Class Test 2a

MATH 3630
Actuarial Mathematics I
Class Test 1 - 3:35-4:50 PM
Wednesday, 15 November 2017
Time Allowed: 1 hour and 15 minutes
Total Marks: 100 points

Please write your name and student number at the spaces provided:

Name: EMIL Student ID: Suggested Solutions

- There are ten (10) written-answer questions here and you are to answer all ten. Each question is worth 10 points.
- Please provide details of your workings in the appropriate spaces provided; partial points will be granted.
- Please write legibly.
- Anyone caught cheating will be subject to university's disciplinary action.

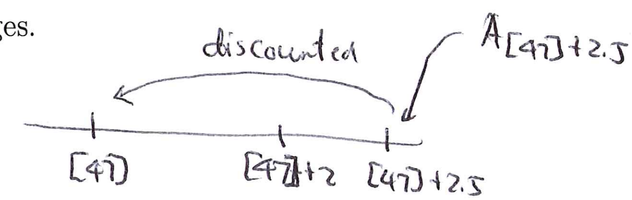
Question No. 1:

You are given:

- The following select-and-ultimate mortality table with a 3-year select period:

$[x]$	$\ell_{[x]}$	$\ell_{[x]+1}$	$\ell_{[x]+2}$	ℓ_{x+3}	x+3
45	1876	1871	1864	1855	48
46	1869	1864	1857	1850	49
47	1862	1856	1849	1841	50

- Deaths are uniformly distributed between integral ages.
- $i = 0.06$
- $1000A_{[47]+2.5} = 496$ when $i = 0.06$.



Calculate $1000\,{}_{2.5|}A_{[47]}$.

$$1000\,{}_{2.5|}A_{[47]} = 1000\, v^{2.5}\, {}_{2.5}p_{[47]}\, A_{[47]+2.5}$$

$$= 1000\, v^{2.5}\, \underbrace{p_{[47]}\, p_{[47]+1}}_{{}_2p_{[47]} = \frac{\ell_{[47]+2}}{\ell_{[47]}}}\, {}_{0.5}p_{[47]+2}\, A_{[47]+2.5}$$

$$1 - 0.5\, q_{[47]+2}$$
$$= 1 - 0.5(1 - p_{[47]+2})$$

$$= 1000 \left(\frac{1}{1.06}\right)^{2.5} \frac{1849}{1862} * \left(1 - 0.5\left(1 - \frac{1841}{1849}\right)\right) * 0.496$$

$$= \underline{\underline{424.8481}}$$

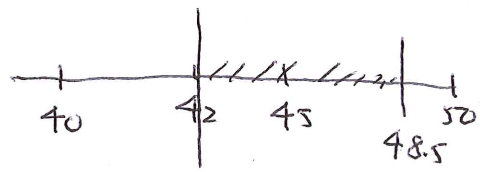
Question No. 2:

For the country of Zoopiter, you are given:

- Zoopiter publishes mortality rates in 5-year intervals, that is mortality rates are of the form: ${}_5q_{5x}$, for $x = 0, 1, 2, \ldots$
- Deaths are assumed to be uniformly distributed between ages $5x$ and $5x + 5$, for $x = 0, 1, 2, \ldots$
- ${}_5p_{40} = 0.97$ ${}_5q_{40} = .03$
- ${}_5p_{45} = 0.94$ ${}_5q_{45} = .06$
- ${}_5p_{50} = 0.90$

Calculate the probability that a person in Zoopiter now age 42 will die the next 6.5 years.

UDD implies uniformly distributing deaths within 5year intervals.



$${}_{6.5}q_{42} = 1 - {}_{6.5}p_{42}$$

$$= 1 - {}_3p_{42}\ {}_{3.5}p_{45}$$

$$= 1 - \frac{{}_5p_{40}}{{}_2p_{40}}\ {}_{3.5}p_{45}$$

$$= 1 - \frac{0.97}{1 - \frac{2}{5}(.03)}\left(1 - \frac{3.5}{5}(.06)\right)$$

$$= 0.05945344$$

$${}_5p_{40} = {}_2p_{40}\ {}_3p_{42}$$

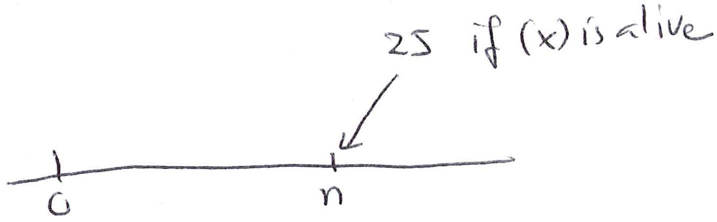
$$\Rightarrow {}_3p_{42} = \frac{{}_5p_{40}}{{}_2p_{40}}$$

