MATH 3630
Actuarial Mathematics I
Final Examination
Tuesday, 13 December 2016
Time Allowed: 2 hours (1:00 - 3:00 pm)
Room: AUST 110
Total Marks: 120 points

Please write your name and student number at the spaces provided:

Name: EMIL **Student ID:** Suggested Solutions

- **There are twelve (12) written-answer questions here and you are to answer all twelve. Each question is worth 10 points. Your final mark will be divided by 120 to convert to a unit of 100%.**
- **Please provide details of your workings in the appropriate spaces provided; partial points will be granted.**
- **Please write legibly.**
- **Anyone caught writing after time has expired will be given a mark of zero.**
- **Best of luck.**
- **Have a Happy and Healthy Christmas and New Year!**

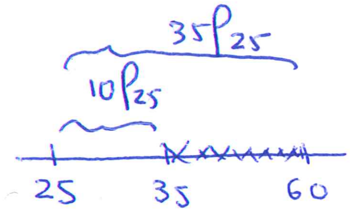
Question	Worth	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
Total	120	
%	$\div$ 120	

Question No. 1:

You are given the force of mortality:

$$\mu_x = x^{-1/4}, \quad \text{for } x \geq 0.$$

Calculate ${}_{10|25}q_{25}$ and interpret this probability.



Recall $${}_tp_x = e^{-\int_0^t \mu_{x+s}\,ds} = e^{-\int_0^t (x+s)^{-1/4}\,ds} = e^{-\frac{4}{3}\left[(x+t)^{3/4} - x^{3/4}\right]}$$

$${}_{10|25}q_{25} = {}_{10}p_{25} - {}_{35}p_{25}$$

$$= e^{-\frac{4}{3}\left[35^{3/4} - 25^{3/4}\right]} - e^{-\frac{4}{3}\left[60^{3/4} - 25^{3/4}\right]}$$

$$= e^{\frac{4}{3}25^{3/4}}\left[e^{-\frac{4}{3}35^{3/4}} - e^{-\frac{4}{3}60^{3/4}}\right]$$

$$= \underline{\underline{0.01385393}}$$

This is the probability that a life now aged 25 will die between ages 35 and 60.

Question No. 2:

Mortality for a population consisting of females and males follow a select-and-ultimate table, an extract of which is given below. Females have a 3-year select period while males have a 2-year select period. Assume mortality follows the Uniform Distribution of Death (UDD) between integral ages.

Females						Males				
x	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	$x+3$	x	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
50	80960	79827	78522	77025	53	50	70764	69124	67224	52
51	79530	78334	76958	75382	54	51	68823	67118	65146	53
52	78021	76760	75312	73655	55	52	66805	65036	62993	54
53	76430	75103	73581	71842	56	53	64711	62879	60768	55
54	74756	73362	71765	69944	57	54	62544	60651	58475	56
55	72998	71535	69863	67958	58	55	60305	58354	56117	57

At select age 51, the population consists of 60% female and 40% male.

Calculate the probability that a randomly chosen person from this population, at select age 51, will die between the ages of 53.25 and 56.75.

For a female,

$$ {}_{2.25|3.5}q^{\text{female}}_{[51]} = {}_{2.25}p^{\text{female}}_{[51]} - {}_{5.75}p^{\text{female}}_{[51]} $$

$$ = \frac{\ell_{[51]+2.25} - \ell_{56.75}}{\ell_{[51]}} $$

$$ = \frac{(.25\,\ell_{54} + .75\,\ell_{[51]+2}) - (.75\,\ell_{57} + .25\,\ell_{56})}{\ell_{[51]}} $$

$$ = \frac{[.25(75382) + .75(76958)] - [.75(69944) + .25(71842)]}{79530} $$

$$ = 0.07727273 $$

EXTRA PAGE FOR ADDITIONAL OR SCRATCH WORK FOR QUESTION 2

Similarly, for a male

$$_{2.25|3.5}q^{male}_{[51]} = \frac{l_{53.25} - l_{56.75}}{l_{[51]}}$$

$$= \frac{[.25(62993) + .75(65146)] - [.75(56117) + .25(58475)]}{68823}$$

$$= 0.1148054$$

For a randomly chosen person, we have

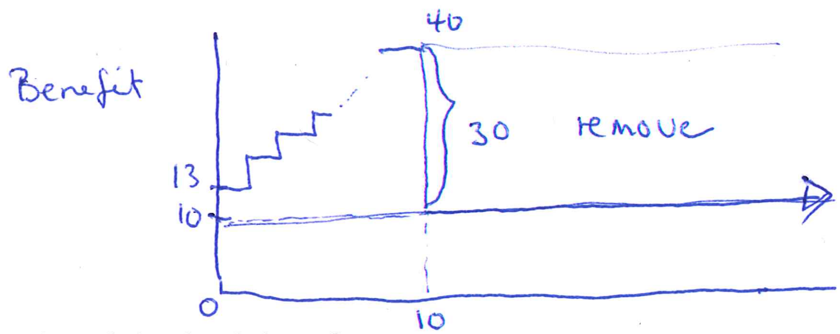
$$_{2.25|3.5}q_{[51]} = 0.60(0.07727273) + 0.40(0.1148054)$$

$$= \underline{\underline{0.09228578}}$$

Question No. 3:

For a special whole life insurance policy issued to (45), you are given:

- Death benefits are payable at the end of the year of death.
- The death benefit in the first year is \$13, increasing by \$3 each year subsequently until reaching \$40, after which the benefit drops to \$10 and remains at that level thereafter.
- $A_{45} = 0.64$
- $A_{55} = 0.77$
- $(IA)_{45} = 9.08$
- $(IA)_{55} = 6.46$
- ${}_{10}E_{45} = 0.68$



Calculate the actuarial present value of the death benefits.

$$\text{APV(death benefits)} = 10A_{45} + 3(IA)_{45} - 3\,{}_{10}E_{45}\,(IA)_{55} - 30\,{}_{10}E_{45}\,A_{55}$$

$$= 10(.64) + 3[9.08 - .68(6.46)] - 30(.68)(.77)$$

$$= \underline{\underline{4.7536}}$$

Question No. 4:

You are given:

- For age prior to 40, mortality follows a constant force with $\mu = 0.01$.
- For ages 40 and later, mortality is uniformly distributed with $\omega = 100$.
- $\delta = 5\%$
- Y is the present value random variable for a whole life annuity-due of \$1 per year issued to (30).

Calculate the probability that Y will be less than 10.

Suppose $T = T_{30}$ is the future lifetime of (30). Density is given by

$$f_T(t) = \begin{cases} 0.01\, e^{-.01t}, & 0 \leq t < 10 \\ e^{-.10} \frac{1}{60}, & 10 \leq t < 70 \end{cases}$$

Thus, the event $Y = \frac{1 - v^{K+1}}{d} < 10$ is equivalent to

$d = 1 - v = 1 - e^{-.05}$

$\delta = .05$

$$v^{K+1} > 1 - 10d \Rightarrow K < \underbrace{\frac{\log(1-10d)}{-\delta} - 1}_{12.37712}$$

$$\Pr[Y < 10] = \Pr[K < 12.37712]$$

$$= \Pr[K \leq 12] = {}_{13}q_{30} = 1 - {}_{13}p_{30}$$

$$= 1 - {}_{10}p_{30} \cdot {}_{3}p_{40}$$

$$= 1 - e^{-.10}\left(1 - \frac{3}{60}\right)$$

$$= \underline{\underline{0.1404045}}$$

Question No. 5:

For a 3-year deferred whole life annuity-due of \$1 per year issued to (95), you are given:

1st payment

0 1 2 3

- $v = 0.90$
- The following extract from a mortality table:

x	95	96	97	98	99	100
ℓ_x	100	75	50	25	10	0

25 25 25 15 10

- Y is the present value random variable for this deferred annuity.

Calculate the standard deviation of Y. Let $K = K_{95}$

K	$\Pr[K=k]$	Y	$Y \Pr[K=k]$	$Y^2 \Pr[K=k]$
0	25/100	0	–	–
1	25/100	0	–	–
2	25/100	0	–	–
3	15/100	v^3	0.10935	0.07971615
4	10/100	v^3+v^4	0.13851	0.1918502
5+	–	–	–	–

$$E[Y] = 0.10935 + 0.13851 = 0.24786$$

$$E[Y^2] = 0.07971615 + 0.1918502 = 0.2715664$$

$$Var[Y] = 0.2715664 - (0.24786)^2 = 0.2101318$$

$$SD[Y] = \sqrt{0.2101318} = \underline{\underline{0.4584013}}$$

Question No. 6:

For a fully discrete whole life insurance of $1 issued to (x), you are given:

- L_a is the loss at issue random variable with the premium determined such that $\mathrm{E}[L_a] = c$, for some constant c.
- L_b is the loss at issue random variable with the premium determined such that $\mathrm{E}[L_b] = 2c$, for the same constant c as above.
- $\mathrm{Var}[L_a] = 0.22$
- $\mathrm{Var}[L_b] = 0.24$

Calculate c.

