_{3.5}p_{[50]}^{female} = \frac{l_{53.5}}{l_{[50]}} = \frac{.5(l_{53}+l_{54})}{l_{[50]}} = \frac{.5(77025+75382)}{80960}$$

$$= 0.9412488$$

Male: $${}_{3.5}p_{[50]}^{male} = \frac{l_{53.5}}{l_{[50]}} = \frac{.5(65146+62993)}{70764} = 0.9053968$$

$${}_{3.5}p_{[50]} = 0.65\ {}_{3.5}p_{[50]}^{female} + 0.35\ {}_{3.5}p_{[50]}^{male}$$

$$= 0.65(0.9412488) + 0.35(0.9053968)$$

$$= \underline{\underline{0.9287006}}$$

Question No. 3:

For a whole life insurance policy issued to (45), you are given:

- Death benefits are payable at the end of the year of death.
- The death benefit is 2.
- Mortality follows the `Illustrative Life Table`.
- $i = 0.06$
- Z is the present value of the benefit random variable.

Calculate $\mathrm{Var}[Z]$.

$$\mathrm{Var}[Z] = E[Z^2] - (E[Z])^2$$

$$= 4\,{}^2A_{45} - (2A_{45})^2$$

$$= 4(0.06802) - (2(0.20120))^2$$

$$= \underline{\underline{0.11015424}}$$

Question No. 4:

You are given:

- For age prior to 50, mortality follows a constant force with $\mu = 0.01$.
- For ages 50 and later, mortality is uniformly distributed with $\omega = 120$.
- $\delta = 5\%$
- Z is the present value random variable for a whole life insurance of 1 payable at the moment of death issued to (40).

Calculate the probability that Z will be greater than 0.5.

Let $T = T_{40}$ = future lifetime of (40). Its density is

$$f_T(t) = \begin{cases} .01e^{-.01t}, & 0 \le t < 10 \\ \frac{1}{70} e^{-.10}, & 10 \le t < 80 \end{cases}$$

$$\Pr[Z > 0.5] = \Pr[v^T > 0.5] = \Pr\Big[T < \underbrace{\frac{\log 0.5}{-\delta}}_{13.86294}\Big] \qquad \delta = .05$$

$$= \int_0^{10} .01e^{-.01t}\,dt + e^{-.10}\int_{10}^{13.86294} \frac{1}{70}\,dt$$

$$= 1 - e^{-.10} + e^{-.10}\,\frac{3.86294}{70}$$

$$= \underline{\underline{0.2743584}}$$

Question No. 5:

For a 4-year deferred whole life annuity-due of 1 per year issued to (95), you are given:

- The following extract from a mortality table:

x	95	96	97	98	99	100
ℓ_x	100	70	40	20	5	0

- $v = 0.90$
- Y is the present value random variable for this deferred annuity.

Calculate Var[Y].

K	Prob	Y	Y × Prob	Y^2 × Prob
≤ 3	95/100	0	0	0
4	5/100	v^4	$v^4\,5/100$	$v^8\,5/100$
5^+	—			

$$\text{Var}[Y] = E[Y^2] - (E[Y])^2$$

$$= v^8 \frac{5}{100} - \left(v^4 \frac{5}{100}\right)^2$$

$$= v^8 \frac{5}{100}\left(1 - \frac{5}{100}\right)$$

$$= \underline{\underline{0.02044719}}$$

Question No. 6:

Get-a-Life Insurance Company issues a special insurance policy to (45) with the following benefits:

- a death benefit of 2000, payable at the end of year of death, provided death occurs before age 65, plus
- an annuity benefit that pays 5000 annually starting immediately when the policyholder reaches age 65.

You are given:

- Annual level premiums of P are paid for the first 20 years only (nothing, thereafter) and are determined according to the actuarial equivalence principle.
- $i = 0.05$
- $\ddot{a}_{45} = 13.96$
- $\ddot{a}_{65} = 11.34$
- ${}_{20}E_{45} = 0.26$

Calculate P.