Question No. 3:

You are given:

- Z is the present value random variable at issue for an n-year pure endowment of 25 on (x).
- ${}_np_x = 0.63$
- $\mathrm{E}[Z] = 3.67$

Calculate $\mathrm{Var}[Z]$.



$$Z = 25v^n I(T_x > n)$$

$$E[Z] = 25\, v^n \underbrace{E[I(T_x > n)]}_{{}_np_x} = 25\, v^n\, {}_np_x = 3.67$$

$$\Rightarrow \quad v^n = \frac{3.67}{25(0.63)}$$

$$= 0.21330159$$

$$\mathrm{Var}[Z] = 25^2 v^{2n} \underbrace{\mathrm{Var}[I(T_x > n)]}_{{}_np_x(1 - {}_np_x)}$$

$$= 25^2 (0.21330159)^2 (0.63)(1 - 0.63)$$

$$= \underline{\underline{7.910306}}$$

Question No. 4:

Becky is age 65 and just newly retired. She has a total personal savings of 1,000,000.

She wants guaranteed income while alive. In exchange for a single payment of 1,000,000, an insurance company promised her an annual payment (at the beginning of each year) of B with:

- the first 10 payments guaranteed, whether she is alive or not, and
- the subsequent payments made provided she is alive.

$V = 1/1.05$

You are given:

- $i = 0.05$
- $\ddot{a}_{65} = 10.263$
- $\ddot{a}_{75} = 7.448$
- $\ddot{a}_{65:\overline{10}|} = 7.095$

Calculate B.

$$1000000 = B\left[\ddot{a}_{\overline{10}|} + {}_{10}E_{65}\,\ddot{a}_{75}\right]$$

$$B = \frac{1000000}{\ddot{a}_{\overline{10}|} + {}_{10}E_{65}\,\ddot{a}_{75}}$$

$$\ddot{a}_{\overline{10}|} = \frac{1-V^{10}}{1-V} = 8.107822$$

Since $\ddot{a}_{65} = \ddot{a}_{65:\overline{10}|} + {}_{10}E_{65}\,\ddot{a}_{75}$, then

$${}_{10}E_{65}\,\ddot{a}_{75} = \ddot{a}_{65} - \ddot{a}_{65:\overline{10}|} = 10.263 - 7.095$$

$$= 3.168$$

$$B = \frac{1000000}{8.107822 + 3.168}$$

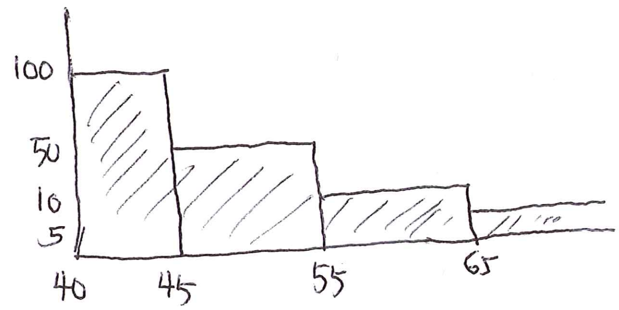
$$= \underline{\underline{88,685.33}}$$

Question No. 5:

For a whole life insurance on (40) with varying benefits, you are given:

- Death benefits are payable at the end of the year of death.
- The benefit amount is:
  - (i) 100 in the first 5 years of death,
  - (ii) decreasing to 50 for the following 10 years,
  - (iii) decreasing further to 10 for the following 10 years, and
  - (iv) decreasing even further to 5 after that until death.
- Mortality follows the `Illustrative Life Table`.
- $i = 0.06$

Calculate the actuarial present value for this insurance.



$$APV = 100 A_{40} - 50\, {}_5E_{40}\, A_{45}$$
$$- 40\, {}_5E_{40}\, {}_{10}E_{45}\, A_{55}$$
$$- 5\, {}_5E_{40}\, {}_{10}E_{45}\, {}_{10}E_{55}\, A_{65}$$
$$= 100(.16132) - 50(.73529)(.20120)$$
$$- 40(.73529)(.52652)(.30514)$$
$$- 5(.73529)(.52652)(.48686)(.43980)$$
$$= \underline{\underline{3.595168}}$$

