MATH 3630
Actuarial Mathematics I
Class Test 1 - 3:35-4:50 PM
Wednesday, 15 November 2017
Time Allowed: 1 hour and 15 minutes
Total Marks: 100 points

Class Test 2C

Please write your name and student number at the spaces provided:

Name: EMIL Student ID: Suggested Solutions

- There are ten (10) written-answer questions here and you are to answer all ten. Each question is worth 10 points.
- Please provide details of your workings in the appropriate spaces provided; partial points will be granted.
- Please write legibly.
- Anyone caught cheating will be subject to university's disciplinary action.

constant μ, δ

Question No. 1: ${}_nE_x @ 2\delta = v^n {}_nE_x$ $A_x = e^{-\delta}(1-e^{-\mu}) / (1-e^{-(\mu+\delta)})$

For a special whole life insurance of 1 issued to (30) with benefits payable at the end of the year of death, you are given:

$$A^{1}_{x:\overline{n}|} = e^{-\delta}(1-e^{-\mu})\frac{(1-e^{-(\mu+\delta)n})}{1-e^{-(\mu+\delta)}}$$

- Mortality follows the `Illustrative Life Table` except for:
  - ages between 35 and 45 where mortality has a constant force of 0.001.
- $i = 0.06$ $\Rightarrow \delta = \log(1.06)$
- Z is the present value random variable for this insurance.

Calculate Var[Z].

$$E[Z] = A^{1}_{30:\overline{5}|} + {}_5E_{30}\, A^{1}_{35:\overline{10}|} + {}_5E_{30}\; {}_{10}E_{35}\, A_{45}$$

${}_5E_{30} = 0.74091$, ${}_{10}E_{35} = e^{-(.001+\log(1.06))(10)}$, $A_{45} = 0.20120$

$$A^{1}_{30:\overline{5}|} = \underbrace{A_{30}}_{.10248} - {}_5E_{30}\underbrace{A_{35}}_{.12872} = .007110065$$

$$A^{1}_{35:\overline{10}|} = \underbrace{e^{-\log(1.06)}(1-e^{-.001})\frac{(1-e^{-(.001+\log(1.06))(10)})}{1-e^{-(.001+\log(1.06))}}}_{.00732691}$$

Plug the values to get, $E[Z] = 0.09495091$

Similarly, evaluating at 2δ, we get

$$E[Z^2] = E[Z] @ 2\delta = {}^2A^{1}_{30:\overline{5}|} + v^5\, {}_5E_{30}\; {}^2A^{1}_{35:\overline{10}|} + v^{15}\, {}_5E_{30}\; {}_{10}E_{35}\; {}^2A_{45}$$

${}^2A_{45} = .06802$

$${}^2A^{1}_{30:\overline{5}|} = \underbrace{{}^2A_{30}}_{.02531} - v^5\, {}_5E_{30}\, \underbrace{{}^2A_{35}}_{.03488} = .005998651$$

$${}^2A^{1}_{35:\overline{10}|} = \underbrace{e^{-2\log(1.06)}(1-e^{-.001})\frac{(1-e^{-(.001+2\log(1.06))(10)})}{1-e^{-(.001+\log(1.06))}}}_{.005545385}$$

Plug the values to get, $E[Z^2] = 0.02069438$

$$\mathrm{Var}[Z] = 0.02069438 - (0.09495091)^2 = \underline{\underline{0.01167871}}$$

when you have constant force μ, δ

$$A_x = \sum_{k=0}^{\infty} e^{-\delta(k+1)} e^{-\mu k} (1-e^{-\mu})$$

$$= e^{-\delta}(1-e^{-\mu}) \underbrace{\sum_{k=0}^{\infty} e^{-(\mu+\delta)k}}_{1/1-e^{-(\mu+\delta)}}$$

$$A^{1}_{x:\overline{n}|} = e^{-\delta}(1-e^{-\mu}) \underbrace{\sum_{k=0}^{n-1} e^{-(\mu+\delta)k}}_{\frac{1-e^{-(\mu+\delta)n}}{1-e^{-(\mu+\delta)}}}$$

Question No. 2:

$\Pr[K=k] = {}_kp_x\, q_{x+k} = e^{-.02k}(1-e^{-.02})$

You are given:

- Mortality follows a constant force of $\mu = 0.02$.
- $i = 0.05$
- Y is the present value random variable for a 3-year temporary life annuity-immediate of 1 per year on (x).