$13.96 - 0.26(11.34) = 11.0116$

$$\text{APV(premiums)} = P\,\ddot{a}_{45:\overline{20}|}$$

$$\text{APV(benefits)} = 2000\,A^{1}_{45:\overline{20}|} + 5000\,{}_{20}E_{45}\,\ddot{a}_{65}$$

0.26 11.34

$$A_{45} - {}_{20}E_{45}\,A_{65}$$

$$1 - d\ddot{a}_{45} - {}_{20}E_{45}(1 - d\ddot{a}_{65})$$

$$= 15173.28$$

$$P = \frac{15173.28}{11.0116} = \underline{\underline{1377.936}}$$

Question No. 7:

A fully discrete whole life policy of 10,000 issued to (35) with level annual premiums is priced with the following expense assumptions:

	% of Premium	Per 1,000	Per Policy
First year	20%	1.0	15
Renewal years	10%	0.3	5

You are given:

- $i = 0.05$
- $\ddot{a}_{35} = 15.0$

Calculate the annual gross premium.

$$G\ddot{a}_{35} = 1000A_{35} + 0.10G + 0.10G\ddot{a}_{35} + .7(10) + .3(10)\ddot{a}_{35} + 10 + 5\ddot{a}_{35}$$

$$\underbrace{G(0.90\ddot{a}_{35} - 0.10)}_{13.4} = \underbrace{1000\underset{\swarrow}{A_{35}} + 17 + 8\ddot{a}_{35}}_{2994.143}$$

$$1 - d\ddot{a}_{35}$$

$$G = \frac{2994.143}{13.4} = \underline{\underline{223.4435}}$$

Question No. 8:

For a fully discrete whole life insurance policy of 1,000,000 on (50), you are given:

- Expenses consist of f in the first year and 50 thereafter.
- Mortality follows the `Illustrative Life Table`.
- $i = 0.06$
- The annual gross premium, calculated using the actuarial equivalence principle, is 18,850.

Calculate f.

APV(premiums) = APV(benefits) + APV(expenses)

$$G\ddot{a}_{50} = 1000000\, A_{50} + (f-50) + 50\,\ddot{a}_{50}$$

Solving for f, we get

$$f = \underbrace{18850(13.2668) - 1000000(0.24905) + 50 - 50(13.2668)}$$

$$= \underline{\underline{415.84}}$$

If you converted $\ddot{a}_{50} = \frac{1-A_{50}}{d} = 13.2666$,

you get 412.08 (some rounding)

Question No. 9:

You are given the following extract of ultimate mortality rates from a two-year select and ultimate mortality table:

x	q_x
50	0.045
51	0.050
52	0.055
53	0.060

The select mortality rates satisfy the following:

- $q_{[x]} = 0.70\, q_x$
- $q_{[x]+1} = 0.80\, q_{x+1}$

You are also given that $i = 0.05$.

$v = \frac{1}{1.05}$

Calculate $A_{[50]:\overline{3}|}$.

$$A_{[50]:\overline{3}|} = v\, q_{[50]} + v^2 p_{[50]}\, q_{[50]+1} + v^3 p_{[50]}\, p_{[50]+1}\, q_{52} + v^3 p_{[50]}\, p_{[50]+1}\, p_{52}$$

$$= v^3 p_{[50]}\, p_{[50]+1}$$

$$= \frac{1}{1.05}\, .7(.045) + \frac{1}{1.05^2}(1-.7(.045))(.8(.05)) + \frac{1}{1.05^3}(1-.7(.045))(1-.8(.05))$$

$$= \underline{\underline{0.8683}}$$

Question No. 10:

The pricing actuary for an insurance company calculates the premium for a fully discrete whole life insurance of 100 on (65) using the equivalence principle and the assumptions that the force of mortality is constant at 0.10 and $i = 0.06$.

The pricing actuary's supervisor believes that the `Illustrative Life Table` is a better mortality assumption.

Calculate the insurance company's expected loss at issue if the premium is not changed and the supervisor is indeed correct.

pricing actuary: $P = 100 \dfrac{A_{65}}{\ddot{a}_{65}}$ where $\ddot{a}_{65} = \underbrace{\sum_{k=0}^{\infty} \left(\frac{1}{1.06}\right)^k e^{-.10k}}$

$$= \frac{1}{1 - \frac{e^{-.10}}{1.06}}$$

$$= 100 \frac{1 - d\ddot{a}_{65}}{\ddot{a}_{65}} \qquad = 6.831544$$

$$= 100 \times \left(\frac{1}{6.831544} - \frac{.06}{1.06} \right) = \underline{8.977602}$$

supervisor: $L_0 = 100 v^{K+1} - P \ddot{a}_{\overline{K+1|}}$

$$E[L_0] = 100 A_{65}^{ILT} - P \ddot{a}_{65}^{ILT}$$

$$= 100(0.43980) - 8.977602(9.8969)$$

$$= \underline{\underline{-44.87043}}$$

Question No. 11:

For a fully discrete 20-year endowment life insurance of 2 issued to (45), you are given:

- Level annual gross premiums are calculated according to the equivalence principle.
- The first year expense is 10% of the gross annual premium.
- Expenses in subsequent years are 4% of the gross annual premium.
- $i = 0.04$
- $A_{45:\overline{20}|} = 0.20$
- ${}^2A_{45:\overline{20}|} = 0.15$
- L_0^g is the gross loss at issue random variable.