Question No. 6: $_nE_x @ 2\delta = v^n {_nE_x}$

constant μ, δ

$A_x = e^{-\delta}(1-e^{-\mu})/(1-e^{-(\mu+\delta)})$

For a special whole life insurance of 1 issued to (30) with benefits payable at the end of the year of death, you are given:

- Mortality follows the `Illustrative Life Table` except for:
  - ages 65 and beyond where mortality has a constant force of 0.01.
- $i = 0.06$ $\Rightarrow$ $\delta = \log(1.06)$
- Z is the present value random variable for this insurance.

Calculate Var[Z].

$$E[Z] = A^{\,1}_{30:\overline{35|}} + \underbrace{{_{35}E_{30}}}_{.29374(.51081)(.68756)} A_{65}$$

$$A_{65} = \overbrace{\frac{e^{-\log(1.06)}(1-e^{-.01})}{1-e^{-(.01+\log(1.06))}}}^{0.1422465}$$

From ILT, $A_{30} - {_{35}E_{30}} A_{65} = \underbrace{.10248 - .29374(.51081)(.68756)(.43980)}$

$= 0.0571079 6$

Plug the values to get,

$E[Z] = 0.07178284$

$$E[Z^2] = \underbrace{E[Z] @ 2\delta}_{\text{evaluated } @ 2\delta} = {^2A^{\,1}_{30:\overline{35|}}} - v^{35} {_{35}E_{30}} {^2A_{65}}$$

$${^2A_{65}} = \overbrace{\frac{e^{-2\log(1.06)}(1-e^{-.01})}{1-e^{-(.01+2\log(1.06))}}}^{.07450508}$$

From ILT, $\underset{.62531}{{^2A_{30}}} - v^{35} {_{35}E_{30}} \underset{.23603}{{^2A_{65}}} = 0.0221419 3$

Plug the values to get,

$E[Z^2] = 0.02314196$

$\text{Var}[Z] = 0.02314196 - (0.07178284)^2 = \underline{\underline{0.01798918}}$

when you have constant force μ, δ

$$A_x = \sum_{k=0}^{\infty} e^{-\delta(k+1)} e^{-\mu k} (1-e^{-\mu})$$

$$= e^{-\delta}(1-e^{-\mu}) \underbrace{\sum_{k=0}^{\infty} e^{-(\mu+\delta)k}}_{1/1-e^{-(\mu+\delta)}}$$

$$A_{x:\overline{n}|}^{1} = e^{-\delta}(1-e^{-\mu}) \underbrace{\sum_{k=0}^{n-1} e^{-(\mu+\delta)k}}_{\frac{1-e^{-(\mu+\delta)n}}{1-e^{-(\mu+\delta)}}}$$

Question No. 7:

For a group of 100 lives, each age 65, with independent future lifetimes, you are given:

- Each life is to be paid 1 per month at the beginning of each month, if alive.
- To fund these payments, each life will contribute an amount of c to a fund to support these payments. This contribution is to be made immediately today and only once.
- Y is the present value random variable today of total annuity payments to the 100 lives.
- $i^{(12)} = 0.06$ $\Rightarrow d^{(12)} = 12\left[1 - \left(1 + \frac{i^{(12)}}{12}\right)^{-1}\right] = 0.05970149$
- $A_{65}^{(12)} = 0.2965$
- ${}^2A_{65}^{(12)} = 0.1191$
- The 95$^{\text{th}}$ percentile of a standard normal distribution is 1.645.

Using the normal approximation, calculate c such that $\Pr[100c > Y] = 0.95$.

Let Y_i = PV random variable of a life annuity of 1 per month for each life i, $i = 1, \dots, 100$

$$E[Y_i] = 12\,\ddot{a}_{65}^{(12)} = 12 * \frac{1 - A_{65}^{(12)}}{d^{(12)}} = 141.4035$$

$$Var[Y_i] = (12)^2 \frac{{}^2A_{65}^{(12)} - \left(A_{65}^{(12)}\right)^2}{\left(d^{(12)}\right)^2} = 1260.016$$

aggregate payment $Y = 100\,Y_i$, $\quad E[Y_i] = 100(141.4035) = 14140.35$

$Var[Y_i] = 100(1260.016) = 126001.6$

With normal approximation, $\Pr[Y < 100c] \approx \Pr\left[N \leq \frac{100c - 14140.35}{\sqrt{126001.6}}\right] = 0.95$

