MATH 3630
Actuarial Mathematics I
Class Test 2 - 3:35-5:25 PM
Monday, 18 November 2019
Time Allowed: 110 minutes
Total Marks: 100 points

Please write your name and student number at the spaces provided:

Name: Emil

Student ID: Suggested Solutions

- There are twelve (12) written-answer questions here and you are to answer all ten. Each question is worth 10 points.
- Please provide details of your workings in the appropriate spaces provided; partial points will be granted. Work with not enough details will get reduced marks even if final answer is correct.
- Please write legibly.
- Anyone caught cheating will be subject to university's disciplinary action.

## Question No. 1:

You are given:

- The following extract from a select-and-ultimate life table with a 2-year select period:

| $[x]$ | $\ell_{[x]}$ | $\ell_{[x]+1}$ | $\ell_{[x]+2}$ | $x+2$ |
|---|---|---|---|---|
| 65 | 100,000 | 99,618 | 97,785 | 67 |
| 66 | 98,154 | 97,243 | 95,816 | 68 |
| 67 | 96,217 | 95,219 | 93,740 | 69 |

- Deaths are uniformly distributed between integer ages.

Calculate ${}_{0.90}q_{[65]+0.75}$.

$$= 1 - {}_{0.90}p_{[65]+0.75} = 1 - \frac{\ell_{[65]+1.65}}{\ell_{[65]+.75}}$$

$$= 1 - \frac{0.65\,\ell_{67} + 0.35\,\ell_{[65]+1}}{0.75\,\ell_{[65]+1} + 0.25\,\ell_{[65]}}$$

$$= 1 - \frac{0.65(97785) + 0.35(99618)}{0.75(99618) + 0.25(100{,}000)}$$

$$= 1 - \frac{98426.55}{99713.50}$$

$$= 1 - 0.9870935 = 0.013$$

## Question No. 2:

Two life insurance policies are actuarially equivalent if they have equal actuarial present values. The following policies issued to (45) are actuarially equivalent:

- A whole life insurance of 250 payable at the end of the year of death.
- A special whole life insurance, also payable at the end of the year of death, that pays 100 for the first 10 years and $B$ thereafter.

You are given:

- $i = 0.05$
- $A_{45} = 0.30$
- $A_{55} = 0.35$
- $A_{45:\overline{10}|} = 0.69$

Calculate the value of $B$.

$$\text{APV(first policy)} = 250 \overset{0.30}{A_{45}} = 75$$

$$\text{APV(second policy)} = 100 \underbrace{A_{45}}_{0.30} + (B-100)\, {}_{10}E_{45} \underbrace{A_{55}}_{0.35}$$

$$= 30 + (B-100)(0.35)\, {}_{10}E_{45}$$

Solving for $B$, we get

$$B = 100 + \frac{45}{.35\, \underbrace{{}_{10}E_{45}}_{0.60}} = 100 + \frac{45}{.35(.60)} = 314.2857$$

$$A_{45:\overline{10}|} = A_{45} - {}_{10}E_{45} A_{55} + {}_{10}E_{45} = A_{45} - (A_{55} - 1)\, {}_{10}E_{45}$$

$$\Rightarrow \quad {}_{10}E_{45} = \frac{A_{45} - A_{45:\overline{10}|}}{A_{55} - 1} = \frac{0.30 - 0.69}{0.35 - 1} = \frac{.39}{.65} = .60$$

**Question No. 3:** ${}_nE_x @ 2\delta = v^n {}_nE_x$

For de Moivre's $A_x = a_{\overline{\omega-x}|}/\omega-x$, $A^{\,1}_{x:\overline{n}|} = a_{\overline{n}|}/\omega-x$

For a special whole life insurance of 2 issued to (40) with benefits payable at the end of the year of death, you are given:

$T_x \sim U(0, \omega-x)$

$P_r(T_x > t) = 1 - \frac{t}{\omega-x}$

- $Z$ is the present value random variable for this insurance.
- Mortality follows the `Survival Ultimate Life Table` except for ages:
  - between 50 and 55 where mortality follows a deMoivre's law with $\omega = 100$.
- $i = 0.05$

Consider the calculations when benefit is 1, and then in the end will multiply the value by $2^2$

Calculate Var($Z$).