Calculate $\mathrm{Var}[L_0^g]$.

$$G\ddot{a}_{45:\overline{20}|} = 2A_{45:\overline{20}|} + .06G + .04G\ddot{a}_{40:\overline{20}|}$$

$$G = \frac{2\underbrace{A_{45:\overline{20}|}}_{.20}}{.96\underbrace{\ddot{a}_{45:\overline{20}|}}_{\frac{1-.20}{.04/1.04}} - .06} = a\,0.2009243$$

$$L_0^g = PVFB_0 + PVFE_0 - PVFP_0$$

$$= 2v^{\min(K+1,20)} + \underbrace{.06G + .04G\ddot{a}_{\overline{\min(K+1,20)}|} - G\ddot{a}_{\overline{\min(K+1,20)}|}}_{-.96G\left(\frac{1-v^{\min(K+1,20)}}{d}\right)}$$

$$= \left(2 + \frac{.96G}{d}\right)v^{\min(K+1,20)} + \text{constant}$$

$$\mathrm{Var}[L_0^g] = \left(2 + \frac{.96G}{d}\right)^2\left(\underbrace{{}^2A_{45:\overline{20}|}}_{.15} - \underbrace{(A_{45:\overline{20}|})^2}_{(.20)^2}\right)$$

$$= \underline{\underline{0.6883291}}$$

Question No. 12:

For a fully discrete whole life insurance of 1000 to (50), you are given:

- Expenses consist of 10% of the annual gross premium in the first year and 5% of the annual gross premium in subsequent years.
- Mortality follows deMoivre's law with $\omega = 120$.
- $i = 0.05$
- The annual gross premium is 17.50.

Calculate the probability of a positive loss at issue.

$$L_0^g = PVFB_0 + PVFE_0 - PVFP_0$$

$$= 1000 v^{K+1} + .05G + .05G\, \ddot{a}_{\overline{K+1}|} - G\, \ddot{a}_{\overline{K+1}|}$$

$$= \underbrace{\left(1000 + \frac{.95G}{d}\right)}_{a} v^{K+1} + \underbrace{.05G - .95G/d}_{b}$$

$$\Pr[L_0^g > 0] = \Pr[a v^{K+1} > -b]$$

$$= \Pr[v^{K+1} > -b/a] = \Pr\Big[K < \underbrace{\frac{\log(-b/a)}{-\delta} - 1}_{26.75746}\Big]$$

$$\Pr[K \leq 26] = {}_{27}q_{50} = \frac{27}{70} = \underline{\underline{0.3857143}}$$

