MATH 3630

Actuarial Mathematics I

Class Test 1 - 3:35-4:50 PM

Wednesday, 15 November 2017

Time Allowed: 1 hour and 15 minutes

Total Marks: 100 points

Class Test 2b

Please write your name and student number at the spaces provided:

Name: EMIL Student ID: Suggested Solutions

- There are ten (10) written-answer questions here and you are to answer all ten. Each question is worth 10 points.
- Please provide details of your workings in the appropriate spaces provided; partial points will be granted.
- Please write legibly.
- Anyone caught cheating will be subject to university's disciplinary action.

Question No. 1:

You are given:

- Z is the present value random variable at issue for an 25-year pure endowment of 10 on (x).
- ${}_{25}p_x = 0.60$
- $\mathrm{Var}[Z] = 1.6\,\mathrm{E}[Z]$

Calculate the annual effective interest rate.

$$Z = 10 v^{25} I(T_{x} > 25)$$

$$E[Z] = 10 v^{25} {}_{25}p_x$$

$$Var[Z] = 100 v^{50} {}_{25}p_x (1 - {}_{25}p_x)$$

$$100 v^{\cancel{50}\,25} \cancel{0.60} (0.40) = 1.6 (10) \cancel{v^{25}} \cancel{(0.60)}$$

$$v^{25} = \frac{16}{100(.4)} = 0.40$$

$$1+i = 0.40^{-1/25}$$

$$\Rightarrow i = 0.40^{-1/25} - 1 = \underline{\underline{0.03733158}}$$

Question No. 2:

Based on the same mortality and interest assumptions, you are given:

- $i = 0.04$
- $\ddot{a}_{65}^{(2)} = 7.7266$ using the Woolhouse's approximation with three terms.
- $\ddot{a}_{65}^{(12)} = 7.5165$ using the Woolhouse's approximation with three terms.

Calculate $\ddot{a}_{65}^{(4)}$ using the Woolhouse's approximation with three terms.

Woolhouse: $\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu_x)$

$$\ddot{a}_{65}^{(2)} = \ddot{a}_{65} - \frac{1}{4} - \frac{3}{48}(\delta + \mu_{65}) = 7.7266$$

$$\ddot{a}_{65}^{(12)} = \ddot{a}_{65} - \frac{11}{24} - \frac{143}{1728}(\delta + \mu_{65}) = 7.5165$$

Deduct the two equations to get, $\delta + \mu_{65} = \dfrac{(7.7266 - 7.5165) - \left(\frac{11}{24} - \frac{1}{4}\right)}{\left(\frac{143}{1728} - \frac{3}{48}\right)}$

$= 0.08722286$

$$\ddot{a}_{65} = \underbrace{\frac{1}{4} + \frac{3}{48}(0.08722286) + 7.7266}_{7.982051}$$

$$\text{W3: } \ddot{a}_{65}^{(4)} = \ddot{a}_{65} - \frac{3}{8} - \frac{15}{192}(\delta + \mu_{65})$$

$$= 7.982051 - \frac{3}{8} - \frac{15}{192}(0.08722286)$$

$$= \underline{\underline{7.600237}}$$

Question No. 3:

Ron is age 65 and just newly retired. He has a total personal savings of 150,000.

He wants guaranteed income while alive. In exchange for a single payment of 150,000, an insurance company promised him an annual payment (at the beginning of each year) of 13,300 with:

- the first 10 payments guaranteed, whether he is alive or not, and
- the subsequent payments made provided he is alive.

You are given:

- $i = 0.05$
- $\ddot{a}_{65} = 10.263$
- $\ddot{a}_{75} = 7.448$

Calculate $\ddot{a}_{\overline{65:\overline{10}|}}$

$$150{,}000 = 13{,}300\left(\ddot{a}_{\overline{10}|} + {}_{10}E_{65}\,\ddot{a}_{75}\right)$$

$$v = \frac{1}{1.05} \Rightarrow \ddot{a}_{\overline{10}|} = \frac{1-v^{10}}{1-v} = 8.107822$$