$$E[Z] = A^{\,1}_{40:\overline{10}|} + {}_{10}E_{40}\, A^{\,1}_{50:\overline{5}|} + {}_{10}E_{40}\, {}_5E_{50}\, A_{55}$$

$A_{55}$: $.23524$

${}_{10}E_{40}$: $.60920$

$A^{\,1}_{50:\overline{5}|}$: $a_{\overline{5}|}/50$, $.009999998$

${}_5E_{50}$: $v^5(1 - 5/50)$

$A^{\,1}_{40:\overline{10}|}$: $A_{40:\overline{10}|} - {}_{10}E_{40}$, $\underbrace{.61494 - .60920}_{.00574}$

$= .006349231$

$$E[Z^2] = {}^2A^{\,1}_{40:\overline{10}|} + v^{10}\, {}_{10}E_{40}\, \underbrace{{}^2A^{\,1}_{50:\overline{5}|}}_{a_{\overline{5}|}/50 \,@\, 2\delta} + v^{10}\, {}_{10}E_{40} \cdot v^{10}\left(1 - \frac{5}{50}\right) \times {}^2A_{55}$$

${}^2A_{55}$: $.07483$

${}^2A^{\,1}_{40:\overline{10}|}$: ${}^2A_{40} - v^{10}\, {}_{10}E_{40}\, {}^2A_{50}$

${}^2A_{40}$: $.12106$, ${}_{10}E_{40}$: $.60920$, ${}^2A_{50}$: $.18931$

$= 0.12106$

$$\mathrm{Var}[Z] = \underbrace{4 * \left[0.12106 - (.006349231)^2\right]}$$

$= \underline{\underline{0.4840787}}$

**Question No. 4:**

A life insurer has a portfolio of insurance policies consisting of 65% male and 35% female, all of the same age $(x)$ at issue. The forces of mortality for males and females, respectively, are given by

$$\mu^{\mathrm{m}}_{x+t} = 0.048, \quad t > 0$$

and

$$\mu^{\mathrm{f}}_{x+t} = 0.015, \quad t > 0.$$

Let $Z$ be the present value random variable for a whole life insurance of 1 to $(x)$ payable at the moment of death. You are given: $\delta = 0.04$.

For a randomly selected policyholder from this portfolio, calculate the probability that $Z$ will be less than (or equal to) 0.50.

$$P_r(Z \leq 0.50) = P_r(Z^M \leq 0.5 | M) P_r(M) + P_r(Z^F \leq 0.5 | F) P_r(F)$$

$$Z^M = e^{-.04 T_M} \leq 0.5 \qquad Z^F = e^{-.04 T_F} \leq 0.5$$

$$-.04 T_M \leq -\log 2 \qquad -.04 T_F \leq -\log 2$$

$$= P_r(T_M > \log 2/.04)\, 0.65 + P_r(T_F > \log 2/.04)\, 0.35$$

$$= e^{-.048(\log 2/.04)} \times 0.65 + e^{-.015(\log 2/.04)} \times 0.35$$

$$= \underline{\underline{0.5528158}}$$

## Question No. 5:

You are given:

- For age prior to 50, mortality follows a constant force with $\mu = 0.01$.
- For ages 50 and later, mortality is uniformly distributed with $\omega = 115$.
- $\delta = 0.05$
- $Z$ is the present value random variable for a whole life insurance of 1 issued to (40), with benefit payable at the end of the year of death.

Calculate $\Pr[Z \leq 0.65]$.

Density of future lifetime of (40) is $f_T(t) = \begin{cases} .01e^{-.01t}, & 0 \leq t < 10 \\ e^{-.10}\frac{1}{65}, & 10 \leq t < 75 \end{cases}$

The event $Z = v^{K+1} \leq 0.65$ is equivalent to

$$(K+1)\log v \leq \log(0.65) \Rightarrow K \geq \underbrace{\frac{\log 0.65}{-\delta} - 1}_{7.615658} \qquad \delta = .05$$

$$\Pr[Z \leq 0.65] = \Pr[K \geq 7.615658]$$
$$= \Pr[K > 7] = \Pr[T > 7]$$
$$= e^{-.01(7)}$$
$$= 0.9323938$$

## Question No. 6:

You and a friend are studying together for Math 3630 midterm exam. One of the practice problems asked to compute the value for $A_{50}$ based on $i = 0.05$.