$T_{50} \sim$ Uniform on $(0, 70)$

EXTRA PAGE FOR ADDITIONAL OR SCRATCH WORK

Illustrative Life Table: Basic Functions and Single Benefit Premiums at i = 0.06

x	l_x	$1000q_x$	$\ddot{a}_x$	$1000A_x$	$1000(^2A_x)$	$1000\,{}_5E_x$	$1000\,{}_{10}E_x$	$1000\,{}_{20}E_x$	x
0	10,000,000	20.42	16.8010	49.00	25.92	728.54	541.95	299.89	0
5	9,749,503	0.98	17.0379	35.59	8.45	743.89	553.48	305.90	5
10	9,705,588	0.85	16.9119	42.72	9.37	744.04	553.34	305.24	10
15	9,663,731	0.91	16.7384	52.55	11.33	743.71	552.69	303.96	15
20	9,617,802	1.03	16.5133	65.28	14.30	743.16	551.64	301.93	20
21	9,607,896	1.06	16.4611	68.24	15.06	743.01	551.36	301.40	21
22	9,597,695	1.10	16.4061	71.35	15.87	742.86	551.06	300.82	22
23	9,587,169	1.13	16.3484	74.62	16.76	742.68	550.73	300.19	23
24	9,576,288	1.18	16.2878	78.05	17.71	742.49	550.36	299.49	24
25	9,565,017	1.22	16.2242	81.65	18.75	742.29	549.97	298.73	25
26	9,553,319	1.27	16.1574	85.43	19.87	742.06	549.53	297.90	26
27	9,541,153	1.33	16.0873	89.40	21.07	741.81	549.05	297.00	27
28	9,528,475	1.39	16.0139	93.56	22.38	741.54	548.53	296.01	28
29	9,515,235	1.46	15.9368	97.92	23.79	741.24	547.96	294.92	29
30	9,501,381	1.53	15.8561	102.48	25.31	740.91	547.33	293.74	30
31	9,486,854	1.61	15.7716	107.27	26.95	740.55	546.65	292.45	31
32	9,471,591	1.70	15.6831	112.28	28.72	740.16	545.90	291.04	32
33	9,455,522	1.79	15.5906	117.51	30.63	739.72	545.07	289.50	33
34	9,438,571	1.90	15.4938	122.99	32.68	739.25	544.17	287.82	34
35	9,420,657	2.01	15.3926	128.72	34.88	738.73	543.18	286.00	35
36	9,401,688	2.14	15.2870	134.70	37.26	738.16	542.11	284.00	36
37	9,381,566	2.28	15.1767	140.94	39.81	737.54	540.92	281.84	37
38	9,360,184	2.43	15.0616	147.46	42.55	736.86	539.63	279.48	38
39	9,337,427	2.60	14.9416	154.25	45.48	736.11	538.22	276.92	39
40	9,313,166	2.78	14.8166	161.32	48.63	735.29	536.67	274.14	40
41	9,287,264	2.98	14.6864	168.69	52.01	734.40	534.99	271.12	41
42	9,259,571	3.20	14.5510	176.36	55.62	733.42	533.14	267.85	42
43	9,229,925	3.44	14.4102	184.33	59.48	732.34	531.12	264.31	43
44	9,198,149	3.71	14.2639	192.61	63.61	731.17	528.92	260.48	44
45	9,164,051	4.00	14.1121	201.20	68.02	729.88	526.52	256.34	45
46	9,127,426	4.31	13.9546	210.12	72.72	728.47	523.89	251.88	46
47	9,088,049	4.66	13.7914	219.36	77.73	726.93	521.03	247.08	47
48	9,045,679	5.04	13.6224	228.92	83.06	725.24	517.91	241.93	48
49	9,000,057	5.46	13.4475	238.82	88.73	723.39	514.51	236.39	49
50	8,950,901	5.92	13.2668	249.05	94.76	721.37	510.81	230.47	50
51	8,897,913	6.42	13.0803	259.61	101.15	719.17	506.78	224.15	51
52	8,840,770	6.97	12.8879	270.50	107.92	716.76	502.40	217.42	52
53	8,779,128	7.58	12.6896	281.72	115.09	714.12	497.64	210.27	53
54	8,712,621	8.24	12.4856	293.27	122.67	711.24	492.47	202.70	54
55	8,640,861	8.96	12.2758	305.14	130.67	708.10	486.86	194.72	55
56	8,563,435	9.75	12.0604	317.33	139.11	704.67	480.79	186.32	56
57	8,479,908	10.62	11.8395	329.84	147.99	700.93	474.22	177.53	57
58	8,389,826	11.58	11.6133	342.65	157.33	696.85	467.12	168.37	58
59	8,292,713	12.62	11.3818	355.75	167.13	692.41	459.46	158.87	59
60	8,188,074	13.76	11.1454	369.13	177.41	687.56	451.20	149.06	60
61	8,075,403	15.01	10.9041	382.79	188.17	682.29	442.31	139.00	61
62	7,954,179	16.38	10.6584	396.70	199.41	676.56	432.77	128.75	62
63	7,823,879	17.88	10.4084	410.85	211.13	670.33	422.54	118.38	63
64	7,683,979	19.52	10.1544	425.22	223.34	663.56	411.61	107.97	64
65	7,533,964	21.32	9.8969	439.80	236.03	656.23	399.94	97.60	65