Solving for ${}_{10}E_{65}\,\ddot{a}_{75}$, we get

$${}_{10}E_{65}\,\ddot{a}_{75} = (150000/13300) - 8.107822$$

$$= 3.170374$$

$$\ddot{a}_{65:\overline{10}|} = \ddot{a}_{65} - {}_{10}E_{65}\,\ddot{a}_{75}$$

$$= 10.263 - 3.170374$$

$$= \underline{\underline{7.092626}}$$

Question No. 4:

For a group of 200 lives, each age 65, with independent future lifetimes, you are given:

- Each life is to be paid 2 per month at the beginning of each month, if alive.
- To fund these payments, each life will contribute an amount of c to a fund to support these payments. This contribution is to be made immediately today and only once.
- Y is the present value random variable today of total annuity payments to the 200 lives.
- $i^{(12)} = 0.09$ $\Rightarrow d^{(12)} = 12\left[1 - \left(1 + \frac{i^{(12)}}{12}\right)^{-1}\right] = .08933002$
- $A_{65}^{(12)} = 0.1814$
- ${}^2A_{65}^{(12)} = 0.0624$
- The 95$^{\text{th}}$ percentile of a standard normal distribution is 1.645.

Using the normal approximation, calculate c such that $\Pr[200c > Y] = 0.95$.

Let Y_i = PV random variable for each life $i = 1, 1, \ldots, 200$

$$E[Y_i] = 24\, \ddot{a}_{65}^{(12)} = 24 * \frac{1 - A_{65}^{(12)}}{d^{(12)}} = 219.9305$$

$$Var[Y_i] = (24)^2 * \frac{{}^2A_{65}^{(12)} - \left(A_{65}^{(12)}\right)^2}{\left(d^{(12)}\right)^2} = 2128.932$$

$$Y = 100 Y_i \Rightarrow E[Y] = 200 \times E[Y_i] = 200(219.9305)$$

$$Var[Y] = 200 * Var[Y_i] = 200(2128.932)$$

With normal approximation, $\Pr[200c > Y] = 0.95$

$$\Rightarrow \Pr\left[N \leq \underbrace{\frac{200c - 200(219.9305)}{\sqrt{200(2128.932)}}}_{=1.645}\right]$$

Thus, we have

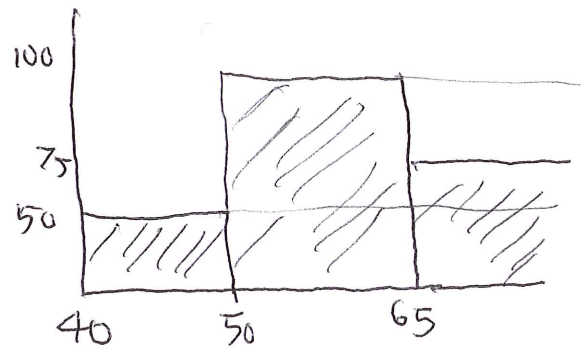
$$c = \frac{200(219.9305) + 1.645\sqrt{200(2128.932)}}{200} = \underline{\underline{225.2975}}$$

Question No. 5:

For a whole life insurance on (40) with varying benefits, you are given:

- Death benefits are payable at the end of the year of death.
- The benefit amount is:
  - (i) 50 in the first 10 years of death,
  - (ii) increasing to 100 for the following 15 years, but
  - (iii) decreasing to 75 after that until death.
- Mortality follows the `Illustrative Life Table`.
- $i = 0.06$

Calculate the actuarial present value for this insurance.



$$APV = 50A_{40} + 50\ {}_{10}E_{40}\ A_{50} - 25\ {}_{10}E_{40}\ {}_{15}E_{50}\ A_{65}$$

$$= 50(.16132) + 50(.53667)(.24905) - 25(.53667)(.51081)(.68756)(.43980)$$

$$= \underline{\underline{12.67649}}$$

Question No. 6: ${}_nE_x \,@\, 2\delta = v^n\,{}_nE_x$

Constant μ, δ

$A_x = e^{-\delta}(1-e^{-\mu})/(1-e^{-(\mu+\delta)})$

For a special whole life insurance of 1 issued to (30) with benefits payable at the end of the year of death, you are given:

$A^{1}_{x:\overline{n}|} = e^{-\delta}(1-e^{-\mu})\dfrac{(1-e^{-(\mu+\delta)n})}{1-e^{-(\mu+\delta)}}$

- Mortality follows the `Illustrative Life Table` except for:
  - ages between 45 and 55 where mortality has a constant force of 0.005.
- $i = 0.06$ $\Rightarrow$ $\delta = \log(1.06)$
- Z is the present value random variable for this insurance.