$= 1.645$

Thus, we have

$$c = \frac{14140.35 + 1.645\sqrt{126001.6}}{100} = \underline{\underline{147.2427}}$$

Question No. 8:

$$\Pr[K=k] = {}_{k|}q_x = {}_kp_x - {}_{k+1}p_x$$
$$= .92^k - .92^{k+1}$$
$$= .92^k(1-.92) = .92^k(.08)$$

You are given:

- For a fixed age x, ${}_kp_x = (0.92)^k$ for $k \geq 0$.
- $i = 0.05$
- Y is the present value random variable for a 3-year temporary life annuity-immediate of 1 per year on (x).

Calculate Var[Y]. evaluate this using $Y = a_{\overline{K}|} = v + v^2 + \dots + v^K$

Make a table like:

K	$\Pr[K=k] = .92^k(.08)$	$Y = a_{\overline{K}|}$	$Y \times \Pr[K=k]$	$Y^2 \times \Pr[K=k]$
0	.08	0	0	0
1	$.92(.08) = .0736$	$a_{\overline{1}|} = .952381$	.07009524	.06675737
2	$.92^2(.08) = .067712$	$a_{\overline{2}|} = 1.859410$	.12590440	.23410795
≥ 3	$\underbrace{1-.08-.0736-.067712}_{= .778688}$	$a_{\overline{3}|} = 2.723248$	2.12056056	5.77481237

$$E[Y] = \sum Y \times \Pr[K=k] = 2.31656$$

$$E[Y^2] = \sum Y^2 * \Pr[K=k] = 6.075678$$

$$\mathrm{Var}[Y] = E[Y^2] - (E[Y])^2 = \underbrace{6.075678 - (2.31656)^2}_{= 0.7092265}$$

Question No. 9: Note: $P_r[K \le k] = {}_{k+1}q_x$

For a whole life annuity-due of 5 payable at the beginning of each year on (x), you are given:

- Mortality follows a constant force of $\mu = 0.05$. $\longrightarrow {}_tp_x = e^{-.05t}$
- $i = 0.035$
- Y is the present value random variable for this annuity.

Calculate the probability that Y exceeds 60.

$$Y = 5\,\ddot{a}_{\overline{K+1}|}$$

$$P_r[Y > 60] = P_r[5\,\ddot{a}_{\overline{K+1}|} > 60] = P_r\left[\frac{1-v^{K+1}}{d} > 12\right]$$

$$= P_r[(K+1)\underbrace{\log v}_{-1/\delta} < \log(1-12d)]$$

$$= P_r[K > \underbrace{-\tfrac{1}{\delta}\log(1-12d) - 1}_{14.13119}]$$

$$= P_r[K \geq 15]$$

$$= 1 - P_r[K \le 14] = 1 - {}_{15}q_x$$

$$= {}_{15}p_x = e^{-.05(15)} = \underline{\underline{.4723666}}$$

Question No. 10:

Based on the same mortality and interest assumptions, you are given:

- $i = 0.055$
- $\ddot{a}_{50}^{(4)} = 11.3470$ using the Woolhouse's approximation with three terms.
- $\ddot{a}_{50}^{(12)} = 11.2633$ using the Woolhouse's approximation with three terms.

Calculate $\ddot{a}_{50}^{(6)}$ using the Woolhouse's approximation with two terms.

Woolhouse : $\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu_x)$

$\ddot{a}_{50}^{(4)} = \ddot{a}_{50} - \frac{3}{8} - \frac{15}{192}(\delta + \mu_{50}) = 11.3470$

$\ddot{a}_{50}^{(12)} = \ddot{a}_{50} - \frac{11}{24} - \frac{143}{1728}(\delta + \mu_{50}) = 11.2633$

Reduct the two equations to get, $\delta + \mu_{50} = \dfrac{(11.3470 - 11.2633) - (\frac{11}{24} - \frac{3}{8})}{(\frac{143}{1728} - \frac{15}{192})}$

$= .0792$

To solve $\ddot{a}_{50} = 11.3470 + \frac{3}{8} + \frac{15}{192}(.0792) = 11.72819$

W2: $\ddot{a}_{50}^{(6)} \approx \ddot{a}_{50} - \frac{5}{12} = \underline{\underline{11.31152}}$

EXTRA PAGE FOR ADDITIONAL OR SCRATCH WORK