Illustrative Life Table: Basic Functions and Single Benefit Premiums at $i = 0.06$

x	l_x	$1000q_x$	$\ddot{a}_x$	$1000A_x$	$1000(^2A_x)$	$1000\,{}_5E_x$	$1000\,{}_{10}E_x$	$1000\,{}_{20}E_x$	x
66	7,373,338	23.29	9.6362	454.56	249.20	648.27	387.53	87.37	66
67	7,201,635	25.44	9.3726	469.47	262.83	639.66	374.36	77.38	67
68	7,018,432	27.79	9.1066	484.53	276.92	630.35	360.44	67.74	68
69	6,823,367	30.37	8.8387	499.70	291.46	620.30	345.77	58.54	69
70	6,616,155	33.18	8.5693	514.95	306.42	609.46	330.37	49.88	70
71	6,396,609	36.26	8.2988	530.26	321.78	597.79	314.27	41.86	71
72	6,164,663	39.62	8.0278	545.60	337.54	585.25	297.51	34.53	72
73	5,920,394	43.30	7.7568	560.93	353.64	571.81	280.17	27.96	73
74	5,664,051	47.31	7.4864	576.24	370.08	557.43	262.31	22.19	74
75	5,396,081	51.69	7.2170	591.49	386.81	542.07	244.03	17.22	75
76	5,117,152	56.47	6.9493	606.65	403.80	525.71	225.46	13.04	76
77	4,828,182	61.68	6.6836	621.68	421.02	508.35	206.71	9.61	77
78	4,530,360	67.37	6.4207	636.56	438.42	489.97	187.94	6.88	78
79	4,225,163	73.56	6.1610	651.26	455.95	470.57	169.31	4.77	79
80	3,914,365	80.30	5.9050	665.75	473.59	450.19	151.00	3.19	80
81	3,600,038	87.64	5.6533	680.00	491.27	428.86	133.19	2.05	81
82	3,284,542	95.61	5.4063	693.98	508.96	406.62	116.06	1.27	82
83	2,970,496	104.28	5.1645	707.67	526.60	383.57	99.81	0.75	83
84	2,660,734	113.69	4.9282	721.04	544.15	359.79	84.59	0.42	84
85	2,358,246	123.89	4.6980	734.07	561.57	335.40	70.56	0.22	85
86	2,066,090	134.94	4.4742	746.74	578.80	310.56	57.83	0.11	86
87	1,787,299	146.89	4.2571	759.03	595.79	285.44	46.50	0.05	87
88	1,524,758	159.81	4.0470	770.92	612.51	260.21	36.61	0.02	88
89	1,281,083	173.75	3.8442	782.41	628.92	235.11	28.17	0.01	89
90	1,058,491	188.77	3.6488	793.46	644.96	210.36	21.13	0.00	90
91	858,676	204.93	3.4611	804.09	660.61	186.21	15.41	0.00	91
92	682,707	222.27	3.2812	814.27	675.83	162.90	10.91	0.00	92
93	530,959	240.86	3.1091	824.01	690.59	140.69	7.47	0.00	93
94	403,072	260.73	2.9450	833.30	704.86	119.79	4.93	0.00	94
95	297,981	281.91	2.7888	842.14	718.61	100.43	3.13	0.00	95
96	213,977	304.45	2.6406	850.53	731.83	82.78	1.90	0.00	96
97	148,832	328.34	2.5002	858.48	744.50	66.97	1.10	0.00	97
98	99,965	353.60	2.3676	865.99	756.60	53.09	0.60	0.00	98
99	64,617	380.20	2.2426	873.06	768.13	41.14	0.31	0.00	99
100	40,049	408.12	2.1252	879.70	779.08	31.12	0.15	0.00	100
101	23,705	437.28	2.0152	885.93	789.44	22.91	0.07	0.00	101
102	13,339	467.61	1.9123	891.76	799.21	16.37	0.03	0.00	102
103	7,101	498.99	1.8164	897.19	808.41	11.33	0.01	0.00	103
104	3,558	531.28	1.7273	902.23	817.02	7.56	0.00	0.00	104
105	1,668	564.29	1.6447	906.90	825.06	4.86	0.00	0.00	105
106	727	597.83	1.5685	911.22	832.53	2.99	0.00	0.00	106
107	292	631.64	1.4984	915.19	839.46	1.76	0.00	0.00	107
108	108	665.45	1.4341	918.82	845.84	0.98	0.00	0.00	108
109	36	698.97	1.3755	922.14	851.69	0.52	0.00	0.00	109
110	11	731.87	1.3223	925.15	857.04	0.26	0.00	0.00	110