Calculate Var[Z].

$e^{-10(.005+\log(1.06))}$

$.54733 * .73529$ $\quad$ $.30514$

$$E[Z] = A^{1}_{30:\overline{15}|} + {}_{15}E_{30}\, A^{1}_{45:\overline{10}|} + {}_{15}E_{30}\,{}_{10}E_{45}\, A_{55}$$

$$A^{1}_{45:\overline{10}|} = \underbrace{e^{-\log(1.06)}(1-e^{-.005})\frac{\left(1-e^{-(.005+\log(1.06))(10)}\right)}{1-e^{-(.005+\log(1.06))}}}_{=\ .03598139}$$

From ILT,

$$A_{30} - {}_{15}E_{30}A_{45} = \underbrace{.10248 - .54733(.73529)(.20120)}_{=\ .02150781}$$

Plug the values to get,

$E[Z] = 0.1012174$

Similarly, evaluating at 2δ, we get

$.13067$

$$E[Z^2] = E[Z]\,@\,2\delta = {}^2A^{1}_{30:\overline{15}|} + v^{15}\,{}_{15}E_{30}\,{}^2A^{1}_{45:\overline{10}|} + v^{25}\,{}_{15}E_{30}\,{}_{10}E_{45}\,{}^2A_{55}$$

$${}^2A^{1}_{45:\overline{10}|} = \underbrace{e^{-2\log(1.06)}(1-e^{-.005})\frac{\left(1-e^{-(.005+2\log(1.06))(10)}\right)}{1-e^{-(.005+2\log(1.06))}}}_{=\ .02728284}$$

$${}^2A^{1}_{30:\overline{15}|} = \underbrace{.02531 - v^{15}(.54733)(.73529)(.06802)}_{=\ .01388778}$$

Plug the values to get,

$E[Z^2] = 0.02497754$

$$\mathrm{Var}[Z] = 0.02497754 - (0.1012174)^2 = \underline{\underline{0.01473257}}$$

when you have constant force μ, δ

$$A_x = \sum_{k=0}^{\infty} e^{-\delta(k+1)} e^{-\mu k} (1-e^{-\mu})$$

$$= e^{-\delta}(1-e^{-\mu}) \underbrace{\sum_{k=0}^{\infty} e^{-(\mu+\delta)k}}_{1/\left(1-e^{-(\mu+\delta)}\right)}$$

$$A^{1}_{x:\overline{n|}} = e^{-\delta}(1-e^{-\mu}) \underbrace{\sum_{k=0}^{n-1} e^{-(\mu+\delta)k}}_{\frac{1-e^{-(\mu+\delta)n}}{1-e^{-(\mu+\delta)}}}$$

Question No. 7:

should have been 966

You are given:

- The following select-and-ultimate mortality table with a 3-year select period:

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	x+3
55	987	980	973	~~976~~	58
56	978	993	966	959	59
57	971	965	958	950	60

- Deaths are uniformly distributed between integral ages.
- $i = 0.05$
- $1000A_{[55]+2.5} = 518$

Calculate $1000\,{}_{2.5|}A_{[55]}$.

$$1000\,{}_{2.5|}A_{[55]} = 1000 \times v^{2.5}\,{}_{2.5}p_{[55]}\,A_{[55]+2.5}$$

$$= 1000 \times \left(\frac{1}{1.05}\right)^{2.5}\,{}_{2}p_{[55]}\,{}_{0.5}p_{[55]+2}\,\cdot\,\overset{0.518}{A_{[55]+2.5}}$$

$$\frac{973}{987} \qquad 1-0.5\left(1-\frac{966}{973}\right)$$

$$= \cancel{452.7112} \qquad \underline{\underline{450.3884}}$$

Question No. 8: use ILT to evaluate $Pr[K=k] = {}_kp_{65}\, q_{65+k}$

You are given:

- Mortality follows the `Illustrative Life Table`.
- $i = 0.05$
- Y is the present value random variable for a 3-year temporary life annuity-immediate of 1 per year on (65).