Calculate $\text{Var}[Y]$.

evalute this using $Y = a_{\overline{K}|} = v + v^2 + \cdots + v^K$

Make a table like:

K	$\Pr[K=k]$	$Y = a_{\overline{K}\|}$	$Y \times \Pr[K=k]$	$Y^2 \times \Pr[K=k]$
0	$(1-e^{-.02}) = .01980133$	0	0	0
1	$(1-e^{-.02})e^{-.02} = .01940923$	.952381	.01848	.01760
2	$(1-e^{-.02})e^{-.04} = .01902491$	1.859410	.03538	.06578
≥ 3	$\underbrace{1-\sum_{K=0}^{2}\Pr[K=k]}_{= .94176453}$	2.723248	2.56466	6.98420

$EY = \sum Y \times \Pr[K=k] = 2.61852$

$EY^2 = \sum Y^2 \times \Pr[K=k] = 7.06758$

$\text{Var}[Y] = EY^2 - (E[Y])^2 = 7.06758 - (2.61852)^2 = \underline{0.21094}$

Question No. 3:

For a group of 500 lives, each age 65, with independent future lifetimes, you are given:

- Each life is to be paid 5 per month at the beginning of each month, if alive.
- To fund these payments, each life will contribute an amount of c to a fund to support these payments. This contribution is to be made immediately today and only once.
- Y is the present value random variable today of total annuity payments to the 500 lives.
- $i^{(12)} = 0.12$ $\Rightarrow d^{(12)} = 12\left[1 - \left(1 + \frac{i^{(12)}}{12}\right)^{-1}\right] = 0.1188119$
- $A_{65}^{(12)} = 0.1196$
- ${}^2A_{65}^{(12)} = 0.0395$
- The 95th percentile of a standard normal distribution is 1.645. $Y = \sum_{i=1}^{500} Y_i$

Using the normal approximation, calculate c such that $\Pr[500c > Y] = 0.95$.

Let Y_i = PV random variable for each life $i = 1, 2, \ldots, 500$

$$E[Y_i] = 5(12)\ddot{a}_{65}^{(12)} = 60 * \frac{1 - A_{65}^{(12)}}{d^{(12)}} = 219.9305$$

$$Var[Y_i] = (60)^2 \frac{{}^2A_{65}^{(12)} - \left(A_{65}^{(12)}\right)^2}{\left(d^{(12)}\right)^2} = 6425.569$$

$$E[Y] = 500\,E[Y_i] = 500(219.9305)$$

$$Var[Y] = 500\,Var[Y_i] = 500(6425.569)$$

With normal approximation, $\Pr[500c > Y] = 0.95$

$$\Rightarrow \Pr\left[N \leq \underbrace{\frac{500c - 500(219.9305)}{\sqrt{500(6425.569)}}}_{1.645}\right]$$

Thus, we have

$$c = \frac{500(219.9305) + 1.645\sqrt{500(6425.569)}}{500} = \underline{\underline{450.4991}}$$

Question No. 4:

Based on the same mortality and interest assumptions, you are given:

- $i = 0.06$
- $\ddot{a}^{(4)}_{35} = 13.9178$ using the Woolhouse's approximation with three terms.
- $\ddot{a}^{(6)}_{35} = 13.8759$ using the Woolhouse's approximation with three terms.

Calculate μ_{35}.

Woolhouse: $\ddot{a}^{(m)}_x \approx \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu_x)$

$$\ddot{a}^{(4)}_{35} = \ddot{a}_{35} - \frac{3}{8} - \frac{15}{192}(\delta + \mu_{35}) = 13.9178$$

$$\ddot{a}^{(6)}_{35} = \ddot{a}_{35} - \frac{5}{12} - \frac{35}{432}(\delta + \mu_{35}) = 13.8759$$

Deduct the two equations, we get

$$\delta + \mu_{35} = \frac{(13.9178 - 13.8759) - \left(\frac{5}{12} - \frac{3}{8}\right)}{\left(\frac{35}{432} - \frac{15}{192}\right)}$$

$$= 0.08064$$

$$\Rightarrow \mu_{35} = 0.08064 - \delta$$

$$= 0.08064 + \log\left(\frac{1}{1.06}\right)$$

$$= \underline{\underline{.02237109}}$$

Question No. 5: Note: $\Pr[K \leq k] = {}_{k+1}q_x$

For a whole life annuity-due of 1 payable at the beginning of each year on (45), you are given:

- Mortality follows de Moivre's law with $\omega = 110$.
- $i = 0.10$
- Y is the present value random variable for this annuity.