Your friend calculated the value and came up with 0.357. But the correct answer turned out to be 0.353. She asked you to review her work and you discovered that she used the following set of mortality assumptions:

$$p_{50} = 0.990 \quad \text{and} \quad p_{51} = 0.980.$$

You realized then that all assumptions in her calculations were correct, except for $p_{50}$.

What must have been the correct value for $p_{50}$?

Using recursion, $A_{50} = vq_{50} + vp_{50}A_{51}$

$= v - vp_{50}(1 - A_{51})$

friend: $0.357 = v - v(.99)(1 - A_{51})$

correct: $0.353 = v - v\,p_{50}(1 - A_{51})$

deduct $0.004 = v(1 - A_{51})(p_{50} - .99)$

So that $1 - A_{51} = \dfrac{.004(1.05)}{p_{50} - .99}$

$\Rightarrow v(.99)(1 - A_{51}) = v - .357$

divide

$\rightarrow 1 - A_{51} = \dfrac{v - .357}{v(.99)}$

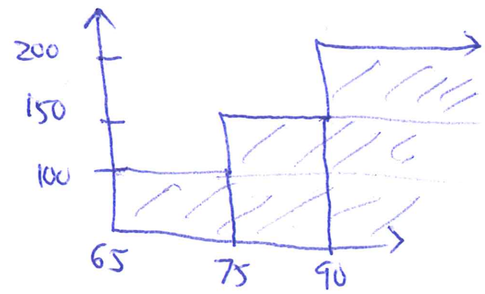
$\dfrac{v - .357}{\cancel{v}(.99)} \cdot \dfrac{p_{50} - .99}{.004(\cancel{1.05})} = 1 \Rightarrow p_{50} = \dfrac{.99(.004)}{v - .357} + .99$

$= \underline{\underline{.9966512}}$

## Question No. 7:

A special whole life annuity-immediate is issued to (65) with the following increasing scale of benefit payments:

| ages | payments |
|---|---|
| [65 – 75) | 100 |
| [75 – 90) | 150 |
| 90 and later | 200 |




You are given:

- The benefits are payable annually.
- Mortality follows the `Survival Ultimate Life Table`.
- $i = 0.05$

Calculate the actuarial present value of this life annuity-immediate.

Since table gives life annuity-due values, better to view payments at age 66 and then discount with life to age 65. In effect, we have

$$APV(\text{annuity}) = {}_1E_{65}\left(100\ddot{a}_{66} + 50\ddot{a}_{76} \cdot {}_{10}E_{66} + 50\ddot{a}_{91}\ {}_{25}E_{66}\right)$$

$\ddot{a}_{66} = 13.2557 \qquad \ddot{a}_{76} = 9.9674 \qquad \ddot{a}_{91} = 4.8858$

$${}_1E_{65} = v\,p_{65} = \frac{1}{1.05}(1-.005915) = 0.9467476$$

$${}_{10}E_{66} = 0.54609$$

$${}_{25}E_{66} = {}_{20}E_{66}\ {}_5E_{86} = (.23112)(.51122) = 0.1273749$$

Plug values, we get

$$APV(\text{annuity}) = \underline{\underline{1542.102}}$$

## Question No. 8:

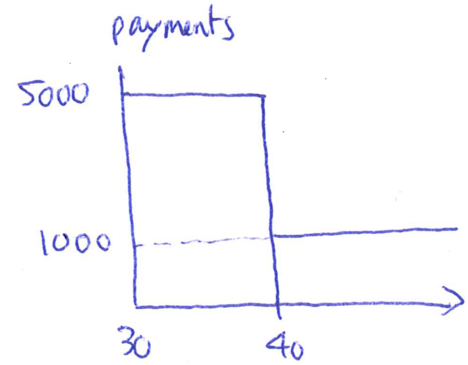
For a special type of whole life annuity-due, payable monthly, issued to (30), you are given:

- Annuity payments are 5,000 per year for the first 10 years and 1,000 thereafter.
- Deaths are uniformly distributed over integral ages.
- $i = 0.05 \qquad \alpha(12) = 1.00020 \qquad \beta(12) = 0.46651$
- The following table of actuarial present values:

| $x$ | $A_x$ | ${}_5E_x$ |
|---|---|---|
| 30 | 0.112 | 0.780 |
| 35 | 0.139 | 0.779 |
| 40 | 0.172 | 0.777 |

$$\ddot{a}_{30} = \frac{1-A_{30}}{d} = 18.648$$

$$\ddot{a}_{40} = \frac{1-A_{40}}{d} = 17.388$$




Calculate the Actuarial Present Value (APV) of this life annuity-due.

$$APV(\text{annuity}) = 5000\,\ddot{a}_{30}^{(12)} - 4000\,{}_{10}E_{30}\,\ddot{a}_{40}^{(12)}$$

$$\ddot{a}_{30}^{(12)} = \alpha(12)\,\ddot{a}_{30} - \beta(12) = \underbrace{1.00020(18.648) - 0.46651}_{18.18522}$$

$$\ddot{a}_{40}^{(12)} = \alpha(12)\,\ddot{a}_{40} - \beta(12) = \underbrace{1.00020(17.388) - 0.46651}_{16.92497}$$

$${}_{10}E_{30} = {}_5E_{30} * {}_5E_{35} = (0.780)(0.779) = 0.60762$$

$$= 5000\,(18.18522) - 4000\,(0.60762)(16.92497)$$

$$= \underline{\underline{49{,}790.30}}$$

**Question No. 9:**

You are given:

- $Y$ is the present value random variable of a (discrete) 2-year deferred 2-year temporary life annuity-due of 1 per year to (95).
- The following extract from a mortality table:

| $x$ | 94 | 95 | 96 | 97 | 98 | 99 | 100 |
|---|---|---|---|---|---|---|---|
| $\ell_x$ | 100 | 85 | 65 | 40 | 15 | 5 | 0 |

- $i = 0.05$

Calculate the standard deviation of $Y$.

payments 0 0 1 1 0

95 96 97 98

$$Y = \begin{cases} v^2, & \text{w.p. } p_{95}\, p_{96}\, q_{97} \\ v^2 + v^3, & \text{w.p. } p_{95}\, p_{96}\, p_{97} \end{cases}$$

$$\frac{l_{97}}{l_{95}}\left(1 - \frac{l_{98}}{l_{97}}\right) = \frac{40}{85}\left(1 - \frac{15}{40}\right) = 0.2941176$$

$$\frac{l_{98}}{l_{95}} = \frac{15}{85} = 0.1764706$$

Thus,

$$E(Y) = 0.2941176\, v^2 + 0.1764706(v^2 + v^3) = 0.5792793$$

$$E(Y^2) = 0.2941176\, v^4 + 0.1764706(v^2 + v^3)^2 = 0.7953778$$

$$SD(Y) = \sqrt{E(Y^2) - [E(Y)]^2}$$

$$= \sqrt{0.7953778 - (0.5792793)^2} = \underline{\underline{0.6780953}}$$

**Question No. 10:**

Based on the same mortality and interest assumptions, you are given:

- $\ddot{a}_{65}^{(4)} = 3.61622$ using the Woolhouse's approximation with three terms.
- $\ddot{a}_{65}^{(6)} = 3.57392$ using the Woolhouse's approximation with three terms.

Calculate $\ddot{a}_{65}^{(12)}$ using the Woolhouse's approximation with three terms.

Recall: $\ddot{a}_{65}^{(m)} = \ddot{a}_{65} - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\mu_{65}+\delta)$

$$\ddot{a}_{65}^{(4)} = \cancel{\ddot{a}_{65}} - \frac{3}{8} - \frac{15}{192}(\mu_{65}+\delta)$$

$$- \quad \ddot{a}_{65}^{(6)} = \cancel{\ddot{a}_{65}} - \frac{5}{12} - \frac{35}{432}(\mu_{65}+\delta)$$

$$\underbrace{(3.61622 - 3.57392)}_{0.0423} = \left(\frac{5}{12} - \frac{3}{8}\right) + \left(\frac{35}{432} - \frac{15}{192}\right)(\mu_{65}+\delta)$$

$$\Rightarrow \mu_{65}+\delta = \frac{0.0423 - \left(\frac{5}{12} - \frac{3}{8}\right)}{\frac{35}{432} - \frac{15}{192}} = 0.21888$$

$$\Rightarrow \ddot{a}_{65} = 3.61622 + \frac{3}{8} + \frac{15}{192}(0.21888) = 4.00832$$

$$\ddot{a}_{65}^{(12)} \approx \underbrace{\ddot{a}_{65}}_{4.00832} - \frac{11}{24} - \frac{143}{1728}(0.21888)$$

$$= \underline{\underline{3.531873}}$$

### Question No. 11:

For a group of $m$ lives age $x$ with independent future lifetimes, you are given:

- Each life is to be paid 1 at the beginning of each year, if alive.
- $i = 0.05$
- $\ddot{a}_x = 9.62$ — $A_x = 1 - d\ddot{a}_x = 0.5419048$
- ${}^2A_x = 0.32$
- $Y$ is the present value random variable of the aggregate payments.
- Using the normal approximation, $F = 2,550$ is the initial size of the fund needed to be 90% certain of being able to make the aggregated payments.
- The 90-th percentile of a standard normal distribution is 1.282.

$$Var(Y_1) = \frac{1}{d^2}\left[{}^2A_x - (A_x)^2\right] = 11.6156$$

Calculate $m$.

$$E(Y) = m\ddot{a}_x = 9.62m$$

$$Var(Y) = m\,Var(Y_1) = 11.6156m$$

$$F = 1.282\sqrt{11.6156m} + 9.62m = 2550$$

$$9.62m + \underbrace{1.282\sqrt{11.6156}}_{4.36927}\sqrt{m} - 2550 = 0$$

Using quadratic formula, $\sqrt{m} = \dfrac{-4.36927 \pm \sqrt{4.36927^2 + 4(9.62)(2550)}}{2(9.62)}$

take positive square root

$$\sqrt{m} = 16.03122$$

$m = 16.03122^2 = \underline{257}$ policies

**Question No. 12:**

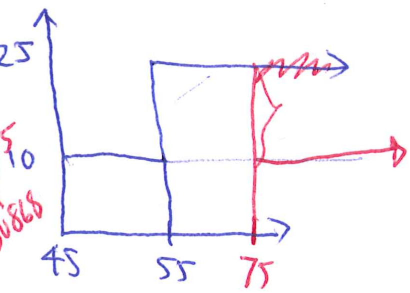
Consider a special whole insurance policy issued to (45). You are given:

$${}_{30}E_{45} = {}_{20}E_{45} \; {}_{10}E_{65}$$

$$.35994 \qquad .55305$$

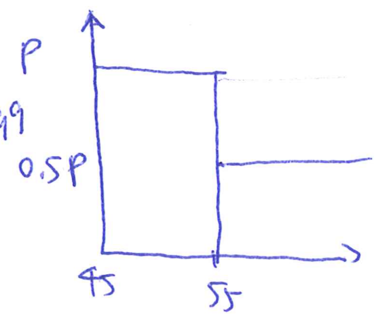
- Death benefit is payable at the end of the year of death.
- The benefit if 10 if death occurs in the first 10 years, 25 if death occurs the following 20 years, and back to 10 for deaths thereafter.
- Level annual premiums $P$ are paid at the beginning of each year for 10 years and decreasing to $0.5P$ per year thereafter.
- Mortality follows the **Survival Ultimate Life Table** with $i = 0.05$.

Calculate $P$ according to the actuarial equivalence principle.




$$APV(\text{benefit}) = 10 \underset{.15161}{A_{45}} + 15 \; \overset{.60655}{{}_{10}E_{45}} \; \underset{.23524}{A_{55}} - 15 \; {}_{30}E_{45} \; \underset{.150868}{A_{75}}$$

$$= \cancel{3.161613712} \quad 2.137468$$




$$APV(\text{premiums}) = P \underset{17.8162}{\ddot{a}_{45}} - 0.5P \; {}_{10}E_{45} \; \overset{16.0599}{\ddot{a}_{55}}$$

Equating by the equivalence principle, we set

$$P = \frac{\cancel{3.161613712} \quad 2.137468}{17.8162 - 0.5(.60655)(16.0599)} = \frac{\overset{2.137468}{\cancel{3.161613712}}}{12.94563}$$

$$= \cancel{0.2442406} \quad 0.1651111 \approx .165$$

EXTRA PAGE FOR ADDITIONAL OR SCRATCH WORK