$$L_a = v^{K+1} - P_a \ddot{a}_{\overline{K+1|}} = \left(1 + \frac{P_a}{d}\right) v^{K+1} - \frac{P_a}{d}$$

$$L_b = v^{K+1} - P_b \ddot{a}_{\overline{K+1|}} = \left(1 + \frac{P_b}{d}\right) v^{K+1} - \frac{P_b}{d}$$

$$E[L_a] = A_x - P_a \ddot{a}_x = c \Rightarrow P_a = \frac{A_x - c}{\ddot{a}_x}$$

$$E[L_b] = A_x - P_b \ddot{a}_x = 2c \Rightarrow P_b = \frac{A_x - 2c}{\ddot{a}_x}$$

$$\mathrm{Var}[L_a] = \left(1 + \frac{P_a}{d}\right)^2 \mathrm{Var}[v^{K+1}] = \left(\frac{1-c}{d\ddot{a}_x}\right)^2 \mathrm{Var}[v^{K+1}]$$

$$1 + \frac{P_a}{d} = 1 + \frac{A_x - c}{d\ddot{a}_x} = \frac{1-c}{d\ddot{a}_x}$$

Similarly

$$\mathrm{Var}[L_b] = \left(\frac{1-2c}{d\ddot{a}_x}\right)^2 \mathrm{Var}[v^{K+1}]$$

So that

$$\frac{0.22}{0.24} = \left(\frac{1-c}{1-2c}\right)^2 \Rightarrow \frac{1-c}{1-2c} = \pm\sqrt{11/12}$$

$$\Rightarrow c = \frac{1 \pm \sqrt{11/12}}{2 \pm \sqrt{11/12}}$$

c has 4 possible values

$c = 0.6\cancel{7}\cancel{5}352$ ✓

$-0.046\cancel{5}3517$ ✓

$-0.01\cancel{4}6055$

$2.\cancel{13}\cancel{96}05$

$2c > 1$

It cannot be > 1

plug, not possible

Only possible answer is $\underline{\underline{-0.04653517 = c}}$

Question No. 7:

For a fully discrete whole life insurance policy of \$10 on (50), you are given:

- Annual benefit premium is calculated according to the equivalence principle.
- Mortality follows deMoivre's law with $\omega = 100$.
- $i = 0.05$

Calculate the probability that the policy makes a positive loss.

$$A_{50} = \sum_{K=0}^{49} v^{k+1} \frac{1}{50} = \frac{1}{50} \frac{1-v^{50}}{d} = 0.3833744$$

$$\ddot{a}_{50} = \frac{1-A_{50}}{d} = \frac{1-0.3833744}{1-1/1.05} = 12.94914$$

$$\text{Thus, } P = 10\, A_{50}/\ddot{a}_{50} = 0.2960618$$

$$L_0 = 10v^{k+1} - P\ddot{a}_{\overline{K+1}|} = \left(10 + \frac{P}{d}\right)v^{K+1} - \frac{P}{d}$$

$$\Pr[L_0 > 0] = \Pr\left[v^{k+1} > \frac{P/d}{10+P/d}\right]$$

$$= \Pr\Bigg[K < \underbrace{\frac{\log\left(\frac{P/d}{10+P/d}\right)}{-\delta} - 1}_{18.65034}\Bigg]$$

$$= \Pr[K \leq 18] = {}_{19}q_{50}$$

$T_{50} \sim$ deMoivre's with $\omega - 50 = 50$

$$= \frac{19}{50} = \underline{\underline{0.38}}$$

Question No. 8:

Get-a-Life Insurance Company issues a special insurance policy to (45) with the following benefits:

- A death benefit of \$10,000, payable at the end of year of death, provided death occurs before age 65.
- An annuity benefit that pays \$50,000 annually starting immediately when the policyholder reaches age 65.

You are given:

- Annual level premiums of P are paid for the first 20 years only (nothing, thereafter) and are determined according to the actuarial equivalence principle.
- $i = 0.05$
- $\ddot{a}_{45} = 13.96$
- $\ddot{a}_{65} = 11.34$
- ${}_{20}E_{45} = 0.26$

Calculate P.

$$\text{APV(benefits)} = 10000\, A^{1}_{45:\overline{20}|} + 50000\, {}_{20}E_{45}\, \ddot{a}_{65}$$

$$\text{APV(premiums)} = P\, \ddot{a}_{45:\overline{20}|}$$

$$A_{45} = 1 - d\ddot{a}_{45} = 0.3352381$$

$$A_{65} = 1 - d\ddot{a}_{65} = 0.46$$

Solving for P, we get

$$P = \frac{10000\left[0.3352381 - 0.26(0.46)\right] + 50000(0.26)(11.34)}{13.96 - 0.26(11.34)}$$

$$= \frac{149576.4}{11.0116}$$

$$= \underline{\underline{13,583.53}}$$

Question No. 9:

For a special two-year endowment insurance issued to (55), you are given:

- An annual gross premium of G is payable at the beginning of each year.
- A benefit of 5 is payable at the end of the year of death, if he dies before age 57.
- A benefit of $2G$ is payable at age 57, if he survives to reach 57.
- Expenses are 0.10G at the beginning of the first year and 0.05G at the beginning of the second year.
- $q_{55} = 0.01$ and $q_{56} = 0.02$
- $i = 0.05$

Calculate G.