Calculate the probability that Y exceeds 7.

$$\Pr[Y > 7] = \Pr\left[\frac{1 - v^{K+1}}{d} > 7\right] = \Pr\left[1 - 7d > v^{K+1}\right] \quad 2$$

$$= \Pr\left[(K+1)\underbrace{\log v}_{-\delta} < \log(1-7d)\right] \quad 2$$

$$= \Pr\Big[K > \underbrace{-\tfrac{1}{\delta}\log(1-7d) - 1}_{9.613776}\Big]$$

$$= \Pr[K \geq 10]$$

$$= 1 - \Pr[K \leq 9] \quad 2$$

$$= 1 - {}_{10}q_{45} \quad 1$$

$$= 1 - \frac{10}{\cancel{55}\ 65} = \frac{\cancel{45}\ 55}{\cancel{55}\ 65} = \cancel{0.818181}\ 0.846154$$

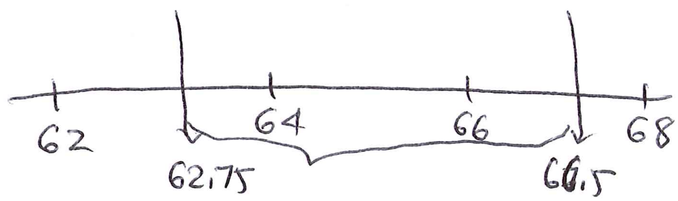
$T_{45} \sim$ Uniform on $(0, 65) \Rightarrow {}_tq_{45} = \frac{t}{65}$ 1

Question No. 6:

For the country of Zooto, you are given:

- Zooto publishes mortality rates in 2-year intervals, that is mortality rates are of the form: ${}_2q_{2x}$, for $x = 0, 1, 2, \ldots$
- Deaths are assumed to be uniformly distributed between ages $2x$ and $2x + 2$, for $x = 0, 1, 2, \ldots$
- ${}_2p_{62} = 0.90$
- ${}_2p_{64} = 0.88$
- ${}_{3.75}p_{62.75} = 0.79097$

Calculate the probability that a person in Zooto now age 66 will die before reaching age 68.



$$\underbrace{{}_{3.75}p_{62.75}}_{0.79097} = \underbrace{{}_{1.25}p_{62.75}}_{\frac{{}_2p_{62}}{{}_{.75}p_{62}}} \; {}_2p_{64} \; {}_{.5}p_{66}$$

since ${}_2p_{62} = {}_{.75}p_{62} \; {}_{1.25}p_{62.75}$

$$0.79097 = \frac{0.90}{\underbrace{1 - \frac{.75}{2}(.10)}_{0.8228571}} \times 0.88 \times \left(1 - \frac{0.5}{2} \, {}_2q_{66}\right)$$

Solving for ${}_2q_{66}$, we get

$${}_2q_{66} = 4\left(1 - 0.79097/0.8228571\right)$$

$$= \underline{\underline{0.1550069}}$$

Question No. 7:

You are given:

- The following select-and-ultimate mortality table with a 3-year select period:

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	x+3
54	977	972	965	958	57
55	970	965	958	951	58
56	963	957	950	942	59

- Deaths are uniformly distributed between integral ages.
- $i = 0.04$
- $1000A_{[55]+2.5} = 535$

Calculate $1000_{2.5|}A_{[55]}$.

$$1000\ {}_{2.5|}A_{[55]} = 1000 \times v^{2.5}\ {}_{2.5}p_{[55]}\ A_{[55]+2.5} \quad 0.535$$

$$= 1000 \times \left(\frac{1}{1.04}\right)^{2.5}\ {}_{2}p_{[55]}\ {}_{0.5}p_{[55]+2}\ A_{[55]+2.5}$$

$$\frac{(958}{970} \qquad 1-0.5\left(1-\frac{951}{958}\right)$$

$$= \underline{\underline{477.2815}}$$

Question No. 8:

Tammy is age 65 and just newly retired. She has a total personal savings of F.