Calculate Var[Y]. → evaluate this using $Y = a_{\overline{K}|} = v + v^2 + \dots + v^K$

Make a table like:

k	$Pr[K=k]$	$Y = a_{\overline{K}\vert}$	$Y \times Pr[K=k]$	$Y^2 \times Pr[K=k]$
0	$q_{65} = .02132$	0	0	0
1	$p_{65}\,q_{66} = \underbrace{(1-.02132)(.02329)}_{.02279346}$	.952381	.02170805	.02067434
2	$p_{65}\,p_{66}\,q_{67} = \underbrace{(1-.02132)(1-.02329)(.02544)}_{\cancel{.02433012}\ .02431775}$	1.859410	.04521668 $\cancel{.0452}$	.08407638
≥ 3	$\underbrace{1-.02132-.02279346-.02431775}_{.93156879}$	2.723248	2.53689287	6.90858851

$E[Y] = \sum Y * Pr[K=k] = 2.603818$

$E[Y^2] = \sum Y^2 \times Pr[K=k] = 7.013339$

$Var[Y] = E[Y^2] - (E[Y])^2 = 7.013339 - (2.603818)^2 = \underline{\underline{0.2334731}}$

Question No. 9:

For the country of Zooturn, you are given:

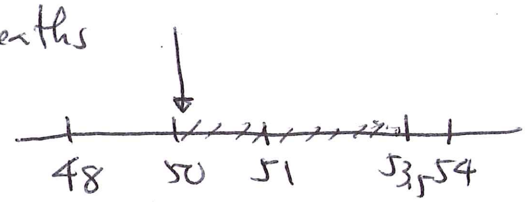
- Zooturn publishes mortality rates in 3-year intervals, that is mortality rates are of the form: ${}_3q_{3x}$, for $x = 0, 1, 2, \ldots$
- Deaths are assumed to be uniformly distributed between ages $3x$ and $3x + 3$, for $x = 0, 1, 2, \ldots$
- ${}_3q_{48} = 0.02$
- ${}_3q_{51} = 0.05$
- ${}_3q_{54} = 0.09$

note: ${}_3p_{48} = {}_2p_{48}\, p_{50}$

$\Rightarrow \quad p_{50} = {}_3p_{48} / {}_2p_{48}$

Calculate the probability that a person in Zooturn now age 50 will die the next 3.5 years.

UDD implies uniformly distributing deaths within 3-year intervals



$$
\begin{aligned}
{}_{3.5}q_{50} &= 1 - {}_{3.5}p_{50} \\
&= 1 - p_{50}\; {}_{2.5}p_{51} \\
&= 1 - \frac{{}_3p_{48}}{{}_2p_{48}}\; {}_{2.5}p_{51} \\
&= 1 - \frac{0.98}{1 - \frac{2}{3}(.02)} \left(1 - \frac{2.5}{3}(.05)\right) \\
&= \underline{\underline{0.04814189}}
\end{aligned}
$$

Question No. 10: Note: $\Pr[K \le k] = {}_{k+1}q_x$

For a whole life annuity-due of 2 payable at the beginning of each year on (50), you are given:

- Mortality follows the `Illustrative Life Table`.
- $i = 0.05$
- Y is the present value random variable for this annuity.

Calculate the probability that Y exceeds 13.5. Let $K = K_{50}$, curtate future lifetime of (50)

$$\Pr[Y > 13.5] = \Pr[2\ddot{a}_{\overline{K+1}|} > 13.5]$$

$$= \Pr\left[\frac{1-v^{K+1}}{d} > 6.75\right]$$

$$= \Pr[(K+1)\underbrace{\log v}_{-\delta} < \log(1-6.75d)]$$

$$= \Pr[K > \underbrace{-\tfrac{1}{\delta}\log(1-6.75d) - 1}_{6.947617}]$$

$$= \Pr[K \ge 7] = 1 - \Pr[K \le 6]$$

$$= 1 - {}_7q_{50} = {}_7p_{50}$$

from ILT $\Rightarrow$ $$= \frac{l_{57}}{l_{50}} = \frac{8479908}{8950901}$$

$$= \underline{\underline{0.9473804}}$$

EXTRA PAGE FOR ADDITIONAL OR SCRATCH WORK