APV(premiums) = $G[1+v(.99)]$

APV(benefits) = $5[v(.01)+v^2(.99)(.02)] + 2G\,v^2(.99)(.98)$

APV(expenses) = $0.10\,G + .05\,G\,v(.99)$

Using equivalence principle to solve for G, we get

$$G\Big[\overbrace{1+v(.99) - 2v^2(.99)(.98) - .10 - .05\,v(.99)}^{0.03571429}\Big]$$

$$= \underbrace{5\left[v(.01) + v^2(.99)(.02)\right]}_{0.137415}$$

$$G = \frac{0.137415}{0.03571429} = \underline{\underline{3.847619}}$$

Question No. 10:

For a fully discrete 20-year endowment life insurance of \$1 issued to (45), you are given:

- Level gross annual premiums are calculated according to the equivalence principle.
- The first year expense is 10% of the gross annual premium.
- Expenses in subsequent years are 2% of the gross annual premium.
- $i = 0.04$
- $A_{45:\overline{20|}} = 0.20$
- ${}^2A_{45:\overline{20|}} = 0.15$
- L_0^g is the gross loss at issue random variable.

Calculate $\mathrm{Var}[L_0^g]$.

$$L_0^g = v^{\min(K+1,20)} - G\,\ddot{a}_{\overline{\min(K+1,20)|}} + 0.08G + 0.02G\,\ddot{a}_{\overline{\min(K+1,20)|}}$$

$$= \left(1 + \frac{.98G}{d}\right) v^{\min(K+1,20)} + \text{constant}$$

$$\mathrm{Var}[L_0^g] = \left(1 + \frac{.98G}{d}\right)^2 \left({}^2A_{45:\overline{20|}} - (A_{45:20})^2\right)$$

where $G = \dfrac{A_{45:\overline{20|}}}{.98\,\ddot{a}_{45:\overline{20|}} - .08} = .009850276$

$A_{45:\overline{20|}} = .20$, $\ddot{a}_{45:\overline{20|}} = \dfrac{1-.20}{.04/1.04}$

$$\mathrm{Var}[L_0^g] = \left(1 + \frac{.98(.009850276)}{.04/1.04}\right)^2 \left[0.15 - (.20)^2\right]$$

$$= \underline{\underline{0.172146}}$$

Question No. 11:

A fully discrete whole life policy of $1,000 issued to (35) with level annual premiums is priced with the following expense assumptions:

	% of Premium	Per 1,000	Per Policy
First year	20%	1.0	5
Renewal years	10%	0.2	2

You are given:

- $i = 0.05$
- $\ddot{a}_{35} = 15.0$

Let G be the gross annual premium and P be the net annual premium.

Calculate $G - P$.

net premium: $P = 1000\dfrac{A_{35}}{\ddot{a}_{35}} = 1000\left(\dfrac{1}{\underbrace{\ddot{a}_{35}}_{15}} - \overbrace{d}^{.05/1.05}\right) = 19.04762$

gross premium: $G\ddot{a}_{35} = 1000A_{35} + .10G + .10G\ddot{a}_{35}$
$+ .8 + .2\ddot{a}_{35}$
$+ 3 + 2\ddot{a}_{35}$

$G(.90\ddot{a}_{35} - .10) = 1000A_{35} + 3.8 + 2.2\ddot{a}_{35}$

$$G = \frac{1000\left(1 - \frac{.05}{1.05}(15)\right) + 3.8 + 2.2(15)}{.90(15) - .10}$$

$= 24.06823$

$G - P = 24.06823 - 19.04762 = \underline{\underline{5.020611}}$

Question No. 12:

The pricing actuary for an insurance company calculates the premium for a fully continuous whole life insurance of \$100 on (65) using the equivalence principle and the assumptions that the force of mortality is constant at 0.08 and $i = 0.06$.

The pricing actuary's supervisor believes that the `Illustrative Life Table` with deaths uniformly distributed over each year of age is a better mortality assumption.

Calculate the insurance company's expected loss at issue if the premium is not changed and the supervisor is indeed correct.

pricing actuary: $P = 100\mu = 100(.08) = 8$

supervisor: $L_0 = 100 v^{T} - 8\bar{a}_{\overline{T}|}$

using ILT, $E[L_0] = 100\bar{A}_{65} - 8\bar{a}_{65}$

$$= 100\frac{i}{\delta}A_{65} - 8\frac{\left(1 - \frac{i}{\delta}A_{65}\right)}{\delta}$$

$$= -29.83191$$

EXTRA PAGE FOR ADDITIONAL OR SCRATCH WORK