She wants guaranteed income while alive. In exchange for a single payment of F, an insurance company promised her an annual payment (at the beginning of each year) of 50,000 with:

- the first 10 payments guaranteed, whether she is alive or not, and
- the subsequent payments made provided she is alive.

You are given:

- $i = 0.05$
- $\ddot{a}_{65} = 10.263$
- $\ddot{a}_{75} = 7.448$
- $\ddot{a}_{65:\overline{10}|} = 7.095$

Calculate F.

$$F = 50000\left[\ddot{a}_{\overline{10}|} + {}_{10}E_{65}\,\ddot{a}_{75}\right]$$

$$v = \frac{1}{1.05} \Rightarrow \ddot{a}_{\overline{10}|} = \frac{1-v^{10}}{1-v} = 8.107822$$

$$\ddot{a}_{65} = \ddot{a}_{65:\overline{10}|} + {}_{10}E_{65}\,\ddot{a}_{75}$$

$$\Rightarrow {}_{10}E_{65}\,\ddot{a}_{75} = \ddot{a}_{65} - \ddot{a}_{65:\overline{10}|}$$

$$= 10.263 - 7.095$$

$$= 3.168$$

Thus, $F = 50000(8.107822 + 3.168)$

$$= 563{,}791.10$$

Question No. 9:

You are given:

- Z is the present value random variable at issue for a 25-year pure endowment of 1 on (x).
- $i = 0.065$
- $\text{Var}[Z] = 0.09\,\text{E}[Z]$

Calculate ${}_{25}p_x$.

$$Z = v^{25}\, I(T_x > 25)$$

$$E[Z] = v^{25}\, {}_{25}p_x$$

$$\text{Var}[Z] = \underbrace{v^{50}\, {}_{25}p_x (1 - {}_{25}p_x)}$$

$$v^{\cancel{50}\,25}\, \cancel{{}_{25}p_x}\, \underbrace{(1 - {}_{25}p_x)} = 0.09\, \cancel{v^{25}}\, \cancel{{}_{25}p_x}$$

$$1 - {}_{25}p_x = 0.09(1+i)^{25}$$

$$\Rightarrow {}_{25}p_x = 1 - 0.09(1.065)^{25}$$

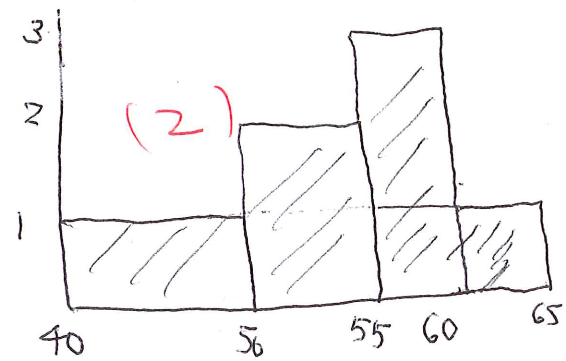
$$= \underline{\underline{0.5655071}}$$

Question No. 10:

For a 25-year term life insurance on (40) with varying benefits, you are given:

- Death benefits are payable at the end of the year of death.
- The benefit amount is:
  - (i) 1 in the first 10 years of death,
  - (ii) increasing to 2 for the following 5 years,
  - (iii) increasing further to 3 for the following 5 years, and
  - (iv) remaining at 1 until reaching age 65.
- Mortality follows the `Illustrative Life Table`.
- $i = 0.06$

Calculate the actuarial present value for this insurance.



$$APV = A^{1}_{40:\overline{25|}} + {}_{10}E_{40}\, A^{1}_{50:\overline{10|}} + {}_{10}E_{40}\, {}_{5}E_{50}\, A^{1}_{55:\overline{5|}} \quad 4$$

0.51081

$$= (A_{40} - {}_{25}E_{40} A_{65}) + {}_{10}E_{40}(A_{50} - {}_{10}E_{50} A_{60}) + {}_{10}E_{40}\, {}_{5}E_{50} (A_{55} - {}_{5}E_{55} A_{60}) \quad 3$$

.51081

$$= (.16132 - (.27414)(.68756)(.43980)) + .53667(.24905 - \cancel{.72137}(.36913)) + .53667(.72137)(.30514 - .70810(.36913))$$

2

$$= \cancel{0.0877003\,4} \quad \underline{0.127829}$$

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EXTRA PAGE FOR ADDITIONAL OR SCRATCH WORK